CONTINUOUS DEPENDENCE OF STATIONARY DISTRIBUTIONS ON PARAMETERS FOR STOCHASTIC PREDATOR–PREY MODELS

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Abstract

This research studies the robustness of permanence and the continuous dependence of the stationary distribution on the parameters for a stochastic predator–prey model with Beddington–DeAngelis functional response. We show that if the model is extinct (resp. permanent) for a parameter, it is still extinct (resp. permanent) in a neighbourhood of this parameter. In the case of extinction, the Lyapunov exponent of predator quantity is negative and the prey quantity converges almost to the saturated situation, where the predator is absent at an exponential rate. Under the condition of permanence, the unique stationary distribution converges weakly to the degenerate measure concentrated at the unique limit cycle or at the globally asymptotic equilibrium when the diffusion term tends to 0.

Keywords: Extinction; robust permanence; stationary distribution; limit cycle

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1. Introduction

Predator–prey relations refer to the relationship between two species where one species is the hunted food source for the other. This relationship plays an important role in the evolution of ecosystems, existing not only in the simplest life forms on Earth like single-celled organisms, but also in complex animal communities. The earliest mathematical model describing this relationship belongs to Lotka [20] and Volterra [28]. Since then, it has become an interesting topic in mathematical biology [8, 17, 24]. In order to describe the predator feeding rate with increasing prey density and quantify the energy transfer across trophic levels, the functional response is added into the standard Lotka–Voltarra equation to make it more realistic. Depending on the characteristics of specific environments, several types of functional

responses have been chosen. The Holling Type II functional response (with the most common form as [14]), where the rate of prey consumption by a predator rises as the prey density increases, but eventually levels off at a plateau (or asymptote) at which the rate of consumption remains constant regardless of the increase in prey density, is introduced in [12]. Similarly, Type III responses are sigmoidal and characterized by a low-density refuge from predation, a mid-density peak in per capita mortality, and then declining mortality owing to predator satiation [23]. Independently, [2, 4] proposed a functional response which is similar to Holling Type II, but containing an extra term describing mutual interference by predators. It became the most common type of functional response and is well documented in empirical studies.

Let $\alpha = (r, K, m, \beta, \gamma, a, b, c)$ be the vector of parameters, whose components are appropriate positive constants. Then a deterministic predator–prey model with the Beddington–DeAngelis functional response has the form

$$\begin{cases} \mathrm{d}x_\alpha(t) = \left(rx_\alpha(t)\left(1 - \dfrac{x_\alpha(t)}{K} \right) - \dfrac{mx_\alpha(t)y_\alpha(t)}{a + by_\alpha(t) + cx_\alpha(t)} \right) \mathrm{d}t, \\ \mathrm{d}y_\alpha(t) = y_\alpha(t)\left(-\gamma + \dfrac{\beta mx_\alpha(t)}{a + by_\alpha(t) + cx_\alpha(t)} \right) \mathrm{d}t, \end{cases} \tag{1.1}$$

where $x_\alpha(t)$ is the prey population and $y_\alpha(t)$ is the predator population at time t (see also [29] and references therein).

It is well known that the quadrants of the plane $\mathbb{R}_+^2 = \{(x, y) : x \geq 0, y \geq 0\}$ and its interior $\mathbb{R}_+^{2,o} = \{(x, y) : x > 0, y > 0\}$ are invariant with respect to (1.1). We denote by $\Phi_\alpha^\phi(t) = (x_\alpha(t), y_\alpha(t))$ the unique solution of (1.1) with initial value $\phi = (x, y) \in \mathbb{R}_+^2$. Let

$$f(\phi, \alpha) = \left(rx\left(1 - \dfrac{x}{K} \right) - \dfrac{mxy}{a + by + cx}; \ -\gamma y + \dfrac{\beta mxy}{a + by + cx} \right)^\top.$$

Consider the Lyapunov function $V(\phi, \alpha) = \beta x + y$. It can be seen that

$$\dot{V}(\phi, \alpha) = V_\phi(\phi, \alpha)f(\phi, \alpha) = \beta rx\left(1 - \dfrac{x}{K} \right) - \gamma y \leq \dfrac{\beta(r+\gamma)^2 K}{4r} - \gamma V(\phi, \alpha).$$

From this inequality, it is easy to prove that the set

$$\mathcal{R}(\alpha) := \left\{ (x, y) \in \mathbb{R}_+^2 : \beta x + y \leq \dfrac{\beta(r+\gamma)^2 K}{4r\gamma} \right\}$$

is also an attractive set with respect to (1.1).

For any vector parameter $\alpha = (r, K, m, \beta, \gamma, a, b, c) \in \mathbb{R}_+^8$, construct the threshold value

$$\lambda_\alpha = -\gamma + \dfrac{\beta mK}{a + Kc}. \tag{1.2}$$

When $\lambda_\alpha > 0$, the system in (1.1) has three non-negative equilibria: $(0, 0)$, $(K, 0)$, and (x_α^*, y_α^*). The long-term behaviour of the model in (1.1) has been classified (see [13, 30], for example) by using the threshold value λ_α as follows.

Lemma 1.1. ([30].) *Let λ_α be a threshold value that is defined by* (1.2).

    (i) *If $\lambda_\alpha \leq 0$ then the boundary equilibrium point $(K, 0)$ is globally asymptotically stable.*

(ii) If $\lambda_\alpha > 0$ and

$$b \geq \min\left\{\frac{c}{\beta}, \frac{m^2\beta^2 - c^2\gamma^2}{\gamma\beta(m\beta - c\gamma) + mr\beta^2}\right\}$$

then the positive equilibrium point $\phi_\alpha^* = (x_\alpha^*, y_\alpha^*)$ of the system in (1.1) is globally asymptotically stable.

(iii) If $\lambda_\alpha > 0$ and

$$b < \min\left\{\frac{c}{\beta}, \frac{m^2\beta^2 - c^2\gamma^2}{\gamma\beta(m\beta - c\gamma) + mr\beta^2}\right\}$$

then the positive equilibrium point $\phi_\alpha^* = (x_\alpha^*, y_\alpha^*)$ is unstable, and there is an exactly stable limit cycle Γ_α.

Furthermore, by the clarity of the positive equilibrium point formula and [9, Theorem 1.2, p. 356], we have the following lemma.

Lemma 1.2 *Let $\alpha = (r, K, m, \beta, \gamma, a, b, c)$ be the parameters of (1.1) such that $\lambda_\alpha > 0$.*

(i) On the set

$$b \geq \min\left\{\frac{c}{\beta}, \frac{m^2\beta^2 - c^2\gamma^2}{\gamma\beta(m\beta - c\gamma) + mr\beta^2}\right\},$$

the mapping $\alpha \to (x_\alpha^*, y_\alpha^*)$ is continuous.

(ii) On

$$b < \min\left\{\frac{c}{\beta}, \frac{m^2\beta^2 - c^2\gamma^2}{\gamma\beta(m\beta - c\gamma) + mr\beta^2}\right\},$$

the mapping $\alpha \to \Gamma_\alpha$ is continuous in Hausdorff distance.

Let $\mathcal{K} \subset \mathbb{R}_+^8$ be a compact set, and write $\mathcal{R}(\mathcal{K}) = \cup_{\alpha \in \mathcal{K}} \mathcal{R}(\alpha)$. It is clear that $\overline{\mathcal{R}(\mathcal{K})}$ is a compact set. Since f has continuous partial derivatives $\nabla_\phi f$ and $\nabla_\alpha f$, these derivatives are uniformly bounded on $\overline{\mathcal{R}(\mathcal{K})} \times \mathcal{K}$. As a consequence, there exists a positive constant L_1 such that

$$\|f(\phi_1, \alpha_1) - f(\phi_2, \alpha_2)\| \leq L_1(\|\phi_1 - \phi_2\| + \|\alpha_1 - \alpha_2\|), \quad \phi_1, \phi_2 \in \mathcal{R}(\mathcal{K}), \ \alpha_1, \alpha_2 \in \mathcal{K}.$$

From this inequality, for $T > 0$ and for any $\phi \in \mathcal{R}(\mathcal{K})$, $\alpha_1, \alpha_2 \in \mathcal{K}$, we have

$$
\begin{aligned}
\sup_{0 \leq t \leq T} \left\| \Phi_{\alpha_1}^\phi(t) - \Phi_{\alpha_2}^\phi(t) \right\| &= \sup_{0 \leq t \leq T} \left\| \int_0^t \left[f\left(\Phi_{\alpha_1}^\phi(s), \alpha_1\right) - f\left(\Phi_{\alpha_2}^\phi(s), \alpha_2\right) \right] \mathrm{d}s \right\| \\
&\leq \int_0^T \left\| f\left(\Phi_{\alpha_1}^\phi(s), \alpha_1\right) - f\left(\Phi_{\alpha_2}^\phi(s), \alpha_2\right) \right\| \mathrm{d}s \\
&\leq L_1 T \|\alpha_1 - \alpha_2\| + L_1 \int_0^T \sup_{0 \leq s \leq t} \left\| \Phi_{\alpha_1}^\phi(s) - \Phi_{\alpha_2}^\phi(s) \right\| \mathrm{d}t.
\end{aligned}
$$

Applying the Gronwall inequality, we obtain

$$\sup_{0 \leq t \leq T} \left\| \Phi_{\alpha_1}^\phi(t) - \Phi_{\alpha_2}^\phi(t) \right\| \leq L_1 T \|\alpha_1 - \alpha_2\| e^{L_1 T} \quad \text{for all } \phi \in \mathcal{R}(\mathcal{K}), \ \alpha_1, \alpha_2 \in \mathcal{K}. \tag{1.3}$$

We now discuss the evolution of the predator–prey model in (1.1) in a random environment, in which some parameters are perturbed by noise (see [7, 15, 16, 22, 25, 27]). Currently, an

important way to model the influence of environmental fluctuations in biological systems is to assume that white noise affects the growth rates. This assumption is reasonable and practicable since there are many small factors involved in the evolution of systems. Hence, the noise must follow a Gaussian distribution by the central limit theorem. Thus, in a random environment, when the parameters r, γ are perturbed, they become $r + \sigma_1 \dot{B}_1$, $\gamma \hookrightarrow \gamma - \sigma_2 \dot{B}_2$, where B_1 and B_2 are two independent Brownian motions. Therefore, (1.1) subjected to environmental white noise can be rewritten as

$$\begin{cases} dx_{\alpha,\sigma}(t) = \left(r x_{\alpha,\sigma}(t) \left(1 - \dfrac{x_{\alpha,\sigma}(t)}{K} \right) - \dfrac{m x_{\alpha,\sigma}(t) y_{\alpha,\sigma}(t)}{a + b y_{\alpha,\sigma}(t) + c x_{\alpha,\sigma}(t)} \right) dt + \sigma_1 x_{\alpha,\sigma}(t) \, dB_1(t), \\ dy_{\alpha,\sigma}(t) = y_{\alpha,\sigma}(t) \left(-\gamma + \dfrac{\beta m x_{\alpha,\sigma}(t)}{a + b y_{\alpha,\sigma}(t) + c x_{\alpha,\sigma}(t)} \right) dt + \sigma_2 y_{\alpha,\sigma}(t) \, dB_2(t), \end{cases}$$

$$(1.4)$$

where $\sigma = (\sigma_1, \sigma_2)$. The existence of a stationary distribution and the stochastic bifurcation for (1.4) was considered in [30], where it was proved that there is a critical point $b^*(\sigma_1; \sigma_2)$ which depends on σ_1 and σ_2 such that the system in (1.4) undergoes a stochastic Hopf bifurcation at $b^*(\sigma_1; \sigma_2)$. The shape of the stationary distribution for the system in (1.4) changes from crater-like to peak-like. However, the conditions imposed on the parameters are rather strict; hence, some of the results in [30] need careful discussion. In [7], a threshold was constructed between distinction and permanence (also the threshold of the existence of a stationary distribution) for the system in (1.4). The extinction and permanence have been considered in a more generalized context in [3, 10, 11] by studying the stochastic permanence of Markov processes via Lyapunov exponents (expected per capita growth rate) and Lyapunov functions. Specifically, they applied these results to give the permanence condition of the stochastic Kolmogorov equations. Also, in [3, 10, 11, 26] the authors have shown that in the case of stochastic persistence there exists a unique invariant probability π^* such that the transition probability $P(t, x, \cdot)$ of the solution (1.4) converges in total variation norm to π^* with an exponential rate. Robust permanence was considered in [26] under the name of δ-perturbation for a system with compact state spaces; [11] studied the robustness of discrete systems for stochastic ecological communities.

The results obtained in [3, 10, 11, 26] are very strong and it is easy to see that some of the results in [7] can be obtained by careful calculations. However, in these works, the authors have mainly dealt with conditions ensuring the permanence of (1.4) but have not focused on studying the robustness of the system.

This paper continues to study this model by considering the robustness of permanence and the continuous dependence of the stationary distribution of (1.4) on the data if it exists. This is important in all mathematical models because in practice, observations do not perfectly reflect biological reality, which causes the threshold estimates to be mere approximations. Precisely, we prove that if the model is extinct (resp. permanent) for a parameter, it is still extinct (resp. permanent) in a neighbourhood of this parameter. In the case of extinction, the Lyapunov exponent of predator quantity is negative and the prey quantity converges almost to the saturated situation, where the predator is absent. Further, if $\lambda_{\alpha_0} > 0$ and $\alpha \to \alpha_0$, $\sigma \to 0$, the stationary distribution $\mu_{\alpha,\sigma}$ of (1.4) will converge weakly to the degenerate distribution concentrated on the critical point $\left(x^*_{\alpha_0}, y^*_{\alpha_0} \right)$ or on the limit cycle Γ_{α_0} of the system in (1.1). Thus, if the model without noise has a critical point $\left(x^*_{\alpha_0}, y^*_{\alpha_0} \right)$ or a limit cycle Γ_{α_0} with parameter α_0, then when the intensity of the noise is small, the long-term population density is almost concentrated at any neighbourhood of $\left(x^*_{\alpha_0}, y^*_{\alpha_0} \right)$ or of the limit cycle Γ_{α_0} for any parameter close to α_0. This

is an important conclusion as the small-noise asymptotics are very relevant in mathematical biology.

The paper is organized as follows. The next section discusses the main results. In section 3, we provide an example to illustrate that when $(\alpha, \sigma) \to (\alpha_0, 0)$, the stationary distribution $\mu_{\alpha, \sigma}$ weakly converges to the degenerate distribution concentrated on $(x_{\alpha_0}^*, y_{\alpha_0}^*)$ or on the limit cycle Γ_{α_0}.

2. Main result

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and let $B_1(t)$ and $B_2(t)$ be two mutually independent Brownian motions. It is well known that both $\mathbb{R}_+^2$ and $\mathbb{R}_+^{2,\mathrm{o}}$ (the interior of $\mathbb{R}_+^2$) are invariant to (1.4), i.e. for any initial value $\phi = (x(0), y(0)) \in \mathbb{R}_+^2$ (resp. in $\mathbb{R}_+^{2,\circ}$), there exists a unique global solution to (1.4) that remains in $\mathbb{R}_+^2$ (resp. $\in \mathbb{R}_+^{2,\circ}$) almost surely [30]. Denote by $\Phi_{\alpha,\sigma}^{\phi}(t) = (x_{\alpha,\sigma}(t), y_{\alpha,\sigma}(t))$ the unique solution of (1.4) with initial value $\phi \in \mathbb{R}_+^2$. For $\alpha = (r, K, m, \beta, \gamma, a, b, c)$ and $\delta > 0$, let

$$\mathcal{U}_\delta(\alpha) = \left\{ \alpha' = (r', K', m', \beta', \gamma', a', b', c') \in \mathbb{R}_+^8 : \|\alpha' - \alpha\| \leq \delta \right\}$$

and $\mathcal{V}_\delta(\phi_0) = \left\{ \phi \in \mathbb{R}_+^2 : \|\phi - \phi_0\| \leq \delta \right\}$ be the balls with radius $\delta > 0$ and center α (resp. ϕ_0). Write $\mathcal{R}_\delta(\alpha) = \bigcup_{\alpha' \in \mathcal{U}_\delta(\alpha)} \mathcal{R}(\alpha')$. For any $R > 0$, write $\mathbf{B}_R = \{ \phi = (x, y) \in \mathbb{R}_+^2 : \|\phi\| \leq R \}$. Let $C^2(\mathbb{R}^2, \mathbb{R}_+)$ be the family of all non-negative functions $V(\phi)$ on $\mathbb{R}^2$ which are twice continuously differentiable in ϕ. For $V \in C^2(\mathbb{R}^2, \mathbb{R}_+)$, define the differential operator $\mathcal{L}V$ associated with (1.4) as

$$\mathcal{L}V(\phi) = V_\phi(\phi)f(\phi, \alpha) + \tfrac{1}{2}\mathrm{trace}\big[g^\top(\phi, \sigma)V_{\phi\phi}(\phi)g(\phi, \sigma)\big],$$

where $V_\phi(\phi)$ and $V_{\phi\phi}(\phi)$ are the gradient and Hessian of $V(\cdot)$, and g is the diffusion coefficient of (1.4) given by

$$g(\phi, \sigma) = \begin{pmatrix} \sigma_1 x & 0 \\ 0 & \sigma_2 y \end{pmatrix}.$$

By virtue of the symmetry of Brownian motions, in the following we are interested only in $\sigma_1 \geq 0$, $\sigma_2 \geq 0$.

Lemma 2.1 *Let $\mathcal{K} \subset \mathbb{R}_+^{8,\mathrm{o}}$ be a compact set and $\overline{\sigma} > 0$. Then, for any $0 \leq p \leq 2\gamma_*/\overline{\sigma}^2$,*

$$\mathbb{E}\big(V\big(\Phi_{\alpha,\sigma}^{\phi}(t)\big)\big) \leq \mathrm{e}^{-H_1 t} V(\phi) + \frac{H_2}{H_1} \quad \text{for all } \alpha \in \mathcal{K}, \ \|\sigma\| \leq \overline{\sigma}, \ t \geq 0, \tag{2.1}$$

where

$$H_1 = \frac{(1+p)}{2}\left(\gamma_* - \frac{p\overline{\sigma}^2}{2}\right), \qquad H_2 = \sup_{\alpha \in \mathcal{K}, \|\sigma\| \leq \overline{\sigma}} \sup_{\phi \in \mathbb{R}_+^2} \{\mathcal{L}V(\phi) + H_1 V(\phi)\},$$

$\gamma_* = \inf\{\beta \wedge \gamma : (r, K, m, \beta, \gamma, a, b, c) \in \mathcal{K}\}$, *and* $V(\phi) = (\beta x + y)^{1+p}$. *As a consequence,*

$$\sup\left\{\mathbb{E}\big\|\Phi_{\alpha,\sigma}^{\phi}(t)\big\|^{1+p} : \alpha \in \mathcal{K}, \ \|\sigma\| \leq \overline{\sigma}, \ t \geq 0\right\} < \infty. \tag{2.2}$$

Proof. The differential operator $\mathcal{L}V(\phi)$ associated with (1.4) is given by

$$
\begin{aligned}
\mathcal{L}V(\phi) &= (1+p)(\beta x + y)^p \left(\beta rx \left(1 - \frac{x}{K} \right) - \gamma y \right) + \frac{p(1+p)}{2} (\beta x + y)^{p-1} \left(\beta^2 \sigma_1^2 x^2 + \sigma_2^2 y^2 \right) \\
&\leq (1+p) \left(-\gamma + \frac{p\|\sigma\|^2}{2} \right) (\beta x + y)^{1+p} + \beta r (1+p) (\beta x + y)^p x \left(\frac{\gamma + r}{r} - \frac{x}{K} \right) \\
&\leq (1+p) \left(-\gamma_* + \frac{p\overline{\sigma}^2}{2} \right) (\beta x + y)^{1+p} + \beta r (1+p) (\beta x + y)^p x \left(\frac{\gamma + r}{r} - \frac{x}{K} \right) \\
&\leq H_2 - H_1 V(\phi),
\end{aligned}
$$

where

$$
H_1 = \frac{1+p}{2} \left(\gamma_* - \frac{p\overline{\sigma}^2}{2} \right), \qquad H_2 = \sup_{\alpha \in \mathcal{K}, \|\sigma\| \leq \overline{\sigma}} \sup_{\phi \in \mathbb{R}_+^2} \{ \mathcal{L}V(\phi) + H_1 V(\phi) \} < \infty.
$$

Thus, $\mathcal{L}V(\phi) \leq H_2 - H_1 V(\phi)$. By a standard argument as in [5, Lemma 2.3], it follows that

$$
\mathbb{E} \left(e^{H_1 t} V \left(\Phi_{\alpha,\sigma}^\phi(t) \right) \right) \leq V(\phi) + \frac{H_2 (e^{H_1 t} - 1)}{H_1}.
$$

Thus, $\mathbb{E} \left(V \left(\Phi_{\alpha,\sigma}^\phi(t) \right) \right) \leq e^{-H_1 t} V(\phi) + H_2/H_1$, i.e. we get (2.1).

By using the inequality $\|\phi\|^{1+p} \leq \max \left\{ 1, \gamma_*^{-(1+p)} \right\} V(\phi)$ and (2.1), we have

$$
\sup \left\{ \mathbb{E} \| \Phi_{\alpha,\sigma}^\phi(t) \|^{1+p} : \alpha \in \mathcal{K}, \ \|\sigma\| \leq \overline{\sigma}, \ t \geq 0 \right\} < \infty.
$$

$\square$

When the predator is absent, the evolution of the prey follows the stochastic logistic equation on the boundary,

$$
\mathrm{d}\varphi_{\alpha,\sigma}(t) = r\varphi_{\alpha,\sigma}(t) \left(1 - \frac{\varphi_{\alpha,\sigma}(t)}{K} \right) \mathrm{d}t + \sigma_1 \varphi_{\alpha,\sigma}(t) \, \mathrm{d}B_1(t). \tag{2.3}
$$

Denote by $\varphi_{\alpha,\sigma}(t)$ the solution of (2.3). By the comparison theorem, it can be seen that $x_{\alpha,\sigma}(t) \leq \varphi_{\alpha,\sigma}(t)$ for all $t \geq 0$ almost surely (a.s.), provided that $x_{\alpha,\sigma}(0) = \varphi_{\alpha,\sigma}(0) > 0$ and $y_{\alpha,\sigma}(0) \geq 0$.

Lemma 2.2 *Let $(x_{\alpha,\sigma}(t), y_{\alpha,\sigma}(t))$ be a solution of (1.4) and $\varphi_{\alpha,\sigma}(t)$ a solution of (2.3).*

(i) *If $r < \sigma_1^2/2$ then the system is exponentially ruined in the sense that the Lyapunov exponents of $\varphi_{\alpha,\sigma}(\cdot)$, $x_{\alpha,\sigma}(\cdot)$ and $y_{\alpha,\sigma}(\cdot)$ are negative.*

(ii) *In the case $r - \sigma_1^2/2 > 0$, (2.3) has a unique stationary distribution $\nu_{\alpha,\sigma}$ with the density $p_{\alpha,\sigma}(x) = Cx^{(2r/\sigma_1^2)-2} e^{-(2r/\sigma_1^2 K)x}$, $x \geq 0$. Further, $\nu_{\alpha,\sigma}$ weakly converges to $\delta_K(\cdot)$ as $\sigma_1 \to 0$, where $\delta_K(\cdot)$ is the Dirac measure with mass at K.*

Proof. It is easy to verify that with the initial condition $\varphi_{\alpha,\sigma}(0) = x_0 > 0$, (2.3) has the unique solution

$$
\varphi_{\alpha,\sigma}(t) = \frac{x_0 \exp \left\{ (r - \sigma_1^2/2)t + \sigma B_1(t) \right\}}{K + rx_0 \int_0^t \exp \left\{ (r - \sigma_1^2/2)s + \sigma B_1(s) \right\} \mathrm{d}s} \tag{2.4}
$$

(see [18], for example).

Therefore, by the law of iterated logarithm, we see that if $r - \sigma_1^2/2 < 0$ then

$$\lim_{t\to\infty} \frac{\ln \varphi_{\alpha,\sigma}(t)}{t} = r - \frac{\sigma_1^2}{2} < 0.$$

Using this estimate and the comparison theorem gets

$$\limsup_{t\to\infty} \frac{\ln x_{\alpha,\sigma}(t)}{t} \leq \lim_{t\to\infty} \frac{\ln \varphi_{\alpha,\sigma}(t)}{t} = r - \frac{\sigma_1^2}{2} < 0.$$

On the other hand, from the second equation in (1.4),

$$\frac{\ln y_{\alpha,\sigma}(t)}{t} = \frac{\ln y_{\alpha,\sigma}(0)}{t} - \gamma - \frac{\sigma_2^2}{2} + \frac{1}{t}\int_0^t \frac{\beta m x_{\alpha,\sigma}(s)}{a + b y_{\alpha,\sigma}(s) + c x_{\alpha,\sigma}(s)}\, ds + \frac{\sigma_2 B_2(t)}{t}. \qquad (2.5)$$

By using the strong law of large numbers and (2.5), we have

$$\limsup_{t\to\infty} \frac{\ln y_{\alpha,\sigma}(t)}{t} = -\gamma - \frac{\sigma_2^2}{2} < 0,$$

which proves item (i).

Consider now the Fokker–Planck equation with respect to (2.3),

$$\frac{\partial p_{\alpha,\sigma}(x, t)}{\partial t} = -\frac{\partial[rx(1 - x/K)p_{\alpha,\sigma}(x, t)]}{\partial x} + \frac{\sigma_1^2}{2}\frac{\partial^2[x^2 p_{\alpha,\sigma}(x)]}{\partial x^2}.$$

It is easy to see that for $r - \sigma_1^2/2 > 0$ this has a unique positive integrable solution $p_{\alpha,\sigma}(x) = Cx^{2r/\sigma_1^2 - 2}e^{-(2r/\sigma_1^2 K)x}$, $x \geq 0$, which is a stationary density of (2.3), where C is the normalizing constant defined by

$$C = \frac{1}{\Gamma\big((2r/\sigma_1^2) - 1\big)}\left(\frac{2r}{\sigma_1^2 K}\right)^{(2r/\sigma_1^2)-1}$$

and $\Gamma(\,\cdot\,)$ is the Gamma function. This means that (2.3) has a stationary distribution whose density is a Gamma distribution $\Gamma\big((2r/\sigma_1^2) - 1, 2r/\sigma_1^2 K\big)$. By direct calculation we have

$$\lim_{\sigma\to 0} \mathbb{E}(\varphi_{\alpha,\sigma}(t)) = \lim_{\sigma_1\to 0}\left[K - \frac{K\sigma_1^2}{2r}\right] = K,$$

$$\lim_{\sigma_1\to 0} \mathrm{Var}(\varphi_{\alpha,\sigma}(t)) = \lim_{\sigma_1\to 0}\left[\frac{K^2\sigma_1^4}{4r^2}\left(\frac{2r}{\sigma_1^2} - 1\right)\right] = 0.$$

These equalities imply that the process $\varphi_{\alpha,\sigma}(t)$ converges to K in L_2 as $\sigma_1 \to 0$, which proves item (ii). □

By Lemma 2.2(i), from now on we are only interested in the case $r > \sigma_1^2/2$. For any $\alpha \in \mathbb{R}_+^{8,o}$ and $\sigma \geq 0$ we define a threshold

$$\lambda_{\alpha,\sigma} = -\gamma - \frac{\sigma_2^2}{2} + \int_0^\infty \frac{m\beta x}{a + cx}p_{\alpha,\sigma}(x)\, dx.$$

We note that when $\sigma = 0$ we have $\lambda_{\alpha,0} = \lambda_\alpha$ as defined by (1.2).

Lemma 2.3 *The mapping $(\alpha, \sigma) \to \lambda_{\alpha,\sigma}$ is continuous on the domain*

$$\mathcal{D} := \left\{ \alpha = (r, K, m, \beta, \gamma, a, b, c) : \alpha \in \mathbb{R}_+^8, \ r > \sigma_1^2/2, \ \sigma_1 \geq 0 \right\}.$$

Proof. The integrability of the function $x^{(2r/\sigma_1^2)-2} e^{-(2r/\sigma_1^2 K)x}$, $x \geq 0$, depends on the two singular points 0 and ∞. For $\alpha_0 = (r_0, K_0, m_0, \beta_0, \gamma_0, a_0, b_0, c_0) \in \mathcal{D}$ and σ_0 with $\sigma_{0,1} > 0$, we can find a sufficiently small $\eta > 0$ such that the function

$$h(x) = \begin{cases} x^{\left((2r_0-\eta)/(\sigma_{0,1}^2+\eta)\right)-2} e^{-\left((2r_0-\eta)/(\sigma_{0,1}^2 K_0+\eta)\right)x} & \text{for } 0 < x < 1, \\ x^{\left((2r_0+\eta)/(\sigma_{0,1}^2-\eta)\right)-2} e^{-\left((2r_0-\eta)/(\sigma_{0,1}^2 K_0+\eta)\right)x} & \text{for } 1 \leq x. \end{cases}$$

is integrable on $\mathbb{R}_+$. Further, for all α to be close to α_0 and σ to be close to σ_0, we have $p_{\alpha,\sigma}(x) \leq h(x)$ for all $x \in \mathbb{R}_+$. As the function $m\beta x/(a+cx)$ is bounded, we can use the Lebesgue dominated convergence theorem to get $\lim_{\alpha \to \alpha_0, \sigma \to \sigma_0} \lambda_{\alpha,\sigma_0} = \lambda_{\alpha,\sigma_0}$.

The case $\sigma_0 = 0$ follows from Lemma 2.2(ii). $\qquad\square$

Theorem 2.1 *If $\lambda_{\alpha,\sigma} < 0$ then $y_{\alpha,\sigma}(t)$ has the Lyapunov exponent $\lambda_{\alpha,\sigma}$ and $x_{\alpha,\sigma}(t) - \varphi_{\alpha,\sigma}(t)$ converges almost surely to 0 as $t \to \infty$ at an exponential rate.*

Proof. Since the function $h(u) = u/(A+u)$ is increasing in $u > 0$, it follows from (2.5) and the comparison theorem that

$$\frac{\ln y_{\alpha,\sigma}(t)}{t} = \frac{\ln y_{\alpha,\sigma}(0)}{t} - \gamma - \frac{\sigma_2^2}{2} + \frac{1}{t}\int_0^t \frac{\beta m x_{\alpha,\sigma}(s)}{a + b y_{\alpha,\sigma}(s) + c x_{\alpha,\sigma}(s)}\, ds + \frac{\sigma_2 B_2(t)}{t}$$

$$\leq \frac{\ln y_{\alpha,\sigma}(0)}{t} - \gamma - \frac{\sigma_2^2}{2} + \frac{1}{t}\int_0^t \frac{\beta m \varphi_{\alpha,\sigma}(s)}{a + c\varphi_{\alpha,\sigma}(s)}\, ds + \frac{\sigma_2 B_2(t)}{t}.$$

Letting $t \to \infty$ and applying the law of large numbers to the process $\varphi_{\alpha,\sigma}$, we obtain

$$\limsup_{t \to \infty} \frac{\ln y_{\alpha,\sigma}(t)}{t} \leq -\gamma - \frac{\sigma_2^2}{2} + \int_0^\infty \frac{\beta m x}{a + cx} p_{\alpha,\sigma}(x)\, dx = \lambda_{\alpha,\sigma}. \tag{2.6}$$

We now prove that the process $x_{\alpha,\sigma}(t) - \varphi_{\alpha,\sigma}(t)$ converges almost surely to 0 by estimating the rate of convergence $\varphi_{\alpha,\sigma}(t) - x_{\alpha,\sigma}(t)$ when $t \to \infty$. Using Itô's formula, we get

$$\ln x_{\alpha,\sigma}(t) = \ln x_0 + \int_0^t \left(r\left(1 - \frac{x_{\alpha,\sigma}(s)}{K}\right) - \frac{m y_{\alpha,\sigma}(s)}{a + b y_{\alpha,\sigma}(s) + c x_{\alpha,\sigma}(s)} - \frac{\sigma_1^2}{2} \right) ds + \sigma_1 B_1(t),$$

$$\ln \varphi_{\alpha,\sigma}(t) = \ln \varphi_{\alpha,\sigma}(0) + \int_0^t \left(r\left(1 - \frac{\varphi_{\alpha,\sigma}(s)}{K}\right) - \frac{\sigma_1^2}{2} \right) ds + \sigma_1 B_1(t).$$

From these and the inequalities

$$\frac{my}{a + by + cx} \leq \frac{my}{a} \quad \text{for all } x, y > 0, \qquad x_{\alpha,\sigma}(t) \leq \varphi_{\alpha,\sigma}(t), \quad t \geq 0,$$

we have

$$0 \leq \limsup_{t \to \infty} \frac{\ln \varphi_{\alpha,\sigma}(t) - \ln x_{\alpha,\sigma}(t)}{t}$$

$$\leq \limsup_{t \to \infty} \frac{1}{t} \int_0^t \left(\frac{r}{K}(x_{\alpha,\sigma}(s) - \varphi_{\alpha,\sigma}(s)) + \frac{m}{a} y_{\alpha,\sigma}(s) \right) ds$$

$$\leq \frac{m}{a} \limsup_{t \to \infty} \frac{1}{t} \int_0^t y_{\alpha,\sigma}(s) \, ds = 0. \tag{2.7}$$

From (2.4), it is easy to see that $\lim_{t \to \infty} (\ln \varphi_{\alpha,\sigma}(t))/t = 0$ a.s. Combining this with (2.7), we obtain $\lim_{t \to \infty} (\ln x_{\alpha,\sigma}(t))/t = 0$ a.s. Hence, from (2.6),

$$\limsup_{t \to \infty} \frac{\ln y_{\alpha,\sigma}(t) - \ln x_{\alpha,\sigma}(t)}{t} \leq \lambda_{\alpha,\sigma} < 0. \tag{2.8}$$

Otherwise,

$$d\left(\frac{1}{x_{\alpha,\sigma}(t)} \right) = \left(\frac{\sigma_1^2 - r}{x_{\alpha,\sigma}(t)} + \frac{r}{K} + \frac{m y_{\alpha,\sigma}(t)}{x_{\alpha,\sigma}(t)(a + b y_{\alpha,\sigma}(t) + c x_{\alpha,\sigma}(t))} \right) dt - \frac{\sigma_1}{x_{\alpha,\sigma}(t)} \, dB_1(t),$$

$$d\left(\frac{1}{\varphi_{\alpha,\sigma}(t)} \right) = \left(\frac{\sigma_1^2 - r}{\varphi_{\alpha,\sigma}(t)} + \frac{r}{K} \right) dt - \frac{\sigma_1}{\varphi_{\alpha,\sigma}(t)} \, dB_1(t).$$

Put

$$z(t) = \frac{1}{x_{\alpha,\sigma}(t)} - \frac{1}{\varphi_{\alpha,\sigma}(t)}.$$

We see that $z(t) \geq 0$ for all $t \geq 0$, and

$$dz(t) = \left((\sigma_1^2 - r)z(t) + \frac{m y_{\alpha,\sigma}(t)}{x_{\alpha,\sigma}(t)(a + b y_{\alpha,\sigma}(t) + c x_{\alpha,\sigma}(t))} \right) dt - \sigma_1 z(t) \, dB_1(t).$$

In view of the variation of constants formula [21, Theorem 3.1, p .96], this yields

$$z(t) = c_1 \Psi(t) \int_0^t \Psi^{-1}(s) \frac{m y_{\alpha,\sigma}(s)}{x_{\alpha,\sigma}(s)(a + b y_{\alpha,\sigma}(s) + c x_{\alpha,\sigma}(s))} \, ds, \tag{2.9}$$

where $\Psi(t) = e^{(\sigma_1^2/2 - r)t - \sigma_1 B_1(t)}$. It is easy to see that

$$\lim_{t \to \infty} \frac{\ln \Psi(t)}{t} = \frac{\sigma_1^2}{2} - r. \tag{2.10}$$

Let $0 < \bar{\lambda} < \max\left\{ (r - \sigma_1^2)/2, -\lambda_{\alpha,\sigma} \right\}$ be arbitrary, and choose $\varepsilon > 0$ such that $0 < \bar{\lambda} + 3\varepsilon < \max\left\{ r - \sigma_1^2/2, -\lambda_{\alpha,\sigma} \right\}$. From (2.10), there are two positive random variables η_1, η_2 such that

$$\eta_1 e^{(\sigma_1^2/2 - r - \varepsilon)t} \leq \Psi(t) \leq \eta_2 e^{(\sigma_1^2/2 - r + \varepsilon)t} \quad \text{for all } t \geq 0 \text{ a.s.} \tag{2.11}$$

Further, it follows from (2.8) that there exists a positive random variable η_3 that satisfies

$$\frac{y_{\alpha,\sigma}(t)}{x_{\alpha,\sigma}(t)} \leq \eta_3 e^{(\lambda_{\alpha,\sigma} + \varepsilon)t} \quad \text{for all } t \geq 0 \text{ a.s.} \tag{2.12}$$

Combining (2.9), (2.11), and (2.12), we get

$$\mathrm{e}^{\overline{\lambda}t} z(t) \le \frac{c_1 m \eta_2 \eta_3}{a \eta_1} \mathrm{e}^{\left(\overline{\lambda} + \sigma_1^2/2 - r + \varepsilon\right)t} \int_0^t \mathrm{e}^{\left(r - \sigma_1^2/2 + \lambda_{\alpha,\sigma} + 2\varepsilon\right)s} \, \mathrm{d}s.$$

Thus,

$$0 \le \lim_{t \to \infty} \mathrm{e}^{\overline{\lambda}t} z(t) \le \frac{c_1 m \eta_2 \eta_3}{a \eta_1 \left(r - \sigma_1^2/2 + \lambda_{\alpha,\sigma} + 2\varepsilon\right)} \lim_{t \to \infty} \left(\mathrm{e}^{(\lambda_{\alpha,\sigma} + \overline{\lambda} + 3\varepsilon)t} - \mathrm{e}^{(\overline{\lambda} + \sigma_1^2/2 - r + \varepsilon)t}\right) = 0.$$

As a result,

$$\lim_{t \to \infty} \mathrm{e}^{\overline{\lambda}t} z(t) = \lim_{t \to \infty} \mathrm{e}^{\overline{\lambda}t} \left(\frac{1}{x_{\alpha,\sigma}(t)} - \frac{1}{\varphi_{\alpha,\sigma}(t)}\right) = 0. \tag{2.13}$$

Since $\lim_{t \to \infty} (\ln \varphi_{\alpha,\sigma}(t))/t = 0$, $\lim_{t \to \infty} \mathrm{e}^{-\overline{\lambda}t/2} \varphi_{\alpha,\sigma}^2(t) = 0$. Using (2.13) we get

$$\lim_{t \to \infty} \mathrm{e}^{\overline{\lambda}t/2} (\varphi_{\alpha,\sigma}(t) - x_{\alpha,\sigma}(t)) = \lim_{t \to \infty} \mathrm{e}^{\overline{\lambda}t/2} \varphi_{\alpha,\sigma}(t) x_{\alpha,\sigma}(t) \left(\frac{1}{x_{\alpha,\sigma}(t)} - \frac{1}{\varphi_{\alpha,\sigma}(t)}\right)$$

$$= \lim_{t \to \infty} \mathrm{e}^{-\overline{\lambda}t/2} \varphi_{\alpha,\sigma}(t) x_{\alpha,\sigma}(t) \mathrm{e}^{\overline{\lambda}t} \left(\frac{1}{x_{\alpha,\sigma}(t)} - \frac{1}{\varphi_{\alpha,\sigma}(t)}\right) = 0.$$

This means that $x_{\alpha,\sigma}(t) - \varphi_{\alpha,\sigma}(t)$ converges almost surely to 0 as $t \to \infty$ at an exponential rate.

We now turn to the estimate of the Lyapunov exponent of $y_{\alpha,\sigma}$. From (2.5) we get

$$\frac{\ln y_{\alpha,\sigma}(t)}{t} = \frac{\ln y_{\alpha,\sigma}(0)}{t} - \gamma - \frac{\sigma_2^2}{2} + \frac{1}{t} \int_0^t \frac{\beta m x_{\alpha,\sigma}(s)}{a + b y_{\alpha,\sigma}(s) + c x_{\alpha,\sigma}(s)} \, \mathrm{d}s + \frac{\sigma_2 B_2(t)}{t}$$

$$= \frac{\ln y_{\alpha,\sigma}(0)}{t} - \gamma - \frac{\sigma_2^2}{2} + \frac{1}{t} \int_0^t \frac{\beta m \varphi_{\alpha,\sigma}(s)}{a + c \varphi_{\alpha,\sigma}(s)} \, \mathrm{d}s + \frac{\sigma_2 B_2(t)}{t}$$

$$+ \frac{1}{t} \int_0^t \beta m \left(\frac{x_{\alpha,\sigma}(s)}{a + b y_{\alpha,\sigma}(s) + c x_{\alpha,\sigma}(s)} - \frac{\varphi_{\alpha,\sigma}(s)}{a + c \varphi_{\alpha,\sigma}(s)}\right) \mathrm{d}s.$$

Since $\lim_{t \to \infty} (x_{\alpha,\sigma}(t) - \varphi_{\alpha,\sigma}(t)) = 0$ and $\lim_{t \to \infty} y_{\alpha,\sigma}(t) = 0$, it is easy to see that

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \beta m \left(\frac{x_{\alpha,\sigma}(s)}{a + b y_{\alpha,\sigma}(s) + c x_{\alpha,\sigma}(s)} - \frac{\varphi_{\alpha,\sigma}(s)}{a + c \varphi_{\alpha,\sigma}(s)}\right) \mathrm{d}s = 0.$$

Thus,

$$\lim_{t \to \infty} \frac{\ln y_{\alpha,\sigma}(t)}{t} = \lim_{t \to \infty} \left(\frac{\ln y_{\alpha,\sigma}(0)}{t} - \gamma - \frac{\sigma_2^2}{2} + \frac{1}{t} \int_0^t \frac{\beta m \varphi_{\alpha,\sigma}(s)}{a + c \varphi_{\alpha,\sigma}(s)} \, \mathrm{d}s + \frac{\sigma_2 B_2(t)}{t}\right) = \lambda_{\alpha,\sigma},$$

which completes the proof. $\qquad\square$

The following lemma is similar to one in [6].

Lemma 2.4 *For T, $\varsigma > 0$ and $\alpha_0 \in \mathbb{R}_+^{8,\mathrm{o}}$, there exists a number κ and δ such that, for all $\alpha \in \mathcal{U}_\delta(\alpha_0)$ and $0 < \|\sigma\| < \kappa$,*

$$\mathbb{P}\big\{\big\|\Phi_{\alpha,\sigma}^\phi(t) - \Phi_{\alpha_0}^\phi(t)\big\| \ge \varsigma \text{ for some } t \in [0, T]\big\} < \exp\left\{-\frac{\kappa}{\|\sigma\|^2}\right\}, \quad \phi \in \mathcal{R}_\delta(\alpha_0). \tag{2.14}$$

Proof. Let $0 < \delta \le \varsigma/2L_1Te^{L_1T}$. From (1.3) we have

$$\sup_{0 \le t \le T} \left\| \Phi_\alpha^\phi(t) - \Phi_{\alpha_0}^\phi(t) \right\| \le \frac{\varsigma}{2} \quad \text{for all } \alpha \in \mathcal{U}_\delta(\alpha_0), \ \phi \in \mathcal{R}_\delta(\alpha_0). \tag{2.15}$$

Let $R > 0$ be large enough that $\mathcal{R}_\delta(\alpha_0) \subset \mathbf{B}_R$, and let $h_R(\,\cdot\,)$ be a twice-differentiable function such that, for $0 \le h_R \le 1$,

$$h_R(\phi) = \begin{cases} 1 & \text{if } \|\phi\| \le R, \\ 0 & \text{if } \|\phi\| > R+1. \end{cases}$$

Put

$$f_h(\phi, \alpha) = h(\phi)f(\phi, \alpha) = \begin{pmatrix} h(\phi)f^{(1)}(\phi, \alpha) \\ h(\phi)f^{(2)}(\phi, \alpha) \end{pmatrix},$$

$$g_h(\phi) = h(x, y)g(x, y) = h(x, y)\begin{pmatrix} \sigma_1 x & 0 \\ 0 & \sigma_2 y \end{pmatrix}.$$

It can be seen that $f_h(\phi, \alpha)$ is a Lipschitz function, i.e. there exists $M > 0$ such that

$$\|f_h(\phi_1, \alpha) - f_h(\phi_2, \alpha)\| \le M\|\phi_2 - \phi_1\| \quad \text{for all } \alpha \in \mathcal{U}_\delta(\alpha_0), \ \phi_1, \phi_2 \in \mathbb{R}_+^2. \tag{2.16}$$

If we choose $M \ge 2R^2$ we also have

$$\left\| g_h(\phi)g_h^\top(\phi) \right\| \le M\|\sigma\|^2 \quad \text{for all } \phi \in \mathbb{R}_+^2. \tag{2.17}$$

Let $\widetilde{\Phi}_{\alpha,\sigma}^\phi(t)$ be the solution starting at $\phi \in \mathcal{R}_\delta(\alpha_0)$ of the equation

$$\mathrm{d}\widetilde{\Phi}(t) = f_h\big(\widetilde{\Phi}(t), \alpha\big)\,\mathrm{d}t + g_h\big(\widetilde{\Phi}(t)\big)\,\mathrm{d}B(t),$$

where $B(t) = (B_1(t), B_2(t))^\top$. Define the stopping time $\tau_R^\phi = \inf\big\{t \ge 0 : \big\|\Phi_{\alpha,\sigma}^\phi(t)\big\| \ge R\big\}$.

It is easy to see that $\widetilde{\Phi}_{\alpha,\sigma}^\phi(t) = \Phi_{\alpha,\sigma}^\phi(t)$ up to time τ_R^ϕ. Since the state space $\mathcal{R}_\delta(\alpha_0)$ of (1.1) is contained in $\mathbf{B}_R$, the solution $\Phi_\alpha^\phi(\,\cdot\,)$ of (1.1) is also the solution of the equation $\mathrm{d}\Phi(t) = f_h(\Phi(t), \alpha)\,\mathrm{d}t$, $t \ge 0$, with initial value $\phi \in \mathcal{R}_\delta(\alpha_0)$. For all $t \in [0, T]$, using Itô's formula we have

$$\begin{aligned} \left\| \widetilde{\Phi}_{\alpha,\sigma}^\phi(t) - \Phi_\alpha^\phi(t) \right\|^2 = {}& 2\int_0^t \big(\widetilde{\Phi}_{\alpha,\sigma}^\phi(s) - \Phi_\alpha^\phi(s)\big)^\top \big(f_h(\widetilde{\Phi}_{\alpha,\sigma}^\phi(s), \alpha) - f_h\big(\Phi_\alpha^\phi(s), \alpha\big)\big)\,\mathrm{d}s \\ & + \int_0^t h\big(\widetilde{\Phi}_{\alpha,\sigma}^\phi(s)\big)\mathrm{trace}\big(g_h\big(\widetilde{\Phi}_{\alpha,\sigma}^\phi(s)\big)g_h^\top\big(\widetilde{\Phi}_{\alpha,\sigma}^\phi(s)\big)\big)\,\mathrm{d}s \\ & + 2\int_0^t \big(\widetilde{\Phi}_{\alpha,\sigma}^\phi(s) - \Phi_\alpha^\phi(s)\big)^\top g_h\big(\widetilde{\Phi}_{\alpha,\sigma}^\phi(s)\big)\,\mathrm{d}B(t). \end{aligned}$$

By the exponential martingale inequality, for $T, \varsigma > 0$ there exists a number $\kappa = \kappa(T, \varsigma)$ such that $\mathbb{P}(\widetilde{\Omega}) \ge 1 - \exp\big\{-\kappa/\|\sigma\|^2\big\}$, where

$$\begin{aligned} \widetilde{\Omega} = \bigg\{ & \sup_{t \in [0,T]} \bigg[\int_0^t \big(\widetilde{\Phi}_{\alpha,\sigma}^\phi(s) - \Phi_\alpha^\phi(s)\big)^\top g_h\big(\widetilde{\Phi}_{\alpha,\sigma}^\phi(s)\big)\,\mathrm{d}B(t) \\ & - \frac{1}{2\|\sigma\|^2}\int_0^t \big(\widetilde{\Phi}_{\alpha,\sigma}^\phi(s) - \Phi_\alpha^\phi(s)\big)^\top g_h\big(\widetilde{\Phi}_{\alpha,\sigma}^\phi(s)\big)g_h^\top\big(\widetilde{\Phi}_{\alpha,\sigma}^\phi(s)\big)\big(\widetilde{\Phi}_{\alpha,\sigma}^\phi(s) - \Phi_\alpha^\phi(s)\big)\,\mathrm{d}s \bigg] \\ & : \kappa, \phi \in \mathcal{R}_\delta(\alpha_0) \bigg\}. \end{aligned}$$

From (2.16) and (2.17), this implies that, for all $\omega \in \widetilde{\Omega}$,

$$
\begin{aligned}
\left\| \widetilde{\Phi}^{\phi}_{\alpha,\sigma}(t) - \Phi^{\phi}_{\alpha}(t) \right\|^2 &\le 2 \int_0^t \left\| \widetilde{\Phi}^{\phi}_{\alpha,\sigma}(s) - \Phi^{\phi}_{\alpha}(s) \right\| \left\| f_h(\widetilde{\Phi}^{\phi}_{\alpha,\sigma}(s), \alpha) - f_h(\Phi^{\phi}_{\alpha}(s), \alpha) \right\| \, \mathrm{d}s \\
&\quad + \int_0^t h\big(\widetilde{\Phi}^{\phi}_{\alpha,\sigma}(s)\big) \mathrm{trace}\big(g_h\big(\widetilde{\Phi}^{\phi}_{\alpha,\sigma}(s)\big) g_h^{\top}\big(\widetilde{\Phi}^{\phi}_{\alpha,\sigma}(s)\big)\big) \, \mathrm{d}s \\
&\quad + \frac{1}{\|\sigma\|^2} \int_0^t \left\| \widetilde{\Phi}^{\phi}_{\alpha,\sigma}(s) - \Phi^{\phi}_{\alpha}(s) \right\|^2 \left\| g_h\big(\widetilde{\Phi}^{\phi}_{\alpha,\sigma}(s)\big) g_h^{\top}\big(\widetilde{\Phi}^{\phi}_{\alpha,\sigma}(s)\big) \right\| \, \mathrm{d}s \\
&\quad + 2 \int_0^t \kappa \, \mathrm{d}s \\
&\le 3M \int_0^t \left\| \widetilde{\Phi}^{\phi}_{\alpha,\sigma}(s) - \Phi^{\phi}_{\alpha}(s) \right\|^2 \, \mathrm{d}s + (M\|\sigma\|^2 + 2\kappa)T.
\end{aligned}
$$

For all $t \in [0, T]$, an application of the Gronwall–Belmann inequality implies that

$$
\left\| \widetilde{\Phi}^{\phi}_{\alpha,\sigma}(t) - \Phi^{\phi}_{\alpha}(t) \right\| \le \sqrt{(M\|\sigma\|^2 + 2\kappa)T \exp\{3MT\}}
$$

in the set $\widetilde{\Omega}$. Choosing κ sufficiently small that $(M\kappa^2 + 2\kappa)T \exp\{3MT\} \le \varsigma^2/4$ implies that

$$
\left\| \widetilde{\Phi}^{\phi}_{\alpha,\sigma}(t) - \Phi^{\phi}_{\alpha}(t) \right\| \le \frac{\varsigma}{2} \quad \text{for all } \alpha \in \mathcal{U}_{\delta}(\alpha_0), \ 0 < \|\sigma\| < \kappa, \ \omega \in \widetilde{\Omega}. \tag{2.18}
$$

From (2.18), (2.15), and the triangle inequality, we have $\|\widetilde{\Phi}^{\phi}_{\alpha,\sigma}(t) - \Phi^{\phi}_{\alpha_0}(t)\| \le \varsigma$ for all $\alpha \in \mathcal{U}_{\delta}(\alpha_0)$, $t \in [0, T]$, $\omega \in \widetilde{\Omega}$. It also follows from this inequality that when $\omega \in \widetilde{\Omega}$ we have $\tau_R > T$, which implies that

$$
\mathbb{P}\big\{ \left\| \Phi^{\phi}_{\alpha,\sigma}(t) - \Phi^{\phi}_{\alpha_0}(t) \right\| < \varsigma \text{ for all } t \in [0, T] \big\} \ge \mathbb{P}(\Omega_1) \ge 1 - \exp\left\{ \frac{\kappa}{\|\sigma\|^2} \right\}
$$

holds for all $0 < \|\sigma\| < \kappa$ and $\alpha \in \mathcal{U}_{\delta}(\alpha_0)$. This completes the proof. $\qquad\square$

Theorem 2.2 *Let $\alpha_0 = (r_0, K_0, m_0, \beta_0, \gamma_0, a_0, b_0, c_0)$ be a vector of parameters of the system in* (1.1) *such that $\lambda_{\alpha_0} > 0$ and*

$$
b_0 < \min\left\{ \frac{c_0}{\beta_0}, \ \frac{m_0^2 \beta_0^2 - c_0^2 \gamma_0^2}{\gamma_0 \beta_0 (m_0 \beta_0 - c_0 \gamma_0) + m_0 r_0 \beta_0^2} \right\}.
$$

Then, there exist $\delta_1 > 0$ and $\overline{\sigma} > 0$ such that, for all $\alpha \in \mathcal{U}_{\delta_1}(\alpha_0)$ and $0 < \|\sigma\| \le \overline{\sigma}$, the Markov process $\Phi^{\phi}_{\alpha,\sigma}(t) = (x_{\alpha,\sigma}(t), y_{\alpha,\sigma}(t))$ has a unique stationary distribution $\mu_{\alpha,\sigma}$. Further, $\mu_{\alpha,\sigma}$ is concentrated on $\mathbb{R}^{2,\circ}_+$ and has a density with respect to the Lebesgue measure. Also, for any open set $\mathcal{V}$ containing Γ_{α_0} we have

$$
\lim_{(\alpha,\sigma) \to (\alpha_0, 0)} \mu_{\alpha,\sigma}(\mathcal{V}) = 1, \tag{2.19}
$$

where Γ_{α_0} is the limit cycle of the system in (1.1) *corresponding to the parameter α_0.*

Proof. Since

$$
\lambda_{\alpha_0} = -\gamma_0 + \frac{K m_0 \beta_0}{a_0 + K c_0} > 0, \qquad b_0 < \min\left\{ \frac{c_0}{\beta_0}, \ \frac{m_0^2 \beta_0^2 - c_0^2 \gamma_0^2}{\gamma_0 \beta_0 (m_0 \beta_0 - c_0 \gamma_0) + m_0 r_0 \beta_0^2} \right\},
$$

we can use Lemma 2.3 to find $\delta_1 > 0$ and $\overline{\sigma}$ such that

$$\lambda_{\alpha,\sigma} = -\gamma - \frac{\sigma_2^2}{2} + \int_0^\infty \frac{m\beta x}{a+cx} p_{\alpha,\sigma}(x)\,\mathrm{d}x > 0, \quad b < \min\left\{\frac{c}{\beta}, \frac{m^2\beta^2 - c^2\gamma^2}{\gamma\beta(m\beta - c\gamma) + mr\beta^2}\right\}$$

hold for all $\alpha \in \mathcal{U}_{\delta_1}(\alpha_0)$ and $0 \leq \|\sigma\| \leq \overline{\sigma}$. By virtue of [7, Theorem 2.3, p. 192], the Markov process $\Phi_{\alpha,\sigma}^\phi(t) = (x_{\alpha,\sigma}(t), y_{\alpha,\sigma}(t))$ has a unique stationary distribution $\mu_{\alpha,\sigma}$ with support $\mathbb{R}^{2,\circ}$. Further, by [1, 19], the stationary distribution $\mu_{\alpha,\sigma}$ has a density with respect to the Lebesgue measure on $\mathbb{R}^{2,\circ}$ by the non-degenerate property of $g(\phi)$.

On the other hand, it follows from (2.2) that for any $\varepsilon > 0$ there exists $R = R(\varepsilon)$ such that $\mu_{\alpha,\sigma}(\mathbf{B}_R) \geq 1 - \varepsilon$ for every $\alpha \in \mathcal{U}_{\delta_1}(\alpha_0)$ and $0 \leq \|\sigma\| \leq \overline{\sigma}$.

By the assumption of Theorem 2.2, it can be seen from Lemma 1.1(ii) that $\phi_{\alpha_0}^*$ is a source point, i.e. two eigenvalues of the matrix $Df(\phi_{\alpha_0}^*, \alpha_0)$ have positive real parts. Therefore, the Lyapunov equation $Df(\phi_{\alpha_0}^*, \alpha_0)^\top P + P Df(\phi_{\alpha_0}^*, \alpha_0) = I$ has a positive definite solution P. Since $Df(\phi, \alpha)$ is continuous in ϕ, α, there exist positive constants $0 < \delta_2 < \delta_1$, $0 < \delta_3$, and c such that

$$\dot{V}(\phi, \alpha) = V_\phi(\phi, \alpha)f(\phi, \alpha) > c\|\phi - \phi_\alpha^*\|,$$
$$\mathrm{trace}\big(g^\top(\phi)Pg(\phi)\big) \geq c\|\sigma\|^2$$

for all $\phi \in \mathcal{V}_{\delta_3}(\phi_{\alpha_0}^*)$, $\alpha \in \mathcal{U}_{\delta_2}(\alpha_0)$, where $V(\phi, \alpha) = (\phi - \phi_\alpha^*)^\top P(\phi - \phi_\alpha^*)$. Hence,

$$\mathcal{L}V(\phi, \alpha) = V_\phi(\phi, \alpha)f(\phi, \alpha) + \mathrm{trace}\big(g^\top(\phi)Pg(\phi)\big) \geq c\|\sigma\|^2$$

for all $\phi \in \mathcal{V}_{\delta_3}(\phi_{\alpha_0}^*)$, $\alpha \in \mathcal{U}_{\delta_2}(\alpha_0)$.

Writing $S = \mathcal{V}_{\delta_3/2}(\phi_{\alpha_0}^*)$ and $\mathcal{Z} = \mathcal{V}_{\delta_3}(\phi_{\alpha_0}^*)$, we now prove that $\lim_{(\alpha,\sigma)\to(\alpha_0,0)} \mu_{\alpha,\sigma}(S) = 0$. First, we note from (2.14) that there is T^* such that if $\phi \in \mathbf{B}_R \setminus S$ then $\Phi_\alpha^\phi(t) \in \mathbf{B}_R \setminus \mathcal{Z}$ for all $t \geq T^*$. Further, for any $T > 0$ we have

$$\mu_{\alpha,\sigma}(S) = \int_{\mathbb{R}_+^2} P_{\alpha,\sigma}(T, \phi, S)\mu_{\alpha,\sigma}(\mathrm{d}\phi)$$
$$= \int_{\mathbf{B}_R \setminus S} P_{\alpha,\sigma}(T, \phi, S)\mu_{\alpha,\sigma}(\mathrm{d}\phi) + \int_S P_{\alpha,\sigma}(T, \phi, S)\mu_{\alpha,\sigma}(\mathrm{d}\phi) + \int_{\mathbf{B}_R^c} P_{\alpha,\sigma}(T, \phi, S)\mu_{\alpha,\sigma}(\mathrm{d}\phi),$$

where $P_{\alpha,\sigma}(T, \phi, \cdot) = \mathbb{P}\big(\Phi_{\alpha,\sigma}^\phi(T) \in \cdot\big)$ is the transition probability of the Markov process $\Phi_{\alpha,\sigma}^\phi$. It is clear that

$$\int_{\mathbf{B}_R^c} P_{\alpha,\sigma}(T, \phi, S)\mu_{\alpha,\sigma}(\mathrm{d}\phi) \leq \mu_{\alpha,\sigma}\big(\mathbf{B}_R^c\big) \leq \varepsilon.$$

For $\phi \in S$, let $\tau_{\alpha,\sigma}^\phi$ be the exit time of $\Phi_{\alpha,\sigma}^\phi(\cdot)$ from $\mathcal{Z}$, i.e. $\tau_{\alpha,\sigma}^\phi = \inf\big\{t \geq 0 : \Phi_{\alpha,\sigma}^\phi(t) \notin \mathcal{Z}\big\}$. By Itô's formula, we have

$$\theta \geq \mathbb{E}V\big(\Phi_{\alpha,\sigma}^\phi\big(\tau_{\alpha,\sigma}^\phi \wedge t\big)\big) - V(\phi) = \mathbb{E}\int_0^{\tau_{\alpha,\sigma}^\phi \wedge t} \mathcal{L}V\big(\Phi_{\alpha,\sigma}^\phi(s)\big)\,\mathrm{d}s$$
$$\geq c\|\sigma\|^2 \mathbb{E}\big[\tau_{\alpha,\sigma}^\phi \wedge t\big] \geq c\|\sigma\|^2 t\mathbb{P}\big(\tau_{\alpha,\sigma}^\phi \geq t\big), \quad t \geq 0,$$

where $\theta = \sup\{V(\phi, \alpha) : \phi \in \mathcal{Z}, \alpha \in \mathcal{U}_{\delta_2}(\alpha_0)\}$. Choosing

$$T_\sigma = \max\left\{\frac{2\theta}{c\|\sigma\|^2}, T^*\right\}$$

means that $\mathbb{P}(\tau_{\alpha,\sigma}^\phi \geq T_\sigma) \leq \frac{1}{2}$ for all $\phi \in S$, $\alpha \in \mathcal{U}_{\delta_2}(\alpha_0)$. Thus,

$$\begin{aligned}
P_{\alpha,\sigma}(T_\sigma, \phi, S) &= \mathbb{P}(\Phi^\phi(T_\sigma) \in S) \\
&= \mathbb{P}(\Phi^\phi(T_\sigma) \in S, \tau_{\alpha,\sigma}^\phi \geq T_\sigma) + \mathbb{P}(\Phi^\phi(T_\sigma) \in S, \tau_{\alpha,\sigma}^\phi < T_\sigma) \\
&\leq \frac{1}{2} + \mathbb{P}(\Phi^\phi(T_\sigma) \in S, \tau_{\alpha,\sigma}^\phi < T_\sigma). \tag{2.20}
\end{aligned}$$

By using the strong Markov property of the solution, we get

$$\mathbb{P}(\Phi^\phi(T_\sigma) \in S, \tau_{\alpha,\sigma}^\phi < T_\sigma) = \int_0^{T_\sigma}\left[\int_{\partial \mathcal{Z}} \mathbb{P}\{\Phi_{\alpha,\sigma}^\psi(T_\sigma - t) \in S\}\mathbb{P}\{\Phi_{\alpha,\sigma}^\phi(t) \in d\psi\}\right]P\{\tau_{\alpha,\sigma}^\phi \in dt\}.$$

Note that when $\psi \in \partial \mathcal{Z}$ we have $\Phi_\alpha^\psi(T_\sigma - t) \notin \mathcal{Z}$ for all $t \geq 0$. Therefore, by Lemma 2.4 with $\varsigma = \delta_3/2$ we can find $\overline{\sigma} > \kappa > 0$ and $0 < \delta_4 \leq \delta_2$ such that $\mathbb{P}(\Phi_{\alpha,\sigma}^\psi(T_\sigma - t) \in S) \leq \exp\{-\kappa/\|\sigma\|^2\}$ for $0 \leq t \leq T_\sigma$, $0 < \|\sigma\| < \kappa$, and $\alpha \in \mathcal{U}_{\delta_4}(\alpha_0)$. Hence,

$$\mathbb{P}\{\Phi^\phi(T_\sigma) \in S, \tau_{\alpha,\sigma}^\phi < T_\sigma\} \leq T_\sigma \exp\left\{-\frac{\kappa}{\|\sigma\|^2}\right\}, \quad \alpha \in \mathcal{U}_{\delta_4}(\alpha_0),\ 0 < \|\sigma\| < \kappa. \tag{2.21}$$

Combining (2.20) and (2.21), we get

$$\int_S P(T_\sigma, \phi, S)\mu_{\alpha,\sigma}(d\phi) \leq \frac{1}{2}\mu_{\alpha,\sigma}(S) + T_\sigma \exp\left\{-\frac{\kappa}{\|\sigma\|^2}\right\}, \quad \alpha \in \mathcal{U}_{\delta_4}(\alpha_0),\ 0 < \|\sigma\| < \kappa.$$

On the other hand, when $\phi \in \mathbf{B}_R \setminus S$ we see that $\Phi_\alpha^\phi(T_\sigma) \notin \mathcal{Z}$. Therefore, using (2.14) again we get

$$\int_{\mathbf{B}_R \setminus S} P(T_\sigma, \phi, S)\mu_{\alpha,\sigma}(d\phi) < \exp\left\{-\frac{\kappa}{\|\sigma\|^2}\right\}, \quad \alpha \in \mathcal{U}_{\delta_4}(\alpha_0),\ 0 < \|\sigma\| < \kappa.$$

Summing up, we have

$$\mu_{\alpha,\sigma}(S) \leq \frac{1}{2}\mu_{\alpha,\sigma}(S) + (T_\sigma + 1)\exp\left\{-\frac{\kappa}{\|\sigma\|^2}\right\} + \varepsilon, \quad \alpha \in \mathcal{U}_{\delta_4}(\alpha_0),\ 0 < \|\sigma\| < \kappa.$$

Noting that $\lim_{(\alpha,\sigma)\to(\alpha_0,0)}(T_\sigma + 1)\exp\{-\kappa/\|\sigma\|^2\} = 0$, we can pass the limit to get

$$\limsup_{(\alpha,\sigma)\to(\alpha_0,0)}\mu_{\alpha,\sigma}(S) \leq \frac{1}{2}\limsup_{(\alpha,\sigma)\to(\alpha_0,0)}\mu_{\alpha,\sigma}(S) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\limsup_{(\alpha,\sigma)\to(\alpha_0,0)}\mu_{\alpha,\sigma}(S) = 0$.

We now prove (2.19). Let $\mathcal{V}$ be an open set containing Γ_{α_0}. It suffices to show that, for any compact set B intersecting neither $\mathcal{V}$ nor S, we have $\lim_{(\alpha,\sigma)\to(\alpha_0,0)}\mu_{\alpha,\sigma}(B) = 0$. Let $3d = \text{dist}(\partial \mathcal{V}, \Gamma_{\alpha_0})$ and $\mathcal{V}_d(\Gamma_{\alpha_0}) = \{x : \text{dist}(x, \Gamma_{\alpha_0}) < d\}$. From Lemma 1.2(ii), there exists $0 < \delta_5 <$

δ_4 such that $\Gamma_\alpha \subset \mathcal{V}_d(\Gamma_{\alpha_0})$ for all $\alpha \in \overline{\mathcal{U}_{\delta_5}}(\alpha_0)$. It is clear that $\mathrm{dist}(B, \mathcal{V}_d(\Gamma_{\alpha_0})) > 2d$. Let $\varepsilon > 0$ and $R = R(\varepsilon) > 0$ be such that $S, B \subset \mathbf{B}_R$ and $\mu_{\alpha,\sigma}(\mathbf{B}_R^c) \leq \varepsilon$. Since Γ_{α_0} is asymptotically stable, we can find an open neighbourhood U of Γ_{α_0} such that $U \subset \mathcal{V}_d(\Gamma_{\alpha_0})$ and if $\phi \in U$ then $\Phi_{\alpha_0}^\phi(t) \in \mathcal{V}_d(\Gamma_{\alpha_0})$ for all $t \geq 0$. Further, the simple property of the limit cycle Γ_{α_0} implies that $\lim_{t \to \infty} \mathrm{dist}(\Phi_{\alpha_0}^\phi(t), \Gamma_{\alpha_0}) = 0$ for all $\phi \notin S$. This means that, for every $\phi \in \overline{\mathbf{B}_R} \setminus S$, there exists a T^ϕ such that $\Phi_{\alpha_0}^\phi(t) \in U$ for all $t \geq T^\phi$.

By the continuity of solutions on the initial condition, there exists $\delta_\phi > 0$ such that $\Phi_{\alpha_0}^z(t) \in U$ for all $z \in \mathcal{V}_{\delta_\phi}(\phi)$ and $t > T^\phi$. Since $\overline{\mathbf{B}_R} \setminus S$ is compact, there are $\phi_1, \phi_2, \ldots, \phi_n$ such that $\overline{\mathbf{B}_R} \setminus S \subset \cup_{k=1}^n \mathcal{V}_{\delta_{\phi_k}}(\phi_k)$. Let $T = \max\left\{T^{\phi_1}, T^{\phi_2}, \ldots, T^{\phi_n}\right\}$. It can be seen that $\Phi_{\alpha_0}^\phi(T) \in U \subset \mathcal{V}_d(\Gamma_{\alpha_0})$ for all $\phi \in \overline{\mathbf{B}_R} \setminus S$. Similar to the above, we have

$$\mu_{\alpha,\sigma}(B) = \int_{\mathbf{B}_R \setminus S} P_{\alpha,\sigma}(T, \phi, B)\mu_{\alpha,\sigma}(\mathrm{d}\phi) + \int_S P_{\alpha,\sigma}(T, \phi, B)\mu_{\alpha,\sigma}(\mathrm{d}\phi)$$
$$+ \int_{\mathbf{B}_R^c} P_{\alpha,\sigma}(T, \phi, B)\mu_{\alpha,\sigma}(\mathrm{d}\phi)$$
$$\leq \int_{\mathbf{B}_R \setminus S} P_{\alpha,\sigma}(T, \phi, B)\mu_{\alpha,\sigma}(\mathrm{d}\phi) + \mu_{\alpha,\sigma}(S) + \mu_{\alpha,\sigma}(\mathbf{B}_R^c)$$
$$\leq \int_{\mathbf{B}_R \setminus S} P_{\alpha,\sigma}(T, \phi, B)\mu_{\alpha,\sigma}(\mathrm{d}\phi) + \mu_{\alpha,\sigma}(S) + \varepsilon.$$

Using Lemma 2.4 again with $\varsigma = d$, we can find $\delta_6 < \delta_5$ and κ_1 such that

$$\mathbb{P}\left\{\left\|\Phi_{\alpha,\sigma}^\phi(T) - \Phi_\alpha^\phi(T)\right\| \geq \varsigma\right\} \leq \exp\left\{-\frac{\kappa_1}{\|\sigma\|^2}\right\}, \quad \alpha \in \mathcal{U}_{\delta_6}(\alpha_0), \ \phi \in \mathbf{B}_R \setminus S, \ 0 < \|\sigma\| < \kappa_1.$$

This inequality implies that

$$\mathbb{P}\left\{\left\|\Phi_{\alpha,\sigma}^\phi(T) \in \mathcal{V}\right\} \geq 1 - \exp\left\{-\frac{\kappa_1}{\|\sigma\|^2}\right\}, \quad \alpha \in \mathcal{U}_{\delta_6}(\alpha_0), \ \phi \in \mathbf{B}_R \setminus S, \ 0 < \|\sigma\| < \kappa_1.$$

Since $B \cap \mathcal{V} = \emptyset$, $P_{\alpha,\sigma}(T, \phi, B) = \mathbb{P}\left\{\left\|\Phi_{\alpha,\sigma}^\phi(T) \in B\right\} \leq \exp\left\{-\kappa_1/\|\sigma\|^2\right\}$ for all $\alpha \in \mathcal{U}_{\delta_6}(\alpha_0), \phi \in \mathbf{B}_R \setminus S, 0 < \|\sigma\| < \kappa_1$. Hence,

$$\mu_{\alpha,\sigma}(B) \leq \exp\left\{-\frac{\kappa_1}{\|\sigma\|^2}\right\} + \mu_{\alpha,\sigma}(S) + \varepsilon,$$

and thus

$$\limsup_{(\alpha,\sigma) \to (\alpha_0, 0)} \mu_{\alpha,\sigma}(B) \leq \lim_{(\alpha,\sigma) \to (\alpha_0, 0)} \left(\exp\left\{-\frac{\kappa}{\|\sigma\|^2}\right\} + \mu_{\alpha,\sigma}(S) + \varepsilon\right) = \varepsilon.$$

Since ε is arbitrary, $\limsup_{(\alpha,\sigma) \to (\alpha_0, 0)} \mu_{\alpha,\sigma}(B) = 0$, which completes the proof. $\qquad\square$

Corollary 2.1 *Suppose that all assumptions of Theorem 2.2 hold. If H is a continuous and bounded function defined on $\mathbb{R}_+^2$, then*

$$\lim_{(\alpha,\sigma) \to (\alpha_0, 0)} \int_{\mathbb{R}_+^2} H(x)\,\mathrm{d}\mu_{\alpha,\sigma}(x) = \frac{1}{T^*}\int_0^{T^*} H(\Phi_{\alpha_0}^{\overline{\phi}}(t))\,\mathrm{d}t,$$

where $\overline{\phi}$ is any point on Γ_{α_0} and T^* is the period of the limit cycle, i.e. $\Phi_{\alpha_0}^{\overline{\phi}}(t + T^*) = \Phi_{\alpha_0}^{\overline{\phi}}(t)$.

Proof. Let $\widehat{\Phi}_{\alpha,\sigma}(\,\cdot\,)$ be the stationary solution of (1.4), whose distribution is $\mu_{\alpha,\sigma}$. We can see that

$$\int_{\mathbb{R}_+^2} H(\phi)\,\mathrm{d}\mu_{\alpha,\sigma}(\phi) = \mathbb{E}H\big(\widehat{\Phi}_{\alpha,\sigma}(t)\big), \quad \text{for all } t \geq 0.$$

In particular,

$$\int_{\mathbb{R}_+^2} H(\phi)\,\mathrm{d}\mu_{\alpha,\sigma}(\phi) = \frac{1}{T^*}\int_0^{T^*} H\big(\widehat{\Phi}_{\alpha,\sigma}(t)\big)\,\mathrm{d}t,$$

where T^* is the period of the cycle. Since the measure $\mu_{\alpha,\sigma}(\,\cdot\,)$ becomes concentrated on the cycle Γ_{α_0} as (α,σ) approaches $(\alpha_0,0)$ and H is a bounded continuous function, we obtain

$$\lim_{(\alpha,\sigma)\to(\alpha_0,0)} \int_{\mathbb{R}_+^2} H(\phi)\,\mathrm{d}\mu_{\alpha,\sigma}(\phi) = \frac{1}{T^*}\int_0^{T^*} H\big(\Phi_{\alpha_0}^{\overline{\phi}}(t)\big)\,\mathrm{d}t,$$

where $\overline{\phi}$ is any point on the limit cycle Γ_{α_0}. This completes the proof. $\qquad\square$

Theorem 2.3 *Let $\alpha_0 = (r_0, K_0, m_0, \beta_0, \gamma_0, a_0, b_0, c_0)$ be a vector of parameters of the system in (1.1) such that $\lambda_{\alpha_0} > 0$ and*

$$b_0 \geq \min\left\{ \frac{c_0}{\beta_0}, \frac{m_0^2\beta_0^2 - c_0^2\gamma_0^2}{\gamma_0\beta_0(m_0\beta_0 - c_0\gamma_0) + m_0 r_0\beta_0^2} \right\}.$$

Then, there exist $\delta > 0$ and $\overline{\sigma} > 0$ such that, for all $\alpha \in \mathcal{U}_\delta(\alpha_0)$ and $0 < \sigma \leq \overline{\sigma}$, the process $(x_{\alpha,\sigma}(t), y_{\alpha,\sigma}(t))$ has a stationary distribution $\mu_{\alpha,\sigma}$ concentrated on $\mathbb{R}_+^{2,\circ}$. Further, for any open set $\mathcal{V}$ containing the positive equilibrium point $\phi_{\alpha_0}^* = (x_{\alpha_0}^*, y_{\alpha_0}^*)$ of the system in (1.1), $\lim_{(\alpha,\sigma)\to(\alpha_0,0)} \mu_{\alpha,\sigma}(\mathcal{V}) = 1$.

Proof. The proof is quite similar to that of Theorem 2.1, so we omit it here. $\qquad\square$

3. Numerical example

Consider (1.1) having the parameter α_0 with $r_0 = 1$, $K_0 = 5$, $m_0 = 9$, $a_0 = 1.75$, $b_0 = 1$, $c_0 = 1$, $\gamma_0 = 0.6$, and $\beta_0 = 0.5$. Direct calculation shows that $\lambda_\alpha = 2.7582 > 0$ and a positive equilibrium $(x^*, y^*) = (0.3, 0.233)$ with $Df(x^*, y^*)$ has two eigenvalues, $0.0016 \pm 0.66\mathrm{i}$. Further,

$$b_0 \leq \min\left\{ \frac{c_0}{\beta_0}, \frac{m_0^2\beta_0^2 - c_0^2\gamma_0^2}{\gamma_0\beta_0(m_0\beta_0 - c_0\gamma_0) + m_0 r_0\beta_0^2} \right\}.$$

Thus, this system has a limit cycle Γ_0, as shown in figure 1, starting from the point $(0.67, 0.25)$. Let $\mathcal{V}$ be an ε-neighbourhood of Γ_0 with $\varepsilon = 0.01$. For $\|\sigma\| \leq 1$, we have $\lambda_{\alpha,\sigma} > 0$. This means that (1.4) has a unique stationary distribution $\mu_{\alpha,\sigma}$.

We estimate the probability $\mu_{\alpha,\sigma}(\mathcal{V})$ as $\sigma \to 0$. To simplify the simulation, we fix all the other parameters, except for the variation of a, and list the results in table 1. The simulation phase pictures of the solution (1.4) are presented in figures 1 and 2.

4. Discussion

Robustness plays a very important role in studying mathematical models in biology and other fields since it ensures the output and forecasts are consistently accurate even if one or more of the input variables or assumptions vary. In fact, we know that the parametric model

TABLE 1. Simulation results for numerical example.

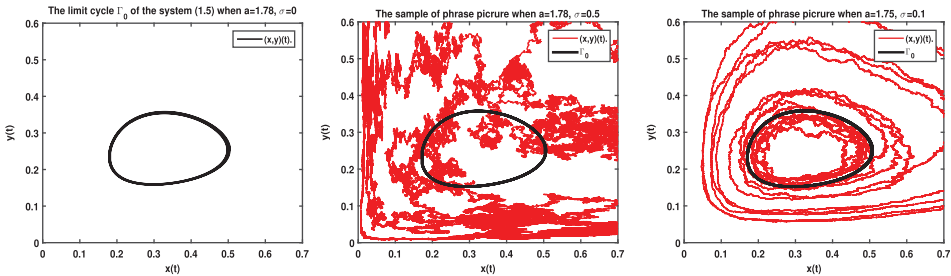| σ	(0.5, 0.5)	(0.1, 0.1)	(0.01, 0.01)	(0.001, 0.001)
a	1.8000	1.7500	1.7200	1.7100
$\lambda_{\alpha,\sigma}$	2.3975	2.7200	2.7450	2.7500
$\mu_{\alpha,\sigma}(\mathcal{V})$	0.1575	0.3380	0.7001	0.8960

FIGURE 1. Left: Limit cycle Γ_0 of (1.1). Centre: A sample path of (1.4) when $a = 1.78$, $\sigma = 0.5$. Right: A sample path of (1.4) when $a = 1.75$, $\sigma = 0.1$.
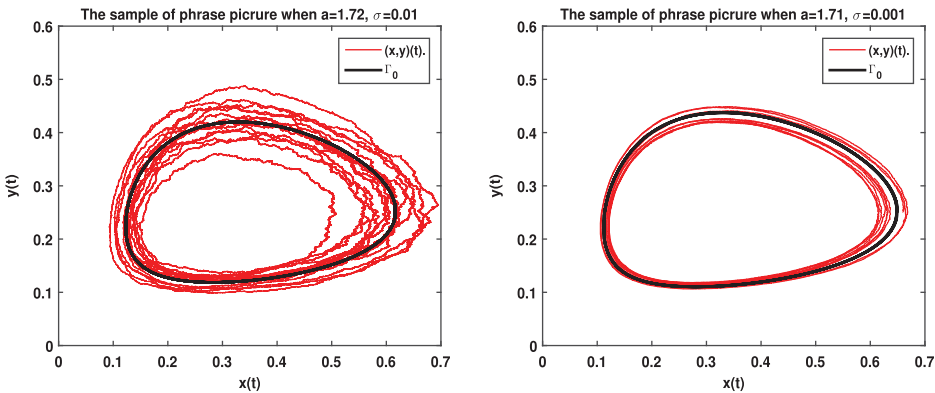

FIGURE 2. Left: A sample path of (1.4) when $a = 1.2$, $\sigma = 0.01$. Right: A sample path of (1.4) when $a = 1.71$, $\sigma = 0.001$.

is not quite true, and therefore we require that the distribution of the estimator changes only slightly if the distribution of the observations is slightly altered from that of the strict parametric model with certain parameter values. In this paper we have proved that the predator–prey model perturbed by white noise with Beddington–DeAngelis functional response (1.4) is robustly permanent and the stationary distribution depends (in weak topology) continuously on the data if it exists. The fact that the long-term population density concentrated at any neighbourhood of $(x_{\alpha_0}^*, y_{\alpha_0}^*)$ or of Γ_{α_0} when the parameter α approaches α_0 is significant in practical problems since it allows us to know that the long-term behaviour of the population is very close to the evolution of the corresponding deterministic system.

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