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CERTAIN CHARACTERISTICS OF THE CURVATURE TENSOR OF AN AFFINELY CONNECTED MANIFOLD

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ABSTRACT. We show how flatness and symmetry of an affinely connected manifold relate themselves to the symmetry of the curvature tensor and its covariant derivative in certain slots. Then we reduce Bianchi first identity to the same form as when the connexion is symmetric; even when the connexion is non-symmetric by considering other conditions.

1. **Introduction.** The curvature tensor K of a differentiable manifold M_n with an affine connexion D is defined by

$$K(X, Y, Z) = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z, \tag{1.1}$$

where X, Y, Z, are arbitrary smooth vector fields on M_n and $[X, Y]$ is the Lie-bracket of X and Y. The torsion tensor S of D is a $(1, 2)$-tensor defined by $S(X, Y) = D_X Y - D_Y X - [X, Y]$. D is called symmetric if $S = 0$. K satisfies Bianchi first identity

$$K(X, Y, Z) + K(Y, Z, X) + K(Z, X, Y) = (D_X S)(Y, Z) + (D_Y S)(Z, X) + (D_Z S)(X, Y) - S(X, S(Y, Z)) - S(Y, S(Z, X)) - S(Z, S(X, Y)) \tag{1.2}$$

and Bianchi second identity

$$(D_X K)(Y, Z, W) + (D_Y K)(Z, X, W) + (D_Z K)(X, Y, W) + K(S(X, Y), Z, W) + K(S(Y, Z), X, W) + K(S(Z, X), Y, W) = 0. \tag{1.3}$$

If M is the $(0, 2)$-tensor obtained by contracting K at the third slot then from [2] we have

$$(D_X M)(Y, Z) + (D_Y M)(Z, X) + (D_Z M)(X, Y) + M(S(Y, Z), X) + M(S(Z, X), Y) + M(S(X, Y), Z) = 0.$$

M_n is called flat iff $K = 0$ and symmetric iff $D_X K = 0$.

2. **Symmetry and flatness.** We have the following

THEOREM 2.1. *If D is symmetric then M_n is flat iff, K is symmetric in the second and third slots.*

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Proof. Suppose $K(X, Y, Z)=K(X, Z, Y)$. Also it is known that $K(X, Z, Y)=-K(Z, X, Y)$. Combining them we find $K(X, Y, Z)+K(Z, X, Y)=0$ which implies in virtue of (1.2) that $K(Y, Z, X)=0$, because $S=0$. The converse follows straightaway.

THEOREM 2.2. *If D is symmetric, then M_n is symmetric iff $(D_X K)(Y, Z, W)$ is symmetric in X and Z.*

Proof follows by the fact that $(D_X K)(Y, Z, W)$ is skew-symmetric in Y, Z and identity (1.3) for $S=0$.

3. **Bianchi first identity**. For a symmetric connection ($S=0$), Bianchi first identity (1.2) reduces to

$$K(X, Y, Z)+K(Y, Z, X)+K(Z, X, Y)=0. \tag{3.1}$$

We generalize this situation in the form of

THEOREM 3.1. *For a differentiable manifold M_n with a connexion D whose torsion and curvature tensors are S and K respectively, Bianchi first identity reduces to* (3.1), *provided M_n has a vector field T parallel with respect to D such that*

$$S=M\otimes T. \tag{3.2}$$

Proof. Let T be a vector field on M_n such that $D_X T=0$, as in an affinely cosymplectic manifold [1] with non-symmetric (ϕ, ξ, η)-connexion [3]. The tensor M is skew-symmetric because K is skew-symmetric in first and second slots. Hence we are justified in assuming (3.2). Therefore we find $D_X S=(D_X M)\otimes T$. Consequently equation (1.2) assumes the form

$$\begin{aligned} K(X, Y, Z)+K(Y, Z, X)+K(Z, X, Y)=(D_X M)(Y, Z)T+(D_Y M) \\ (Z, X)T+(D_Z M)(X, Y)T-M(Y, Z)M(X, T)T-M(Z, X)M(Y, T) \\ T-M(X, Y)M(Z, T)T. \end{aligned} \tag{3.3}$$

Inserting the value of S from (3.2) in (1.4) we find

$$\begin{aligned} (D_X M)(Y, Z)+(D_Y M)(Z, X)+(D_Z M)(X, Y)-M(Y, Z)M \\ (X, T)-M(Z, X)M(Y, T)-M(X, Y)M(Z, T)=0. \end{aligned} \tag{3.4}$$

Exploiting (3.4) in (3.3) we find (3.1). This completes the proof.

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