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Casimir Operators and Nilpotent Radicals

J. C. Ndogmo

Abstract. It is shown that a Lie algebra having a nilpotent radical has a fundamental set of invariants consisting of Casimir operators. A different proof is given in the well known special case of an abelian radical. A result relating the number of invariants to the dimension of the Cartan subalgebra is also established.

1 Introduction

An important problem arising in the representation theory of Lie groups is the determination of invariants of a given representation. If ρ is a representation of the Lie group G in a finite-dimensional vector space V, the invariants of ρ are elements $\mathbf{v}$ of V for which the equality $\rho(g)\mathbf{v} = \mathbf{v}$ holds for all $g \in G$. A map ρ is a representation of a connected Lie group G in the finite-dimensional vector space V if and only if its differential $d\rho$ is a representation of the Lie algebra L of G in V. Moreover, for any $\mathbf{v} \in V$ we have $\rho(G)\mathbf{v} = \mathbf{v}$ if and only if $d\rho(L)\mathbf{v} = 0$, and the latter condition defines $\mathbf{v}$ as an invariant of $d\rho$. Invariants of G and L are therefore the same, and in general they are more easily analyzed on Lie algebras.

When ρ is the adjoint representation, Ad, of G in its Lie algebra L, the invariants of the corresponding representation of G in the symmetric algebra $S(L)$ are called the polynomial invariants of L. Note that $S(L)$ is itself isomorphic to the ring $\mathbb{K}[X_1, \dots, X_n]$ of polynomials in n indeterminates over the base field $\mathbb{K}$ of L, where $n = \dim L$. On the other hand the algebra S^I generated in $S(L)$ by invariant polynomial functions is algebraically isomorphic to the algebra of Casimir operators ([7]), which is the center $\mathcal{Z}(L)$ of the universal enveloping algebra $\mathfrak{A}(L)$ of L. The transcendence degree of S^I over $\mathbb{K}$, which is the cardinality r of a maximal set of algebraically independent elements of S^I, is therefore the same as for $\mathcal{Z}(L)$.

More generally, if we denote by $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ a basis of L and by $\psi(\mathbf{v}_i)$ the associated infinitesimal generators (see [15, p. 52]) of the adjoint action of G on L, then the solutions F of the systems of linear first order partial differential equations

$$\psi(\mathbf{v}_i) \cdot F = 0, \qquad \text{for } i = 1, \dots, n, \tag{1.1}$$

defined by the vector fields $\psi(\mathbf{v}_i)$ are termed (formal) *invariants* of L. A maximal set of functionally independent solutions to (1.1) is referred to as a *fundamental set of invariants* of L, and its elements are called *fundamental invariants*. By the number of invariants of L we shall mean the number of fundamental invariants. It is clear that

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polynomial invariants of L also satisfy (1.1), so that $r \leq \mathcal{N}$, where $\mathcal{N}$ is the number of fundamental invariants and equals the index of L ([9, p. 64]). However, for a general Lie algebra, and especially for solvable ones, the equality $r = \mathcal{N}$ does not hold ([1, 14]). Over the last decade, the study of invariant functions of Lie algebras has been the subject of intensive research undertaken by various researchers, and the problem of existence of polynomial invariants has always been inherent in these studies (see [3–5, 14, 16, 17, 20, 21] and the references therein).

In this paper we show that a Lie algebra having a nilpotent radical must have a fundamental set of invariants consisting of Casimir operators, and we give a different proof of this fact in the well-known case of an abelian radical ([17]). We also derive a result relating the number of invariants to the dimension of the Cartan subalgebra of L. We shall assume that the base field of all Lie algebras is of characteristic zero, and all Lie algebras considered are assumed to be finite-dimensional.

2 Characterization of Invariants

We write the Levi decomposition of any given finite-dimensional Lie algebra L in the form $L = \mathcal{S} \dotplus \mathcal{R}$, where $\mathcal{S}$ is the Levi factor and the ideal $\mathcal{R}$ is the radical of L. This decomposition is said to be nontrivial if $[\mathcal{S}, \mathcal{R}] \neq 0$, and in such case L is neither semisimple nor nilpotent.

Lemma 2.1 *A Lie algebra L having a nilpotent radical is isomorphic to an algebraic Lie algebra.*

Proof By Ado's theorem, L has a faithful representation ϕ in a finite-dimensional vector space V in which elements in the nilradical of L are represented by nilpotent endomorphisms ([11, p. 203]). Since the radical $\mathcal{R}$ of L is nilpotent, it is equal to its nilradical. The Lie algebra $\phi(\mathcal{R})$ therefore consists of nilpotent endomorphisms and is consequently algebraic ([6, p. 123, Proposition 14]). Moreover, $\phi(\mathcal{R})$ is the radical of $\phi(L)$, and a subalgebra of the Lie algebra $\mathfrak{gl}(V)$ is algebraic if and only if its radical is algebraic ([6, p. 129]). Consequently, $\phi(L)$ is algebraic. ■

All semisimple and nilpotent Lie algebras naturally belong to the class of Lie algebras having a nilpotent radical. This is also the case for all perfect Lie algebras, and more generally for Lie algebras which are derived subalgebras of finite-dimensional Lie algebras, because they all possess a nilpotent radical [11, p. 91, Corollary 2].

By a result of [1], every perfect Lie algebra, *i.e.*, a Lie algebras L for which $[L, L] = L$, has a fundamental set consisting of polynomial invariants. However, we notice that this property also holds for Lie algebras with an abelian radical. Indeed, write the Levi decomposition of L in the form

$$L = \mathcal{S} \oplus_{\pi} \mathcal{R}, \tag{2.1}$$

where π is the restriction to the semisimple Lie algebra $\mathcal{S}$ of the adjoint representation of L in the radical $\mathcal{R}$. If $\mathcal{R}^{\mathcal{S}} = \{v \in \mathcal{R} : \pi(\mathcal{S})v = 0\}$ is the set of invariants of this representation, then, because π is semisimple, we have the direct sum of vector spaces $\mathcal{R} = \mathcal{R}^{\mathcal{S}} \dotplus [\mathcal{S}, \mathcal{R}]$. More precisely, we have the following result proved in [13].

Lemma 2.2 *Let $L = \mathcal{S} \oplus_\pi \mathcal{R}$ be a nontrivial Levi decomposition of L with Levi factor $\mathcal{S}$, and suppose that the representation π defines the $[\mathcal{S}, \mathcal{R}]$-type commutation relations.*

(i) *If π does not possess a copy of the trivial representation, then L is perfect, and it has therefore a fundamental set of invariants consisting of polynomials.*

(ii) *The representation π does not possess a copy of the trivial representation if and only if $\pi(\mathcal{S})\mathcal{R} = \mathcal{R}$.*

Using this lemma, we can readily prove the following result.

Theorem 2.3 *Let $L = \mathcal{S} \oplus_\pi \mathcal{R}$ be a Levi decomposition of L.*

(i) *The Lie algebra L is perfect if and only if $\mathcal{R}^{\mathcal{S}} \subseteq [\mathcal{R}, \mathcal{R}]$.*

(ii) *If the radical $\mathcal{R}$ of L is abelian, then L has a fundamental set of invariants consisting of polynomial functions.*

Proof It is clear from the definitions that L is perfect if and only if

$$\mathcal{R} = [\mathcal{S}, \mathcal{R}] + [\mathcal{R}, \mathcal{R}].$$

Rewriting the right-hand side of this last equality as a direct sum $[\mathcal{S}, \mathcal{R}] \dotplus W \cap [\mathcal{R}, \mathcal{R}]$ of vector subspaces, where W is a complement subspace of $[\mathcal{S}, \mathcal{R}]$ in $\mathcal{R}$, we see that L is perfect if and only if $\mathcal{R}^{\mathcal{S}} \subseteq [\mathcal{R}, \mathcal{R}]$, which proves part (i). For part (ii), we first note that by Lemma 2.2(i), if the representation π does not possess a copy of the trivial representation, then L is perfect, and the result follows. If π does have a copy of the trivial representation, then $\mathcal{R}^{\mathcal{S}} \neq 0$, and by part (i) above, L is not perfect. However, L is in this case a direct sum of the perfect ideal $\mathcal{S} \dotplus [\mathcal{S}, \mathcal{R}]$ and the abelian ideal $\mathcal{R}^{\mathcal{S}}$. It then follows again that L has a fundamental set of invariants consisting of polynomials. ■

Lemma 2.4 *A Lie algebra with a nilpotent radical has a fundamental set of invariants consisting of rational functions.*

Proof Indeed, by Lemma 2.1 any such Lie algebra is isomorphic to an algebraic Lie algebra, and the lemma follows from a result of J. Dixmier [8], asserting that any algebraic Lie algebra has a fundamental set of invariants consisting of rational functions. ■

In the sequel we shall denote by $\mathrm{Fract}(A)$ the field of fractions of a Noetherian and integral ring A. Set $\mathfrak{K}(L) = \mathrm{Fract}(\mathfrak{A}(L))$, and $\mathscr{K}(L) = \mathrm{Fract}(S(L))$, and for each $x \in L$, denote by $\mathrm{ad}_{\mathscr{K}(L)}\, x$ the derivation of $\mathscr{K}(L)$ that extends $\mathrm{ad}_L\, x$ and thus defines a representation of L in $\mathscr{K}(L)$. The invariants of this representation are called the rational invariants of L. Similarly, for each $x \in L$, denote by $\mathrm{ad}_{\mathfrak{K}(L)}\, x$ the derivation of $\mathfrak{K}(L)$ that extends $\mathrm{ad}_L\, x$. Finally, denote by $\mathcal{C}(L)$ and $\mathscr{C}(L)$ the center of $\mathfrak{K}(L)$ and $\mathscr{K}(L)$ respectively, when they are endowed with the adjoint representation. By a result of [19], $\mathcal{C}(L)$ and $\mathscr{C}(L)$ are isomorphic fields. Moreover, we have the following result ([9]), in which L^* denotes the dual vector space of L.

Lemma 2.5 *We have $f \in \mathscr{C}(L)$ if and only if there exists some weight $\lambda \in L^*$ of the adjoint representation of L in $S(L)$ such that $f = p_1/p_2$, where $p_1, p_2 \in S(L)_\lambda = \{p \in S(L) : \mathrm{ad}_{S(L)}\, x\,(p) = \lambda(x)p,\ \text{for all } x \in L\}$.*

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If λ is in L^*, an element of $S(L)_\lambda$ is called a λ-semi-invariant of L in $S(L)$. It is clear that if the weight space of any such λ is not reduced to zero, then λ defines a one-dimensional representation of L in $\mathbb{K}$, which vanishes on any perfect subalgebra of L.

Theorem 2.6 *A Lie algebra L with a nilpotent radical has a fundamental set of invariants consisting of Casimir operators.*

Proof By Lemma 2.1 L is isomorphic to an algebraic Lie algebra, since it has a nilpotent radical. It has therefore a fundamental set of invariants that consists of rational invariants, by Lemma 2.4. In this case, a sufficient condition for L to have only polynomial invariants is for the equality $\mathscr{C}(L) = \operatorname{Fract}(S^I)$ to hold, by a result of [9], since S^I is the center of $S(L)$. Because of Lemma 2.5, it suffices to verify that the only weight of the adjoint representation of L in $S(L)$ is zero. Consider the Levi decomposition $L = \mathcal{S} \dotplus \mathcal{R}$, and let λ be any weight of $\operatorname{ad}_{S(L)}$. If $x \in \mathcal{S}$, then clearly $\lambda(x) = 0$. On the other hand if $x \in \mathcal{R}$, $\operatorname{ad}_L x$ is nilpotent and $\operatorname{ad}_{S(L)} x$ is locally nilpotent, and hence $\lambda(x) = 0$. The rest of the theorem follows from the isomorphism between $\mathscr{C}(L)$ and $\mathcal{C}(L)$. ■

Special cases of this result are known for L semisimple ([9, 18]), L nilpotent ([2]), and L perfect ([1]). Theorem 2.6 therefore unifies and extends seemingly unrelated results asserting that fundamental invariants of these special types of Lie algebras can all be chosen as Casimir operators, and shows that the only Lie algebras that may not have a fundamental set of invariants consisting of Casimir operators are to be found only among Lie algebras with a non nilpotent radical, and in particular among solvable non nilpotent Lie algebras.

3 The Number of Invariants

We shall now derive some results concerning the number of invariants of the Lie algebra $L = \mathcal{S} \dotplus \mathcal{R}$ when the radical $\mathcal{R}$ is nilpotent. We assume for this purpose that L has a nontrivial Levi decomposition.

Recall that for a given Lie algebra L, the rank of L is the minimum multiplicity of the eigenvalue $\lambda = 0$ over the set of all linear operators $\operatorname{ad}_L x$, as x runs through the entire Lie algebra L. On the other hand, a Cartan subalgebra of L is a nilpotent subalgebra that is self-normalizing, and the rank of L equals the dimension of any of its Cartan subalgebras. In case L is semisimple, the cardinality of a fundamental set of invariants equals the rank of the Lie algebra ([18]). However, for nilpotent Lie algebras, it is clear that the equality between number of invariants and rank of the algebra does not hold. We investigate this equality in the case of a nontrivial Levi decomposition and a nilpotent radical.

Let $\{x, y, h\}$ be the standard basis of $\mathfrak{sl}(2, \mathbb{K})$, so that h generates the maximal toral subalgebra.

Theorem 3.1 *Suppose that $L = \mathcal{S} \oplus_\pi \mathcal{R}$, as in (2.1), where π is the representation of $\mathcal{S}$ in $\mathcal{R}$ defining the $[\mathcal{S}, \mathcal{R}]$-type commutation relations. If $\mathcal{S} = \mathfrak{sl}(2, \mathbb{K})$ and $\mathcal{R}_0$ is the weight space of $\mathcal{R}$ relative to h corresponding to the zero weight, then L has rank* $\dim \mathcal{R}_0 + 1$.

Proof Let $\mathcal{R}_\mu$ denote the weight space of $\mathcal{R}$ corresponding to the weight μ relative to h, so that $\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_*$ where $\mathcal{R}_* = \sum_{\mu\neq 0} \mathcal{R}_\mu$. Let $A = \mathcal{H}_0 \dotplus \mathcal{R}$, where $\mathcal{H}_0$ is the Cartan subalgebra of $\mathcal{S}$ generated by h. Then $\mathcal{H} = \mathcal{H}_0 \dotplus \mathcal{R}_0$ is a Cartan subalgebra of A. Indeed, if $v \in \mathcal{R}_*$, then $[h, v] = \mu v$, with $\mu \neq 0$. Hence $\mathcal{H}$ is self-normalizing in A. Clearly, $\mathcal{R}_0$ is a nilpotent subalgebra of $\mathcal{R}$ and we easily see that $\mathcal{H}^{(n)} = \mathcal{R}_0^{(n)}$ for all n, where $\mathfrak{L}^{(n)}$ denotes the $(n+1)$-th term of the descending central series of a given Lie algebra $\mathfrak{L}$. Therefore, $\mathcal{H}$ is nilpotent, and hence it is a Cartan subalgebra of A. But since A is itself the inverse image of the Cartan subalgebra of $\mathcal{S}$ under the canonical surjection $L \to L/\mathcal{R}$, with $L/\mathcal{R} \simeq \mathcal{S}$, it follows from [10, Lemma 15.3 B] that any Cartan subalgebra of A is also a Cartan subalgebra of L. Hence $\mathcal{H}$ is also a Cartan subalgebra of L, and the result follows from the equality $\dim \mathcal{H}_0 = 1$. ∎

For our first example concerning the connection between the number of invariants and the rank of the algebra in the case of a nilpotent radical, we shall also need the following result, proved in [13], with $\mathcal{S} = \mathfrak{sl}(2, \mathbb{K})$ and π irreducible.

Theorem 3.2 *Suppose that the radical $\mathcal{R}$ has dimension d. Then the number σ of invariants of $L = \mathcal{S} \oplus_\pi \mathcal{R}$ is given by*

(i) $\sigma = 2$, *for* $d = 1$,
(ii) $\sigma = 1, 2$, *for* $d = 2, 3$, *respectively,*
(iii) $\sigma = d - 3$, *for* $d \geq 4$.

Example 1 Suppose that in Theorem 3.1 the representation π defining the $[\mathcal{S}, \mathcal{R}]$-types commutation relations of L is irreducible. In this case the rank of L is clearly $1 + (\dim \mathcal{R} \pmod 2)$, and in particular this rank can only be either 1 or 2. It readily follows from Theorem 3.2 that the number of invariants coincides with the rank of L when $\dim \mathcal{R} < 6$, but this is no longer true when $d \geq 6$, as the number of invariants is $d-3$ for $d \geq 4$. Of course, when π is irreducible, the radical $\mathcal{R}$ is abelian, and hence nilpotent. We then conclude that contrary to the case of semisimple Lie algebras, the number of invariants of Lie algebras with a nilpotent radical is not equal to the rank of the Lie algebra.

More precisely, the following general result follows from Theorem 3.1 and Theorem 3.2.

Corollary 3.3 *Suppose that the Lie algebra L has a Levi decomposition of the form $L = \mathcal{S} \oplus_\pi \mathcal{R}$ as in* (2.1)*, with $\mathcal{S} = \mathfrak{sl}(2, \mathbb{K})$ and π irreducible. Then the number of fundamental invariants of L is equal to the dimension of its Cartan subalgebra if and only if* $\dim \mathcal{R} < 6$.

Despite the fact that, contrary to the semisimple case, the maximal number of functionally independent invariants is not equal to the rank of the Lie algebra for Lie algebras with a nilpotent radical, these Lie algebras have however one similarity with semisimple Lie algebras regarding the number of invariants. Indeed, both types of Lie algebras have each at least one nontrivial Casimir operator ([9, p. 163]). Of course this property does not hold for solvable Lie algebras and the simplest example is given by the two-dimensional Lie algebra with commutation relations $[X, Y] = Y$.

4 Concluding Remarks

Due to the important role played by Casimir operators in representation theory and other branches of mathematics, and in physics, the problem of determination of Lie algebras having a fundamental set of invariants consisting of polynomials has always been inherent in the study of invariants of Lie algebras, from the early years of investigation of these invariant functions [2, 18], and up to the most recent years [4, 5, 14, 16, 17, 20]. We have therefore proved the general and unifying result that Lie algebras with a nilpotent radical admit a fundamental set of invariants consisting of Casimir operators (Theorem 2.6), and although this result is not true for Lie algebras with a non nilpotent radical, it might still be possible, however, to extend the result of Theorem 2.6 to a class to Lie algebras of the latter type. Indeed, there are known solvable non nilpotent Lie algebras having a fundamental set of invariants consisting of Casimir operators. This is, for example, the case for the four-dimensional Lie algebra L with commutation relations $[N_2, N_3] = N_1$, $[N_2, X_1] = N_2$, and $[N_3, X_1] = -N_3$, where the nilradical is generated by N_1, N_2, N_3. This Lie algebra has indeed a fundamental set of invariants consisting of the polynomials $F_1 = n_1$, and $F_2 = n_2 n_3 - n_1 x_1$, where $\{n_1, n_2, n_3, x_1\}$ is a coordinate system associated with the basis $\{N_1, N_2, N_3, X_1\}$ of L. A determination of all solvable non nilpotent Lie algebras satisfying this property is still however to be found.

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References

[1] L. Abellanas and L. Martinez Alonso, *A general setting for Casimir invariants.* J. Mathematical Phys. **16**(1975), 1580–1584. http://dx.doi.org/10.1063/1.522727

[2] P. Bernat, *Sur le corps des quotients de l'algèbre enveloppante d'une algèbre de Lie.* C. R. Acad. Sci. Paris. **254**(1962), 1712–1714.

[3] V. Boyko, J. Patera, and R. Popovych, *Invariants of triangular Lie algebras.* J. Phys. A **40**(2007), no. 27 7557–7572. http://dx.doi.org/10.1088/1751-8113/40/27/009

[4] ______, *Invariants of triangular Lie algebras with one nil-independent diagonal element.* J. Phys. A **40** (2007), no. 32, 9783–9792. http://dx.doi.org/10.1088/1751-8113/40/32/005

[5] R. Campoamor-Stursberg, *Some remarks concerning the invariants of rank one solvable real Lie algebras.* Algebra Colloq. **12**(2005), no. 3, 497–518.

[6] C. Chevalley, *Théorie des groupes de Lie. III.* Hermann, Paris, 1955.

[7] J. Dixmier, *Sur l'algébre enveloppante d'une algébre de Lie nilpotente.* Arch. Math. **10**(1959), 321–326. http://dx.doi.org/10.1007/BF01240805

[8] ______, Sur les représentations unitaires des groupes de Lie nilpotents II, Bull. Soc. Math. France 85 (1957), 325–388.

[9] ______, *Algèbres enveloppantes.* Gauthier-Villars, Paris, 1974.

[10] J. E. Humphreys, *Introduction to Lie algebras and Representation Theory.* Graduate Texts in Mathematics 9. Springer-Verlag, New York 1972.

[11] N. Jacobson, *Lie Algebras.* Interscience Tracts in Pure and Applied Mathematics 10. John Wiley & Sons, New York, 1962.

[12] E. D. Letellier, *Deligne-Lusztig induction for invariant functions on finite Lie algebras of Chevalley's type.* Tokyo J. Math. **28**(2005), no. 1, 265–282. http://dx.doi.org/10.3836/tjm/1244208292

[13] J. C. Ndogmo, *Invariants of a semi-direct sum of Lie algebras.* J. Phys. A **37**(2004), no. 21, 5635–5647. http://dx.doi.org/10.1088/0305-4470/37/21/009

[14] ______, *Properties of the invariants of solvable Lie algebras.* Canad. Math. Bull. **43**(2000), no. 4, 459–471. http://dx.doi.org/10.4153/CMB-2000-054-0

[15] P. J. Olver, *Applications of Lie Groups to Differential Equations.* Second edition. Graduate Texts in Mathematics 107. Springer-Verlag, New York, 1993.
[16] J. N. Pecina-Cruz, *An algorithm to calculate the invariants of any Lie algebra.* J. Math. Phys. **35**(1994), no. 6, 3146–3162. http://dx.doi.org/10.1063/1.530458
[17] M. Perroud, *The fundamental invariants of inhomogeneous classical groups.* J. Math. Phys **24**(1983), no. 6, 1381–1391. http://dx.doi.org/10.1063/1.525870
[18] G. Racah, *Sulla caratterizzazione delle rappresentazioni irriducibli dei gruppi semisemplici di Lie.* Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. **8**(1950), 108–112.
[19] R. Rentschler, *Sur le centre du corps enveloppant d'une algèbre de Lie résoluble.* C. R. Acad. Sci. Paris Sér. A-B **276**(1973), A21–A24.
[20] L. Snobl and P. Winternitz, *A class of solvable Lie algebras and their Casimir invariants.* J. Phys. A **38**(2005), no. 12, 2687–2700. http://dx.doi.org/10.1088/0305-4470/38/12/011
[21] ———, *All solvable extensions of a class of nilpotent Lie algebras of dimension n and degree of nilpotency* $n-1$. J. Phys. A **42** (2009), no. 10, 105201. http://dx.doi.org/10.1088/1751-8113/42/10/105201

School of Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, South Africa
e-mail: jean-claude.ndogmo@wits.ac.za