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GLOBAL ATTRACTIVITY OF A CLASS OF DELAY DIFFERENTIAL EQUATIONS WITH IMPULSES

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Abstract

In this paper, we study the global attractivity of the zero solution of a particular impulsive delay differential equation. Some sufficient conditions that guarantee every solution of the equation converges to zero are obtained.

1. Introduction

Recently, with the rapid development of the theory and applications of impulsive differential equations, the study of the impulsive delay differential equation has attracted the interest of many mathematicians [1–9]. The purpose of this paper is to study the global attractivity of the following impulsive delay differential equation:

$$\begin{cases} x'(t) + a(t)x(t) = p(t)(1 - e^{x(t-\tau)}), & t \geq 0,\ t \neq t_k, \\ x(t_k^+) - x(t_k) = b_k x(t_k), & k \in N, \end{cases} \tag{1.1}$$

where $a(t), p(t) \in C([0, +\infty), [0, +\infty))$, $\tau > 0$, $b_k > -1$, $p(t) > 0$, for all $k \in N$, $t \geq 0$, $0 < t_1 < t_2 < \cdots$, with $t_k \to +\infty$ as $k \to +\infty$.

In the special case where $p(t) = aN_0a(t)$, (1.1) has been used to model the impulsive growth of red blood cells.

As usual, we say that $x(t)$ defined in $[-\tau, +\infty)$ is a solution of (1.1), if $x(t)$ is absolutely continuous at points $t \neq t_k$ and at $t = t_k$, $x(t_k^+)$ exists, $x(t)$ is left-continuous for $t \geq -\tau$, and satisfies (1.1) for $t \geq 0$.

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2. Main results

The main results are as follows.

THEOREM 1. *Suppose that:*

(i) *there is a positive number* p *such that*

$$p(p+1/2) < 1; \tag{2.1}$$

(ii) *for any* $\epsilon > 0$, *there exists an integer* N *such that*

$$\prod_{k=n}^{n+m}(1+b_k) < 1+\epsilon, \quad n > N, m \geq 0; \tag{2.2}$$

(iii)

$$\int_0^{+\infty} p(s) e^{\int_0^s a(u)\,du} \prod_{0 \leq t_k < s} (1+b_k)^{-1}\,ds = +\infty; \tag{2.3}$$

(iv) *for sufficiently large* t, *we have*

$$\int_{t-\tau}^{t} p(s) e^{\int_t^s a(u)\,du} \prod_{t-\tau \leq t_k < s} (1+b_k)^{-1}\,ds \leq p + \frac{1}{2} e^{-\int_{t-\tau}^{t} a(u)\,du} \tag{2.4}$$

and

$$a(t) \geq a(t-\tau), \quad t \geq \tau. \tag{2.5}$$

Then every solution of (1.1) *tends to zero as* $t \to \infty$.

THEOREM 2. *Suppose that* (2.2), (2.3) *and* (2.5) *hold and for sufficiently large* t, *we have*

$$\int_{t-\tau}^{t} p(s) e^{\int_{t-\tau}^{s} a(u)\,du} \prod_{t-\tau \leq t_k < s} (1+b_k)^{-1}\,ds \leq 3/2. \tag{2.6}$$

Then every solution of (1.1) *tends to zero as* $t \to \infty$.

REMARK 1. Condition (2.2) is not critical; it allows the convergence of

$$\prod_{k=1}^{+\infty}(1+b_k)$$

and the possibility that $-1 < b_k \leq 0$ as special cases. Condition (2.5) allows constant functions, nondecreasing functions and τ-periodic functions as special cases.

REMARK 2. If the impulsives disappeared and $a(t) \equiv 0$, Theorem 2 is the main result of [4]. If $a(t) \equiv \lambda$, then the conditions of Theorem 2 are

$$\int_0^{+\infty} p(t)e^{\lambda t}\,dt = +\infty \quad \text{and} \quad \int_{t-\tau}^{t} p(s)e^{(s-t+\tau)\lambda}\,ds \le \frac{3}{2},$$

which improve the conditions in [5].

3. Proofs of the theorems

LEMMA 1. *Suppose that* (2.2) *and* (2.3) *hold. Then every nonoscillatory solution of* (1.1) *tends to zero as* $t \to \infty$.

PROOF. Without loss of generality, suppose that $x(t)$ is eventually positive. Then there exists $T_1 \ge 0$ such that $x(t-\tau) > 0$ for $t \ge T_1$, $t \ne t_k$. Moreover, $x(t)$ is decreasing in $(t_k, t_{k+1}]$ with $t_k \ge T_1$. Let $\liminf_{t\to+\infty} x(t) = \alpha$, then $\alpha \ge 0$. We shall prove that $\alpha = 0$. Since $x(t_k)$ is the left minimum value of $x(t)$, there exists a subsequence $\{x(t_{k_j})\}$ such that $\lim_{j\to+\infty} x(t_{k_j}) = \alpha$. If $\alpha > 0$, choosing $\epsilon > 0$ such that $\alpha - \epsilon > 0$, again there exists $T > T_1$ such that $x(t-\tau) > \alpha - \epsilon$, for $t \ge T$. Then by (1.1), we have

$$\prod_{T\le t_k<t_{k_j}} (1+b_k)^{-1}x(t_{k_j}) - x(T) \le (1-e^{\alpha-\epsilon}) \int_T^{t_{k_j}} p(s)e^{\int_T^s a(u)du} \prod_{T\le t_k<s} (1+b_k)^{-1}ds,$$

which contradicts (2.2) and (2.3), so $\alpha = 0$. Now for any $t \ge T$, there exists t_{k_j} such that $t_{k_j} \le t < t_{k_{j+1}}$ and $t_{k_j} < t_{k_j+1} < \cdots < t_{k_j+l} \le t$. Then

$$\begin{aligned} 0 < x(t) < x(t_{k_j+l}^+) &= (1+b_{k_j+l})x(t_{k_j+l}) \le (1+b_{k_j+l})x(t_{k_j+l-1}^+) \\ &= (1+b_{k_j+l})(1+b_{k_j+l-1})x(t_{k_j+l-1}) \le \cdots \le \prod_{s=0}^{l}(1+b_{k_j+s})x(t_{k_j}). \end{aligned}$$

From (2.3), there exists a constant $A > 0$, such that $\prod_{s=0}^{l}(1+b_{k_j+s}) \le A$ for any l and any k_j. Hence $0 < x(t) \le Ax(t_{k_j})$. Let $t \to +\infty$. Then we obtain $\lim_{t\to+\infty} x(t) = 0$.

LEMMA 2. *Suppose that* (2.2), (2.4) *or* (2.6), *and* (2.5) *hold. Then every oscillatory solution of* (1.1) *is bounded.*

PROOF. From (2.4) and (2.5), or (2.5) and (2.6), we obtain

$$\int_{t-\tau}^{t} p(s)e^{\int_t^s a(u)du} \prod_{s\le t_k<t} (1+b_k)\,ds \le M,$$

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where M is a positive constant. First, we shall prove that $x(t)$ is bounded above. By (1.1),

$$x'(t) + a(t)x(t) \leq p(t), \quad t \geq 0, \ t \neq t_k. \tag{3.1}$$

Choose any sequence $\{c_n\}$ such that $x(c_n) = 0$ and $0 < c_1 < c_2 < \cdots$, with $\lim_{\to +\infty} c_n = +\infty$, $x(t) \geq 0$ for $t \in [c_{2i-1}, c_{2i}]$ and $x(t) \leq 0$ for $t \in [c_{2i}, c_{2i+1}]$. Let

$$\hat{x}_i = \sup_{t \in [c_{2i-1}, c_{2i}]} x(t) \quad \text{and} \quad \tilde{x}_i = \inf_{t \in [c_{2i}, c_{2i+1}]} x(t).$$

We shall prove that $\{\hat{x}_i\}$ and $\{\tilde{x}_i\}$ are bounded. First, we prove that $\{\hat{x}_i\}$ is bounded above; there are two cases to consider.

Case 1: $\hat{x}_i$ is the maximum value of $x(t)$ in $[c_{2i-1}, c_{2i}]$. Then there exists $c \in (c_{2i-1}, c_{2i})$ such that $\hat{x}_i = x(c) > 0$, $x'(c) \geq 0$. By (1.1), $x(c - \tau) \leq 0$, so there exists $\xi \in (c - \tau, c)$ such that $x(\xi) = 0$. Integrating (3.1) from ξ to c, we obtain

$$\hat{x}_i = x(c) \leq \int_\xi^c p(t) e^{\int_c^t a(u)\,du} \prod_{t \leq t_k < c} (1 + b_k)\, dt \leq M.$$

Case 2: $\hat{x}_i$ is not the maximum value of $x(t)$ in $[c_{2i-1}, c_{2i}]$. Then there exists $t_{k+l} \in (c_{2i-1}, c_{2i})$ such that $\hat{x}_i = x(t_{k+l}^+)$. We assume that $c_{2i-1} < t_{k+1} < \cdots < t_{k+l}$. There are two possible cases to consider.

Subcase 2.1: $x(t_{k+l})$ is the left locally maximum value. By Case 1, we have $x(t_{k+l}) \leq M$, so $\hat{x}_i = x(t_{k+l}^+) = (1 + b_{k+l})x(t_{k+l}) \leq (1 + b_{k+l})M$.

Subcase 2.2: $x(t_{k+l})$ is not the left locally maximum value. There are two possible subcases to consider.

Subcase 2.2.1: If $x(t_{k+l}^+) < x(t_{k+l})$, then $x(t)$ has maximum value noted by $x(c)$ in (t_{k+l-1}, t_{k+l}). By Case 1, $x(c) \leq M$, so $\hat{x}_i = x(t_{k+l}^+) = (1 + b_{k+l})x(t_{k+l}) \leq (1 + b_{k+l})x(c) \leq (1 + b_{k+l})M$.

Subcase 2.2.2: If $x(t_{k+l-1}^+) \geq x(t_{k+l})$, we have two possible cases to consider.

Subcase 2.2.2.1: If $x(t_{k+l-1})$ is the left locally maximum value, then, by Case 1, $x(t_{k+l-1}) \leq M$. Thus $\hat{x}_i = x(t_{k+l}^+) = (1 + b_{k+l})x(t_{k+l}) \leq (1 + b_{k+l})(1 + b_{k+l-1})M$.

Subcase 2.2.2.2: $x(t_{k+l-1})$ is not the left maximum of $x(t)$. Repeating this process, at the end, if $x(t_{k+1})$ is the left locally maximum value of $x(t)$, then $x(t_{k+1}) \leq M$. Therefore

$$\hat{x}_i \leq \cdots \leq \prod_{s=1}^{l} (1 + b_{k+s}) x(t_{k+1}) \leq \prod_{s=1}^{l} (1 + b_{k+s}) M.$$

Otherwise, since $x(c_{2i-1}) = 0$, $x(t)$ has maximum value noted by $x(c)$ in (c_{2i-1}, t_{k+1}). By Case 1, $x(c) \leq M$, so

$$\hat{x}_i \leq \prod_{s=1}^{l} (1 + b_{k+s}) x(t_{k+1}) \leq \prod_{s=1}^{l} (1 + b_{k+s}) x(c) \leq \prod_{s=1}^{l} (1 + b_{k+s}) M.$$

Then $\hat{x}_i \le AM$, where A is defined in Lemma 1.

From Cases 1 and 2, we have $\hat{x}_i \le \max\{M, AM\} = B$. Next we shall prove that $\{\tilde{x}_i\}$ is bounded below. By (1.1), we obtain

$$x'(t) + a(t)x(t) \ge (1 - e^B)p(t), \quad t \ge 0,\ t \ne t_k.$$

Using a similar method to the above, we obtain

$$\tilde{x}_i \ge (1 - e^B)M \qquad \text{or} \qquad \tilde{x}_i \ge (1 - e^B)AM.$$

This shows that $\{\tilde{x}_i\}$ is bounded below, and completes the proof of the lemma.

LEMMA 3. *Suppose that* (2.1), (2.2), (2.4) *and* (2.5) *hold. Then every oscillatory solution of* (1.1) *tends to zero as* $t \to +\infty$.

PROOF. Suppose $x(t)$ is any oscillatory solution of (1.1). By Lemma 2, $x(t)$ is bounded, so let $\limsup_{t\to+\infty} x(t) = v$, $\liminf_{t\to+\infty} x(t) = u$, Then $-\infty < u \le 0 \le v < +\infty$, and by (2.2), for any $\epsilon > 0$, there exists N such that

$$\prod_{k=n}^{n+m}(1 + b_k) < 1 + \epsilon, \quad n \ge N,\ m \ge 0.$$

Again for this $\epsilon > 0$, there exists $T \ge t_N$ such that

$$u_1 = u - \epsilon < x(t - \tau) < v + \epsilon = v_1, \quad t \ge T.$$

Then (1.1) gives

$$x'(t) + a(t)x(t) \le (1 - e^{u_1})p(t), \quad t \ge T,\ t \ne t_k, \tag{3.2}$$

$$x'(t) + a(t)x(t) \ge (1 - e^{v_1})p(t), \quad t \ge T,\ t \ne t_k. \tag{3.3}$$

Choose a sequence $\{c_n\}$ such that $x(c_n) = 0$, $T < c_1 < \cdots < c_n \to +\infty$, $n \to +\infty$, $x(t) \ge 0$, for $t \in (c_{2i-1}, c_{2i})$ and $x(t) \le 0$ for $t \in (c_{2i}, c_{2i+1})$. Let

$$\hat{x}_i = \sup_{t\in(c_{2i-1},c_{2i})} x(t) \qquad \text{and} \qquad \tilde{x}_i = \inf_{t\in(c_{2i},c_{2i+1})} x(t).$$

Without loss of generality, we assume that $\limsup_{i\to\infty} \hat{x}_i = v$ and $\liminf_{i\to\infty} \tilde{x}_i = u$. First, we prove that

$$\hat{x}_i \le p(1 - e^{u_1})(1 + \epsilon) \tag{3.4}$$

or

$$\hat{x}_i \le p(1 - e^{u_1})(1 + \epsilon)^2. \tag{3.5}$$

There are two possible cases to consider.

Case 1: $\hat{x}_i$ is the maximum value of $x(t)$ in (c_{2i-1}, c_{2i}). Then there exists $c \in (c_{2i-1}, c_{2i})$ such that $\hat{x}_i = x(c) > 0, x'(c) \geq 0$. By (1.1), $x(c-\tau) \leq 0$, so there exists $\xi \in (c-\tau, c)$ such that $x(\xi) = 0$, if $t \in [\xi, c]$ then $t - \tau \leq \xi$. Integrating (3.2) from $t-\tau$ to ξ, we have

$$-\prod_{t-\tau\leq t_k<\xi}(1+b_k)x(t-\tau)e^{\int_0^{t-\tau} a(u)\,du} \leq (1-e^{u_1})\int_{t-\tau}^{\xi} p(s)e^{\int_0^s a(u)\,du}\prod_{s\leq t_k<\xi}(1+b_k)\,ds. \quad (3.6)$$

Since $1 - e^x \leq -x$ and by (1.1), we have

$$x'(t) + a(t)x(t) \leq -p(t)x(t-\tau), \quad t \geq 0,\ t \neq t_k. \quad (3.7)$$

Then

$$x'(t) + a(t)x(t) \leq (1-e^{u_1})p(t)\int_{t-\tau}^{\xi} p(s)e^{\int_{t-\tau}^{t} a(u)\,du}\prod_{t-\tau\leq t_k<s}(1+b_k)^{-1}\,ds. \quad (3.8)$$

Integrating (3.8) from ξ to c, we get

$$\begin{aligned}
x(c)e^{\int_0^c a(u)\,du} &\leq (1-e^{u_1})\int_{\xi}^{c} p(t)e^{\int_0^t a(u)\,du}\prod_{t\leq t_k<c}(1+b_k)\int_{t-\tau}^{\xi} p(s)e^{\int_{t-\tau}^{t} a(u)\,du} \\
&\quad\times\prod_{t-\tau\leq t_k<s}(1+b_k)^{-1}\,ds\,dt \\
&\leq (1-e^{u_1})\left[\int_{\xi}^{c} p(t)e^{\int_0^t a(u)\,du}\prod_{t\leq t_k<c}(1+b_k)\left(p+\frac{1}{2}e^{-\int_{t-\tau}^{t} a(u)\,du}\right)\right. \\
&\quad\times e^{\int_{t-\tau}^{t} a(u)\,du}\,dt - \int_{\xi}^{c} p(t)e^{\int_0^t a(u)\,du}\prod_{t\leq t_k<c}(1+b_k)\int_{\xi}^{t} p(s)e^{\int_0^s a(u)\,du} \\
&\quad\left.\times\prod_{s\leq t_k<c}(1+b_k)\,ds\,e^{-\int_0^{t-\tau} a(u)\,du}\prod_{t-\tau\leq t_k<c}(1+b_k)^{-1}\,dt\right].
\end{aligned}$$

Using (2.2), (2.4) and (2.5), we obtain

$$\begin{aligned}
x(c)e^{\int_0^c a(u)du} &\leq (1-e^{u_1})\int_{\xi}^{c} p(t)e^{\int_0^t a(u)\,du}\prod_{t\leq t_k<c}(1+b_k)\,dt\left(\frac{1}{2}+p\,e^{\int_{c-\tau}^{c} a(u)du}\right) \\
&\quad-\frac{1-e^{u_1}}{1+\epsilon}e^{-\int_0^{c-\tau} a(u)\,du}\int_{\xi}^{c} p(t)e^{\int_0^t a(u)\,du} \\
&\quad\times\prod_{t\leq t_k<c}(1+b_k)\int_{\xi}^{t} p(s)e^{\int_0^s a(u)\,du}\prod_{s\leq t_k<c}(1+b_k)\,ds\,dt
\end{aligned}$$

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$$= (1 - e^{u_1}) \int_\xi^c p(t) e^{\int_0^t a(u)\,du} \prod_{t \le t_k < c} (1 + b_k)\, dt \left(\frac{1}{2} + p\, e^{\int_{c-\tau}^c a(u)\,du} \right)$$

$$- \frac{1 - e^{u_1}}{1 + \epsilon} e^{-\int_0^{c-\tau} a(u)\,du} \frac{1}{2} \left(\int_\xi^c p(t) e^{\int_0^t a(u)\,du} \prod_{t \le t_k < c} (1 + b_k)\, dt \right)^2 .$$

In the following, we consider two possible subcases.

Subcase 1.1: $\int_\xi^c p(t) e^{\int_0^t a(u)\,du} \prod_{t \le t_k < c} (1 + b_k)\, dt \le (1 + \epsilon) e^{\int_0^{c-\tau} a(u)\,du}$. Since the function

$$\left(1/2 + p\, e^{\int_{c-\tau}^c a(u)\,du}\right) x - (1 + \epsilon)^{-1} e^{-\int_0^{c-\tau} a(u)\,du} x^2/2$$

is increasing, we obtain $x(c) e^{\int_0^c a(u)\,du} \le p(1 - e^{u_1}) e^{\int_0^c a(u)\,du} (1 + \epsilon)$. Then (3.4) holds.

Subcase 1.2: $\int_\xi^c p(t) e^{\int_0^t a(u)\,du} \prod_{t \le t_k < c} (1 + b_k) dt > (1 + \epsilon) e^{\int_0^{c-\tau} a(u)du}$. We choose $\eta \in (\xi, c)$ such that

$$\int_\eta^c p(t) e^{\int_0^t a(u)du} \prod_{t \le t_k < c} (1 + b_k) dt = (1 + \epsilon) e^{\int_0^{c-\tau} a(u)du}.$$

Integrating (3.2) from ξ to η, we have

$$x(\eta) e^{\int_0^\eta a(u)du} \le (1 - e^{u_1}) \int_\xi^\eta p(t) e^{\int_0^t a(u)du} \prod_{t \le t_k < \eta} (1 + b_k) dt.$$

Integrating (3.8) from η to c,

$$x(c) e^{\int_0^c a(u)\,du} - \prod_{\eta \le t_k < c} (1 + b_k) x(\eta) e^{\int_0^\eta a(u)\,du}$$

$$\le (1 - e^{u_1}) \int_\eta^c p(t) e^{\int_0^t a(u)\,du} \prod_{t \le t_k < c} (1 + b_k) \int_{t-\tau}^\xi p(s) e^{\int_{t-\tau}^s a(u)\,du}$$

$$\times \prod_{t - \tau \le t_k < c} (1 + b_k)^{-1}\, ds\, dt.$$

Then we get

$$x(c) e^{\int_0^c a(u)\,du} \le (1 - e^{u_1}) \Bigg[\int_\xi^\eta p(t) e^{\int_0^t a(u)\,du} \prod_{t - \tau \le t_k < c} (1 + b_k)\, dt$$

$$+ \int_\eta^c p(t) e^{\int_0^t a(u)\,du} \prod_{t \le t_k < c} (1 + b_k) \int_{t-\tau}^\xi p(s) e^{\int_{t-\tau}^s a(u)\,du}$$

$$\times \prod_{t - \tau \le t_k < s} (1 + b_k)^{-1}\, ds\, dt \Bigg].$$

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Similarly to the argument we used in Subcase 1.1, we get

$$\begin{aligned} x(c)e^{\int_0^c a(u)\,du} &\le (1-e^{u_1})\int_\xi^\eta p(t)e^{\int_0^t a(u)\,du}\prod_{t\le t_k<c}(1+b_k)\,dt \\ &\quad - \frac{1-e^{u_1}}{2(1+\epsilon)e^{\int_0^{c-\tau} a(u)\,du}}\left[\left(\int_\xi^c p(t)e^{\int_0^t a(u)\,du}\prod_{t\le t_k<c}(1+b_k)\,dt\right)^2\right. \\ &\quad \left. - \left(\int_\xi^\eta p(t)e^{\int_0^t a(u)\,du}\prod_{t-\tau\le t_k<c}(1+b_k)\,dt\right)^2\right] \\ &\quad + (1-e^{u_1})\left(\frac{1}{2}+pe^{\int_{c-\tau}^c a(u)\,du}\right)\int_\eta^c p(t)e^{\int_0^t a(u)\,du}\prod_{t\le t_k<c}(1+b_k)\,dt \\ &= p(1-e^{u_1})(1+\epsilon)e^{\int_0^c a(u)\,du}. \end{aligned}$$

Hence (3.4) is proved.

Case 2: $\hat{x}_i$ is not the maximum value of $x(t)$ in (c_{2i-1}, c_{2i}). Then there exists $t_{k+l} \in (c_{2i-1}, c_{2i})$ such that $\hat{x}_i = x(t_{k+l}^+)$. Suppose that $c_{2i-1} < t_{k+1} < \cdots < t_{k+l}$. Proving that $x(t)$ is bounded, we obtain

$$\hat{x}_i \le \prod_{s=j}^{l}(1+b_k)p(1-e^{u_1})(1+\epsilon), \quad j = 1, 2, \ldots, l.$$

Then $\hat{x}_i \le (1+\epsilon)^2 p(1-e^{u_1})$. From (3.4) and (3.5), let $i \to +\infty$ and $\epsilon \to 0$ to obtain

$$v \le p(1-e^u). \tag{3.9}$$

Next, we shall prove

$$u \ge (p+1/2)(1-e^v). \tag{3.10}$$

There are two cases to consider.

Case 1: $\tilde{x}_i$ is the minimum value of $x(t)$ in $[c_{2i}, c_{2i+1}]$. Then there exists $c \in (c_{2i}, c_{2i+1})$ such that $x(c) = \tilde{x}_i \le 0$, $x'(c) = 0$. By (1.1), $x(c-\tau) \ge 0$. Then there exists $\xi \in [c-\tau, c)$ such that $x(\xi) = 0$. Integrating (3.3) from ξ to c, we obtain

$$x(c)e^{\int_0^c a(u)\,du} \ge (1-e^{v_1})\int_\xi^c p(t)e^{\int_0^t a(u)\,du}\prod_{t\le t_k<c}(1+b_k)\,dt.$$

Then by (2.2) and (2.4), we get

$$\tilde{x}_i \ge (1+\epsilon)(1-e^{v_1})(p+1/2). \tag{3.11}$$

Case 2: $\tilde{x}_i$ is not the minimum value of $x(t)$ in (c_{2i}, c_{2i+1}). Then there exists $t_{k+l} \in (c_{2i}, c_{2i+1})$ such that $\tilde{x}_i = x(t_{k+l}^{+})$. Suppose that $c_{2i} < t_{k+1} < \cdots < t_{k+l}$. Proving that $x(t)$ is bounded, we get

$$\tilde{x}_i \geq \prod_{s=j}^{l}(1 + b_{k+s})(1+\epsilon)(p + 1/2)(1 - e^{v_1}). \tag{3.12}$$

By (2.2), $\tilde{x}_i \geq (1+\epsilon)^2(p+1/2)(1-e^{v_1})$. Let $i \to +\infty$ and $\epsilon \to 0$. By (3.11) and (3.12), we get (3.10). From (2.1), (3.9), (3.10) and the fact that $-\infty < u \leq 0 \leq v < +\infty$, we get $u = v = 0$. Then $x(t)$ tends to zero as $t \to \infty$. By Lemmas 1 and 3, Theorem 1 is proved.

In order to prove Theorem 2, we need the following lemma.

LEMMA 4. *Suppose that* (2.2), (2.5) *and* (2.6) *hold. Then every oscillatory solution of* (1.1) *tends to zero as* $t \to \infty$.

PROOF. From Lemma 2, $x(t)$ is bounded. By the proof of Lemma 3, we get (3.2), (3.3) and (3.6). Choose $\{c_n\}$ satisfing the conditions in Lemma 3, with $\hat{x}_i \to v, \tilde{x}_i \to u$ as $i \to +\infty$. There are two cases to consider.

Case 1: $\hat{x}_i$ is the maximum value of $x(t)$ in (c_{2i-1}, c_{2i}). Substituting (3.6) into (1.1), we have, for $t \in [\xi, c]$, $t \neq t_k$,

$$x'(t) + a(t)x(t) \leq p(t)\left[1 - \exp\left(-A\int_{t-\tau}^{\xi} p(s)e^{\int_{t-\tau}^{s} a(u)\,du} \prod_{t-\tau \leq t_k < s}(1+b_k)^{-1}\,ds\right)\right], \tag{3.13}$$

where $1 - e^{u_1} = A$. Integrating (3.13) from ξ to c, we get

$$\begin{aligned}
x(c)e^{\int_0^c a(u)du} &\leq \int_{\xi}^{c} p(t)\left[1 - \exp\left(-A\int_{t-\tau}^{\xi} p(s)e^{\int_{t-\tau}^{s} a(u)\,du}\prod_{t-\tau\leq t_k<s}(1+b_k)^{-1}\,ds\right)\right] \\
&\quad \times e^{\int_0^t a(u)\,du}\prod_{t\leq t_k<c}(1+b_k)\,dt \\
&\leq \int_{\xi}^{c} p(t)e^{\int_0^t a(u)\,du}\prod_{t\leq t_k<c}(1+b_k)\,dt - \int_{\xi}^{c} p(t)e^{\int_0^t a(u)\,du} \\
&\quad \times \prod_{t\leq t_k<c}(1+b_k)\exp\left(-A\int_{t-\tau}^{t} p(s)e^{\int_{t-\tau}^{s} a(u)\,du}\prod_{t-\tau\leq t_k<s}(1+b_k)^{-1}\,ds\right. \\
&\quad \left. +A\int_{\xi}^{t} p(s)e^{\int_{t-\tau}^{s} a(u)\,du}\prod_{t-\tau\leq t_k<s}(1+b_k)^{-1}\,ds\right)dt
\end{aligned}$$

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$$\leq \int_\xi^c p(t) e^{\int_0^t a(u)\,du} \prod_{t \leq t_k < c} (1+b_k)\,dt$$
$$- e^{-3A/2} \int_\xi^c p(t) e^{\int_0^t a(u)\,du} \prod_{t \leq t_k < c} (1+b_k)$$
$$\times \exp\left(\frac{A \int_\xi^t p(s) e^{\int_0^s a(u)\,du} \prod_{s \leq t_k < c}(1+b_k)\,ds}{e^{\int_0^{c-\tau} a(u)\,du} \prod_{t-\tau \leq t_k < c}(1+b_k)} \right) dt$$
$$\leq \int_\xi^c p(t) e^{\int_0^t a(u)\,du} \prod_{t \leq t_k < c} (1+b_k)\,dt - e^{-3A/2} A^{-1} (1+\epsilon) e^{\int_0^{c-\tau} a(u)\,du}$$
$$\times \left[\exp\left(\frac{A \int_\xi^c p(s) e^{\int_0^t a(u)\,du} \prod_{s \leq t_k < c}(1+b_k)\,ds}{(1+\epsilon) e^{\int_0^{c-\tau} a(u)\,du}} \right) - 1 \right].$$

Case 1.1: $\int_\xi^c p(t) e^{\int_0^t a(u)du} \prod_{t \leq t_k < c}(1+b_k)\,dt \leq -(1/A)\ln(1-A) e^{\int_0^{c-\tau} a(u)\,du}(1+\epsilon)$. Then

$$x(c) e^{\int_0^c a(u)\,du} \leq -\frac{\ln(1-A)}{A} e^{\int_0^{c-\tau} a(u)\,du}(1+\epsilon) - \frac{e^{-3A/2}}{(1-A)(1+\epsilon)^{-1} e^{-\int_0^{c-\tau} a(u)\,du}}$$
$$= -\frac{\ln(1-A)}{A} e^{\int_0^{c-\tau} a(u)\,du}(1+\epsilon) - \frac{1+\epsilon}{1-A} e^{-3A/2} e^{\int_0^{c-\tau} a(u)\,du},$$

so

$$x(c) \leq (1+\epsilon)\left(-\frac{\ln(1-A)}{A} - \frac{e^{-3A/2}}{1-A} \right).$$

By Kuang's method [1, (2.21)], we get

$$\hat{x}_i = x(c) \leq (1+\epsilon)(A - A^2/6). \tag{3.14}$$

Case 1.2:

$$\int_\xi^c p(t) e^{\int_0^t a(u)\,du} \prod_{t \leq t_k < c} (1+b_k)\,dt \leq \frac{3}{2} e^{\int_0^{c-\tau} a(u)\,du}(1+\epsilon)$$
$$< -\frac{\ln(1-A)}{A} e^{\int_0^{c-\tau} a(u)\,du}(1+\epsilon).$$

Then, integrating (3.13) from ξ to c, similarly to Case 1.1, we get

$$\hat{x}_i = x(c) \leq 3(1+\epsilon)/2 + (1+\epsilon)(e^{3A/2} - 1)/A.$$

By a method similar to that used by Kuang in [1, (2.19)], we get (3.14).

Case 1.3: $\int_\xi^c p(t) e^{\int_0^t a(u)du} \prod_{t \leq t_k < c}(1+b_k)\,dt > -(1/A)\ln(1-A)(1+\epsilon) e^{\int_0^{c-\tau} a(u)\,du}$. Choose $\eta \in (\xi, c)$ such that

$$\int_\eta^c p(t) e^{\int_0^t a(u)\,du} \prod_{t \leq t_k < c} (1+b_k)\,dt = -\frac{\ln(1-A)}{A}(1+\epsilon) e^{\int_0^{c-\tau} a(u)\,du}.$$

Integrating (3.2) from ξ to η, we have

$$x(\eta)e^{\int_0^\eta a(u)du} \le A\int_\xi^\eta p(t)e^{\int_0^\eta a(u)du}\prod_{t\le t_k<\eta}(1+b_k)dt.$$

Integrating (3.13) from η to c, we have

$$\begin{aligned}
&x(c)e^{\int_0^c a(u)\,du} - \prod_{\eta\le t_k<c}(1+b_k)x(\eta)e^{\int_0^\eta a(u)\,du}\\
&\quad\le \int_\eta^c p(t)e^{\int_0^t a(u)\,du}\prod_{t\le t_k<c}(1+b_k)\left[1-\exp\left(-A\int_{t-\tau}^{\xi}\frac{p(s)e^{\int_{t-\tau}^{s} a(u)\,du}}{\prod_{t-\tau\le t_k<s}(1+b_k)}\,ds\right)\right]dt.
\end{aligned}$$

Deleting $x(\eta)$, we obtain

$$\begin{aligned}
&x(c)e^{\int_0^c a(u)\,du}\\
&\quad\le A\int_\xi^\eta p(t)e^{\int_0^t a(u)\,du}\prod_{t\le t_k<c}(1+b_k)\,dt + \int_\eta^c p(t)e^{\int_0^t a(u)\,du}\prod_{t\le t_k<c}(1+b_k)\,dt\\
&\qquad - e^{-3A/2}\int_\eta^c p(t)e^{\int_0^t a(u)\,du}\prod_{t\le t_k<c}(1+b_k)\exp\left(A\int_\xi^t\frac{p(s)e^{\int_{t-\tau}^{s} a(u)\,du}}{\prod_{t-\tau\le t_k<s}(1+b_k)}\,ds\right)dt\\
&\quad\le A\int_\xi^\eta p(t)e^{\int_0^t a(u)\,du}\prod_{t\le t_k<c}(1+b_k)\,dt + \int_\eta^c p(t)e^{\int_0^t a(u)\,du}\prod_{t\le t_k<c}(1+b_k)\,dt\\
&\qquad - \frac{1+\epsilon}{e^{3A/2}A}e^{\int_0^{c-\tau} a(u)\,du}\left[\exp\left(\frac{A}{1+\epsilon}\int_\xi^c p(t)e^{\int_{c-\tau}^t a(u)\,du}\prod_{t\le t_k<c}(1+b_k)\,dt\right)\right.\\
&\qquad \left.-\exp\left(\frac{A}{1+\epsilon}\int_\xi^\eta p(t)e^{\int_{c-\tau}^t a(u)\,du}\prod_{t\le t_k<c}(1+b_k)\,dt\right)\right].
\end{aligned}$$

Then

$$\begin{aligned}
\hat{x}_i = x(c) &\le A(1+\epsilon)e^{-\int_{c-\tau}^c a(u)\,du}\int_\xi^\eta p(t)e^{\int_{c-\tau}^t a(u)\,du}\prod_{t\le t_k<c}(1+b_k)\,dt\\
&\quad + e^{-\int_{c-\tau}^c a(u)\,du}\int_\eta^c p(t)e^{\int_{c-\tau}^t a(u)\,du}\prod_{t\le t-k<c}(1+b_k)\,dt\\
&\quad - \frac{1+\epsilon}{e^{3A/2}A}e^{-\int_{c-\tau}^c a(u)\,du}\left(\exp\left(\frac{A}{1+\epsilon}\int_\xi^c p(t)e^{\int_{c-\tau}^t a(u)\,du}\prod_{t\le t_k<c}(1+b_k)\,dt\right)\right.\\
&\quad \left.-\exp\left(\frac{A}{1+\epsilon}\int_\xi^\eta p(t)e^{\int_{c-\tau}^t a(u)\,du}\prod_{t\le t_k<c}(1+b_k)\,dt\right)\right)
\end{aligned}$$

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$$\leq (1+\epsilon)(3A/2 - ((1-A)/A)\ln(1-A) - 1).$$

By Kuang's method in [1, (2.21)], we get (3.14).

Case 2: If $\hat{x}_i$ is not the maximum value of $x(t)$ in (c_{2i-1}, c_{2i}), then there exists $t_{k+l} \in (c_{2i-1}, c_{2i})$ such that $\hat{x}_i = x(t_{k+l})$. Suppose that $c_{2i-1} < t_{k+1} < \cdots < t_{k+l}$. Then we can obtain

$$\hat{x}_i \leq \prod_{s=j}^{l}(1+b_{k+s})(1+\epsilon)(A - A^2/6), \quad j = 1, 2, \ldots, l.$$

Then by (2.2), we get $\hat{x}_i \leq (1+\epsilon)^2(A - A^2/6)$. By (3.13) and (3.14), let $i \to +\infty$ and $\epsilon \to 0$ to obtain $v \leq (1-e^u) - (1-e^u)^2/6$.

Next we prove

$$u \geq (1-e^v) - (1-e^v)^2/6. \tag{3.15}$$

There are two cases to consider.

Case 1: $\tilde{x}_i$ is the minimum value of $x(t)$ in (c_{2i}, c_{2i+1}). Then there exists $c \in (c_{2i}, c_{2i+1})$ such that $x(c) = \tilde{x}_i < 0$, $x'(c) \leq 0$. There exists $\xi \in (c-\tau, c)$ such that $x(\xi) = 0$. If $t \in [\xi, c]$, then $t - \tau \leq \xi$. Integrating (3.3) from $t-\tau$ to ξ, then

$$-\prod_{t-\tau \leq t_k < \xi}(1+b_k)x(t-\tau)e^{\int_0^{t-\tau} a(u)\,du} \geq B\int_{t-\tau}^{\xi} p(s)e^{\int_{t-\tau}^{s} a(u)\,du}\prod_{t-\tau \leq t_k < s}(1+b_k)\,ds,$$

where $B = 1 - e^{v_1}$. By (1.1), we get, for $t \in [\xi, c]$, $t \neq t_k$,

$$x'(t) + a(t)x(t) \geq p(t)\left(1 - \exp\left(-B\int_{t-\tau}^{\xi}\frac{p(s)e^{\int_{t-\tau}^{s} a(u)\,du}}{\prod_{t-\tau \leq t_k < s}(1+b_k)}\,ds\right)\right). \tag{3.16}$$

There are three subcases to consider.

Subcase 1.1: $\int_\xi^c p(t)e^{\int_{t-\tau}^{t} a(u)\,du}\prod_{t \leq t_k < c}(1+b_k)\,dt \leq 1+\epsilon$. Integrating (3.7) from ξ to c, we have

$$\tilde{x}_i = x(c) \geq B\int_\xi^c p(t)e^{\int_c^t a(u)du}\prod_{t \leq t_k < c}(1+b_k)\,dt.$$

Then

$$\tilde{x}_i \geq (1+\epsilon)B \geq (1+\epsilon)(B - B^2/6). \tag{3.17}$$

Subcase 1.2:

$$1+\epsilon < \int_\xi^c p(t)e^{\int_{t-\tau}^{t} a(u)\,du}\prod_{t \leq t_k < c}(1+b_k)\,dt \leq \left(\frac{3}{2} + \frac{\ln(1-B)}{B}\right)(1+\epsilon).$$

Integrating (3.3) from ξ to c, we have

$$\begin{aligned}\tilde{x}_i = x(c) &\geq B\int_\xi^c p(t)e^{\int_c^t a(u)\,du}\prod_{t\leq t_k<c}(1+b_k)\,dt \\ &\geq B\left(\frac{3}{2}+\frac{\ln(1-B)}{B}\right)(1+\epsilon)\geq(1+\epsilon)\left(B-\frac{1}{6}B^2\right).\end{aligned}$$

Hence (3.17) is proved.

Subcase 1.3:

$$\frac{3}{2}(1+\epsilon)\geq\int_\xi^c p(t)e^{\int_{c-\tau}^t a(u)\,du}\prod_{t\leq t_k<c}(1+b_k)\,dt>\left(\frac{3}{2}+\frac{\ln(1-B)}{B}\right)(1+\epsilon).$$

Choose $\eta\in(\xi,c)$ such that

$$\int_\eta^c p(t)e^{\int_{c-\tau}^t a(u)\,du}\prod_{t\leq t_k<c}(1+b_k)\,dt=\left(\frac{3}{2}+\frac{\ln(1-B)}{B}\right)(1+\epsilon).$$

Integrating (3.3) from ξ to η and integrating (3.16) from η to c, we obtain

$$\begin{aligned}\tilde{x}_i &= x(c)\\ &\geq B\int_\xi^\eta p(t)e^{\int_c^t a(u)\,du}\prod_{t\leq t_k<c}(1+b_k)\,dt+\int_\eta^c p(t)e^{\int_c^t a(u)\,du}\prod_{t\leq t_k<c}(1+b_k)\,dt\\ &\quad - e^{-3B/2}\int_\eta^c p(t)e^{\int_{c-\tau}^t a(u)\,du}\prod_{t\leq t_k<c}(1+b_k)\\ &\quad\times\exp\left(\frac{B}{1+\epsilon}\int_\xi^t p(s)e^{\int_{c-\tau}^t a(u)\,du}\prod_{s\leq t_k<c}(1+b_k)\,ds\right)dt\\ &\geq B\int_\xi^\eta p(t)e^{\int_{c-\tau}^t a(u)\,du}\prod_{t\leq t_k<c}(1+b_k)\,dt+\int_\eta^c p(t)e^{\int_{c-\tau}^t a(u)\,du}\prod_{t\leq t_k<c}(1+b_k)\,dt\\ &\quad-\frac{1+\epsilon}{Be^{3B/2}}\left(\exp\left(\frac{B}{1+\epsilon}\int_\xi^c p(t)e^{\int_{c-\tau}^t a(u)\,du}\prod_{t\leq t_k<c}(1+b_k)\,dt\right)\right.\\ &\quad\left.-\exp\left(\frac{B}{1+\epsilon}\int_\xi^\eta p(t)e^{\int_{c-\tau}^t a(u)\,du}\prod_{t\leq t_k<c}(1+b_k)\,dt\right)\right)\\ &\geq(1+\epsilon)\left(\frac{3}{2}B-\frac{1}{B}((1-B)\ln(1-B)+B)\right)\geq(1+\epsilon)\left(B-\frac{1}{6}B^2\right).\end{aligned}$$

Then (3.17) is proved. The last inequality is obtained by the method used by Yu in [6, page 234].

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Case 2: $\tilde{x}_i$ is not the minimum value of $x(t)$ in (c_{2i}, c_{2i+1}). Then there exists $t_{k+l} \in (c_{2i}, c_{2i+1})$ such that $\tilde{x}_i = x(t_{k+l}^+)$. Suppose that $c_{2i} < t_{k+1} < \cdots < t_{k+l}$. Then we can obtain $\tilde{x}_i \geq \prod_{s=j}^{l}(1 + b_{k+s})(1 + \epsilon)(B - B^2/6), j = 1, 2, \ldots, l$. By (2.2), we have

$$\tilde{x}_i \geq (1 + \epsilon)^2(B - B^2/6). \tag{3.18}$$

From (3.17) and (3.18), let $i \to +\infty$ and $\epsilon \to 0$ to obtain (3.15).

Let $1 - e^u = x$, $1 - e^v = -y$. Then (3.15) and (3.16) become

$$\ln(1 + y) \leq x - x^2/6, \quad \ln(1 - x) \geq -y - y^2/6.$$

By [6, Lemma 1.4] , $x = y = 0$, so $u = v = 0$. Then $x(t)$ tends to zero as $t \to \infty$. By Lemmas 1 and 4, we obtain Theorem 2.

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