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# LOCAL FREENESS OF PROFINITE GROUPS

BY
ANDREW PLETCH

ABSTRACT. In this paper we discuss the relationship between local properties such as freeness and projectivity of a group and the freeness or projectivity of its pro-**C**-completion. We show that for certain classes, **C**, of finite groups (e.g. $p$-groups, nilpotent groups, super-solvable groups) the pro-**C**-completion of a locally free pro-**C**-group is a free pro-**C**-group. We also show that under certain circumstances the converse is also true but we leave open the question, for example, of whether a locally free pro-$p$-group is free.

§1. Let $G$ be a group and $\mathcal{G}=\{G_\alpha\}_{\alpha\in\mathcal{A}}$ be the set of all finitely generated subgroups of $G$. To every homomorphism $f$ from $G$ to a group $H$ there corresponds a set $\mathcal{G}(f)=\{f/G_\alpha,$ for all $G_{\alpha\in\mathcal{A}}\}$. Conversely, the set of homomorphisms

$$\mathcal{F}=\{f_\alpha : G_\alpha \to H \mid \text{ if } \quad G_\alpha \le G_\beta \quad \text{then} \quad f_{\beta/G_\alpha}=f_\alpha\}$$

is $\mathcal{G}(f)$ for some $f\in \operatorname{Hom}(G,H)$. $f$ is defined by its restriction to $G_\alpha$ for each $\alpha\in\mathcal{A}$. The correspondence $f\mapsto\mathcal{G}(f)$ is a bijection from $\operatorname{Hom}(G,H)$ to $\varprojlim \operatorname{Hom}(G_\alpha, H)$.

Suppose now that $A$ and $B$ are finite groups and $e:A\to B$ is an epimorphism. We call a group $G$ *f(initely)-projective* if the diagram

$$\begin{array}{ccc} & & G \\ & & \downarrow{\scriptstyle h} \\ A & \xrightarrow{e} & B \end{array}$$

can always be commutatively completed by a morphism $g : G \to A$. Obviously a free group is $f$-projective.

DEFINITION 1.1. A group is called *locally f-projective* if every finitely generated subgroup is $f$-projective.

PROPOSITION 1.2. *A locally f-projective group is f-projective.*

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**Proof.** To say that a group, $G$, is $f$-projective is to say that every epimorphism $e: A \to B$ of finite groups induces a surjection $e': \mathrm{Hom}(G, A) \to \mathrm{Hom}(G, B)$. The sets $\{\mathrm{Hom}(G_\alpha, A)\}_{\alpha\in\mathcal{A}}$ and $\{\mathrm{Hom}(G_\alpha, B)\}_{\alpha\in\mathcal{A}}$ both form inverse systems of finite sets. They are non-empty because the 0-morphism is an element of each and finite because both $A$ and $B$ are finite and each $G_\alpha$ is finitely generated. Since each $G_\alpha$ is $f$-projective the induced function $e'_\alpha: \mathrm{Hom}(G_\alpha, A) \to \mathrm{Hom}(G_\alpha, B)$ is always surjective. It is well-known that $\varprojlim$ is a right exact functor over projective systems of non-empty finite sets [1, III, p. 58] and hence $e': \varprojlim \mathrm{Hom}(G_\alpha, A) \to \varprojlim \mathrm{Hom}(G_\alpha, B)$ is surjective. Hence, due to the correspondence established at the outset of this section, the function $e': \mathrm{Hom}(G, A) \to \mathrm{Hom}(G, B)$ is also surjective and so $G$ is $f$-projective.

Suppose now that $G$ is a Hausdorff topological group. A (topologically) finite generated closed subgroup $G_\alpha$ of $G$ is a subgroup of $G$ closed under the topology of $G$, itself containing an algebraically finitely generated subgroup whose closure in $G$ is $G_\alpha$. Let $\mathcal{G}$ be the set of all such subgroups of $G$.

Definition 1.3. A topological group is *locally some property* $(P)$ if $(P)$ is a property of each $G_\alpha$ in $\mathcal{G}$.

Definition 1.4. $G$ is *t(opologically) f(initely) projective* if the diagram

$$\begin{array}{ccc} & & G \\ & & \downarrow h \\ A & \xrightarrow{e} & B \end{array}$$

can always be commutatively completed by a continuous homomorphism $g: G \to A$ where $A$ and $B$ are finite groups with discrete topology and $h$ is continuous. We use the term $f$-projective for abstract groups or topological groups considered abstractly. We can now prove:

Proposition 1.5. *Let $G$ and $\mathcal{G}$ be as above. Suppose that $G$ has the property that any homomorphism (not necessarily continuous) from $G$ to a finite group is continuous when restricted to any element of $\mathcal{G}$. Then $G$ is $f$-projective if it is locally $tf$-projective.*

**Proof.** The proof is essentially that of Proposition 1.2. The condition placed on $G$ assures that $\mathrm{Hom}_{\mathrm{cont}}(G_\alpha, A)$ for any finite group $A$ with discrete topology, is non-empty and finite since $A$ has a Hausdorff topology. Hence an epimorphism $e: A \to B$ of finite groups induces a surjective function $e': \varprojlim \mathrm{Hom}_{\mathrm{cont}}(G_\alpha, A) \to \varprojlim \mathrm{Hom}_{\mathrm{cont}}(G_\alpha, B)$. By hypothesis there is a bijective correspondence between $\mathrm{Hom}(G, X)$ and $\varprojlim \mathrm{Hom}_{\mathrm{cont}}(G_\alpha, X)$ for any finite group $X$ and hence the result.

REMARKS 1.6. We shall see in the next section that the existence of topological groups which verify the hypothesis of Proposition 1.5 is not uncommon. In the case of profinite groups it is an open question as to whether the condition is always satisfied and is known to be so in certain circumstances.

1.7. EXAMPLES of $f$-projective groups are most easily found when restrictions are placed on the finite groups with respect to which the original group is $f$-projective. For example, torsion-free abelian groups are $f$-projective with respect to finite abelian groups since they are locally free (hence projective) while the restricted Burnside quotients $B^*(p, n)$ (see [3], p. 380 for notation) are $f$-projective with respect to finite groups of exponent $p$ without necessarily being locally $f$-projective.

§2. In this section we assume all groups are profinite (i.e. Hausdorff, compact, totally disconnected) and all morphisms are continuous unless otherwise stated. Profinite groups are all inverse limits of inverse systems of finite groups.

Let $\mathbf{C}$ be a class of finite groups which is assumed to be closed under the formation of subgroups, quotients, and finite direct products. A $\mathbf{C}$-group is a member of $\mathbf{C}$. A pro-$\mathbf{C}$-group is the inverse limit of an inverse system of $\mathbf{C}$-groups. A pro-$\mathbf{C}$-group, $G$, is $\mathbf{C}$-*projective*, (see [2], p. 156), if the diagram

$$\begin{array}{ccc} & & G \\ & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{e} & B \end{array}$$

where $A$ and $B$ are pro-$\mathbf{C}$-groups, and $f$ and $e$ are continuous morphisms, can be commutatively completed by a continuous morphism $g : G \to A$. We extend Proposition 1, [2] to:

PROPOSITION 2.1. *A pro-$\mathbf{C}$-group is $\mathbf{C}$-projective if and only if it is tf-projective with respect to all $\mathbf{C}$-groups.*

A profinite group is called *strongly complete* if every subgroup of finite index is open. For several classes, $\mathbf{C}$, of finite groups it is known that all topologically finitely generated pro-$\mathbf{C}$-groups are strongly complete. For example, if $\mathbf{C}$ is the class of $p$-groups, nilpotent groups, supersolvable groups, and others. We call such a class *strongly complete* as well.

LEMMA 2.2. *Let $G$ be any pro-$\mathbf{C}$-group where $\mathbf{C}$ is strongly complete. Let $A \in \mathbf{C}$ and $f : G \to A$ be any homomorphism (not necessarily continuous). Then $f$, restricted to any finitely generated closed subgroup of $G$ is continuous.*

**Proof.** Since $\mathbf{C}$ is a subgroup-closed, a closed subgroup of a pro-$\mathbf{C}$-group is also a pro-$\mathbf{C}$-group. A morphism from a profinite group to another is continuous if and only if its kernal is closed. Since the kernal of $f$, when restricted to any finitely generated closed subgroup of $G$ is of finite index in the subgroup it is open (hence closed) in the subgroup by the strong completeness of **C**. Hence its restriction is continuous.

THEOREM 2.3. *Let $G$ be a pro-**C**-group where **C** is strongly complete. If $G$ is locally **C**-projective, then $G$ is $f$-projective (as an abstract group) with respect to **C**-groups.*

**Proof.** By Proposition 2.1, $G$ is locally $tf$-projective with respect to **C**-groups. By Lemma 2.2, $G$ verifies the hypothesis of Proposition 1.5. Hence $G$ is $f$-projective with respect to **C**-groups by Proposition 1.5.

For any group, $G$, the pro-**C**-completion, $\hat{G}$, of $G$ is the inverse limit of the projective system of all quotients of $G$ which belong to **C**. There is a canonical homomorphism $i : G \to \hat{G}$ whose kernal is the intersection of all normal subgroups of $G$ such that the quotient found by factoring out the normal subgroup belongs to **C**. This morphism is defined by the universal property of inverse limits.

THEOREM 2.4. *Let $G$ be an abstract group which is $f$-projective with respect to **C**-groups. Then $\hat{G}$ is **C**-projective.*

**Proof.** For any **C**-groups, $A$ and $B$, and any epimorphism $e : A \to B$, $h : \hat{G} \to B$ continuous, the diagram

$$\begin{array}{ccc} G & \xrightarrow{i} & \hat{G} \\ & & \downarrow{\scriptstyle h} \\ A & \xrightarrow{e} & b \end{array}$$

can be commutatively completed by hypothesis by a morphism $k : G \to A$. since $A$ is a **C**-group, the image of $G$ under $k$ is a **C**-group and so the morphism $k$ can be uniquely extended to a continuous morphism $\hat{k} : \hat{G} \to A$ such that $\hat{k} \circ i = k$. Since both $h$ and $e \circ \hat{k}$ extend $\varepsilon \circ k$ and both are continuous the uniqueness of such an extension yields $h = e \circ \hat{k}$. Hence, by Proposition 2.1, $\hat{G}$ is **C**-projective.

COROLLARY 2.5. *Let $G$ be a pro-**C**-group, **C** a strongly complete class, and further suppose $G$ is locally **C**-projective. Then $\hat{G}$ is **C**-projective.*

**Proof.** By Theorem 2.3, $G$ is $f$-projective (as an abstract group) with respect to **C**-groups. Hence the theorem applies.

If **C** is the class of $p$-groups we write pro-$p$ for pro-**C**. A function $f : X \to G$ from a set X to a profinite group $G$ is said to *converge to zero* if every open neighbourhood of the identity contains almost all of $f(X)$. A pro-$p$-group $F$ is said to be *free on a set X* if there is a function $e : X \to F$ converging to zero such that every function $f: X \to G$ to a pro-$p$-group $G$ which converges to zero can be uniquely extended to a continuous homomorphism $f': F \to G$ such that $f' \circ e = f$.

COROLLARY 2.6. *The pro-p-completion of a free pro-p-group is a free pro-p-group.*

**Proof.** By Gruenberg, [2], p. 164, for pro-$p$-groups, freeness and projectivity are equivalent. All closed subgroups of a free pro-$p$-group are free, [4] p. 236, and the class of $p$-groups is strongly complete, [5]. Hence all free pro-$p$-groups verify the conditions of Corollary 2.5.

In general one does not have that a pro-**C**-group which is **C**-projective is also locally **C**-projective. Hence one cannot say that the pro-**C**-completion of a **C**-projection group is **C**-projective. In [2], p. 159, a class **C** of finite groups is defined as being *saturated* if $A \in \mathbf{C}$ whenever $A/\text{Frat}(A) \in \mathbf{C}$. Frat$(A)$ being the Frattini subgroup of $A$. For saturated classes, **C**, a pro-**C**-group, $G$, is **C**-projective if and only if the cohomological dimension of $G$ with respect to coefficient modules in **C** is $\leq 1$, [2, Theorem 4]. Since closed subgroups of pro-**C**-groups never have larger cohomological dimension, [4], p. 204, one has that **C**-projective pro-**C**-groups are locally **C**-projective and hence

COROLLARY 2.7. *Let* **C** *be a saturated strongly complete class of finite groups and G a pro-***C***-group which is* **C***-projective. Then* $\hat{G}$ *is* **C***-projective.*

§3. In this section we restrict ourselves to pro-$p$-groups and consider the relationship between freeness and local freeness of such groups. If $G$ is a pro-$p$-group, $\hat{G}$ will denote its pro-$p$-completion. It has been noted earlier that for pro-$p$-groups, freeness and $p$-projectivity are equivalent. Corollary 2.5 states, in particular, that the pro-$p$-completion of a locally free pro-$p$-group is a free pro-$p$-group. The converse is also true.

PROPOSITION 3.1. *Let G be a pro-p-group such that* $\hat{G}$ *is free, then G is locally free.*

**Proof.** $G$ canonically embeds in $\hat{G}$, call the embedding $h$. $h(G)$ is dense in $\hat{G}$ and the relative topology on $h(G)$ has as open neighbourhoods of the identity of $h(G)$ all subgroups of finite index in $h(G)$. For any finitely generated closed subgroup $H$ of $G$, the morphism $h$ maps $H$ homeomorphically to $h(H)$ since all subgroups of finite index in $H$ are already open by the strong completeness of the class of $p$-groups. Hence $h(H)$ is a compact subgroup of $\hat{G}$ and therefore closed. Since all closed subgroups of a free pro-$p$-group are free as well, we have that $h(H)$, and hence $H$, is free.

The previous proposition leads to the question: Is a locally free pro-$p$-group free? We have the following partial result:

THEOREM 3.2. *Let $G$ be a locally free pro-$p$-group and further suppose that $G \cong \hat{G}_0$ for some group $G_0$. Then $G$ is free.*

**Proof.** Without loss of generality we may assume that $G_0$ is a subgroup of $G$. The embedding $h : G \to \hat{G}$ therefore also embeds $G_0$ in $\hat{G}$. Let $\overline{h(G_0)}$ be the closure of $h(G_0)$ in $\hat{G}$. Since $\hat{G}$ is free, $\overline{h(G_0)}$ is free as well. Moreover, since $G \cong \hat{G}_0$, there is a continuous epimorphism $h' : G \to \overline{h(G_0)}$ defined as the extension of the inclusion $h : G_0 \to \overline{h(G_0)}$.

There exists, as well, another continuous epimorphism $\pi : \hat{G} \to G$ and it is trivial to show that $\pi \circ h' = 1_G$. Hence $h'$ is a monomorphism and hence a homeomorphism. Therefore $G$ is also a free pro-$p$-group.

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INSTITUTO DE MATHEMÁTICA
UNIVERSIDADE DE SÃO PAULO
SÃO PAULO, BRASIL

present address:

DEPARTAMENTO DE MATEMÁTICAS
UNIVERSIDAD DE PUERTO RICO
MAYAGÜEZ, PUERTO RICO, 00708