Canad. Math. Bull. Vol. 64 (4), 2023, pp. 1213–1230
http://dx.doi.org/10.4153/S0008439523000292

Composition operators on weighted analytic spaces

Marouane Bourass, Ibrahim Marrhich, and Fouzia Mkadmi

Abstract. We characterize the membership in the Schatten ideals $\mathcal{S}_p$, $0 < p < \infty$, of composition operators acting on weighted Dirichlet spaces. Our results concern a large class of weights. In particular, we examine the case of perturbed superharmonic weights. Characterization of composition operators acting on weighted Bergman spaces to be in $\mathcal{S}_p$ is also given.

1 Introduction

Let $\text{Hol}(\mathbb{D})$ be the set of holomorphic functions on the open unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$. For an analytic self-map φ on $\mathbb{D}$, we consider the composition operator

$$C_\varphi f := f \circ \varphi, \quad f \in \text{Hol}(\mathbb{D}).$$

For general information on composition operators on spaces of analytic functions, we refer the reader to the monographs by Shapiro [26] and Cowen and MacCluer [9]. Boundedness, compactness, and membership in Schatten ideals of composition operators are the goals of several papers on various spaces of analytic functions (see, for instance, [10, 16, 21, 24, 27]). Recall that, for $p > 0$, the Schatten p-ideal of a separable Hilbert space $\mathcal{H}$, denoted by $\mathcal{S}_p(\mathcal{H})$, consists of compact operators T on $\mathcal{H}$ for which the sequence of singular values $s_n(T)$ belongs to ℓ^p.

For $\alpha > -1$, let $\mathcal{H}_\alpha$ be the weighted analytic space given by

$$\mathcal{H}_\alpha = \left\{ f \in \text{Hol}(\mathbb{D}) : \int_{\mathbb{D}} |f'(z)|^2 \, dA_\alpha(z) < \infty \right\},$$

where $dA_\alpha(z) = (1-|z|^2)^\alpha \, dA(z)$. As usual, $dA(z) := dx dy/\pi$, $z = x + iy$, is the normalized Lebesgue area measure on $\mathbb{D}$. For $\alpha \in (-1, 1)$, $\mathcal{H}_\alpha$ is the standard Dirichlet space and is denoted by $\mathcal{D}_\alpha$. $\mathcal{H}_1$ is the classical Hardy space H^2. For $\alpha > 1$, $\mathcal{H}_\alpha$ is the standard Bergman space $A^2_{\alpha-2}$. Recall that

$$A^2_\beta := \left\{ f \in \text{Hol}(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|^2 \, dA_\beta(z) < \infty \right\}, \quad \beta > -1.$$

Received by the editors January 16, 2023; revised April 3, 2023; accepted April 5, 2023.
Published online on Cambridge Core April 17, 2023.
AMS subject classification: **47B32**, **30H20**, **47B10**, 47B33, 47B35.
Keywords: Composition operators, Toeplitz operators, weighted Bergman spaces, weighted Dirichlet spaces, Schatten p-ideals.

Littlewood's subordination principle guarantees the boundedness of C_φ on the Hardy space H^2 (see, for example, [26]). The compactness of C_φ on H^2 has been characterized in [25] by Shapiro. For $\alpha \geq 1$, Luecking and Zhu obtained in [19] a characterization for a composition operator C_φ to be in $S_p(\mathcal{H}_\alpha)$ for $p > 0$. Pau and Pérez in [21] gave an analogous characterization for the standard Dirichlet spaces $\mathcal{D}_\alpha$, $\alpha \in (0,1)$.

Let φ be an analytic self map of $\mathbb{D}$, and let $\alpha > -1$. The Nevanlinna counting function $N_{\varphi,\alpha}$ of φ associated with $\mathcal{H}_\alpha$ is defined by

$$N_{\varphi,\alpha}(w) = \sum_{w=\varphi(z)} \left(1-|z|^2\right)^\alpha \quad \text{if } w \in \varphi(\mathbb{D}); \qquad N_{\varphi,\alpha}(w) = 0 \quad \text{if } w \notin \varphi(\mathbb{D}).$$

We summarize the results obtained in [19, 21] as follows. Let $\alpha > 0$ and $p > 0$. Then

$$C_\varphi \in S_p(\mathcal{H}_\alpha) \Longleftrightarrow \int_{\mathbb{D}} \left(\frac{N_{\varphi,\alpha}(w)}{(1-|w|)^\alpha}\right)^{p/2} d\lambda(w) < \infty, \tag{1.1}$$

where $d\lambda(z) := dA(z)/\left(1-|z|^2\right)^2$ is the Möbius invariant measure on $\mathbb{D}$.

A weight on $\mathbb{D}$ is a function $\omega : \mathbb{D} \to (0,+\infty)$ which is integrable with respect to dA. If $\omega : [0,1) \to (0,+\infty)$ is a radial weight, then we extend it to $\mathbb{D}$ by setting $\omega(z) = \omega(|z|)$. The weighted Dirichlet space $\mathcal{D}_\omega$ associated with a weight ω on $\mathbb{D}$ is defined by

$$\mathcal{D}_\omega = \left\{ f \in \operatorname{Hol}(\mathbb{D}) : \mathcal{D}_\omega(f) := \left(\int_{\mathbb{D}} |f'(z)|^2 \omega(z) dA(z)\right)^{1/2} < \infty \right\}.$$

The space $\mathcal{D}_\omega$ endowed with the norm $\|f\|^2_{\mathcal{D}_\omega} := |f(0)|^2 + \mathcal{D}_\omega(f)^2$ is a Hilbert space (see Lemma 2.1).

For $p > 1$, let $\mathcal{C}_p$ be the set of weights ω such that, for some $\alpha \in (0,1)$ (or equivalently for all $\alpha \in (0,1)$, see [18]), we have

$$\left(\int_{\Delta(z,\alpha)} \omega dA\right)^{1/p} \left(\int_{\Delta(z,\alpha)} \omega^{-p'/p} dA\right)^{1/p'} \lesssim |\Delta(z,\alpha)|, \ z \in \mathbb{D},$$

where $\Delta(z,\alpha) = \{w \in \mathbb{D} : |z-w| < \alpha(1-|z|^2))\}$ and $1/p + 1/p' = 1$. Here and throughout the paper, for a Borel set Δ of $\mathbb{D}$, $|\Delta|$ denotes the Lebesgue measure of Δ. As usual, for real positive quantities A and B, $A \lesssim B$ means that there is an absolute constant $C > 0$ such that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$ both hold, then we write $A \asymp B$.

We denote

$$\tilde{\omega}(z) = \frac{1}{(1-|z|^2)^2} \int_{\Delta(z,1/2)} \omega(\zeta) dA(\zeta), \ z \in \mathbb{D},$$

and, for $t \geq 0$, let

$$\omega_t(z) = \int_{\mathbb{D}} \frac{\omega(w)(1-|z|^2)^t}{|1-\overline{w}z|^{2+t}} dA(w), \ z \in \mathbb{D}.$$

For $p > 1$ and $t \geq 0$, a weight ω is said to belong to the class $\mathcal{C}_{p,t}$ if $\omega \in \mathcal{C}_p$ and $\omega_t \lesssim \tilde{\omega}$. The class of such weights is introduced by Bourass and Marrhich in [7]. In this paper, based on results obtained in [7], we obtain characterizations of boundedness,

https://doi.org/10.4153/S0008439523000292 Published online by Cambridge University Press

compactness, and Schatten classes membership for composition operators on $\mathcal{D}_\omega$, $\omega \in \mathcal{C}_{p,t}$. Our results cover Bekollé–Bonami weights, superharmonic weights, and the radial admissible weights introduced by Kellay and Lefèvre in [15].

In particular, we are interested in perturbed superharmonic weights on $\mathbb{D}$. Let $u \in \mathcal{C}^2(\mathbb{D})$ be a positive superharmonic function on $\mathbb{D}$. Recall that u admits the representation

$$u(z) = \int_{\mathbb{D}} \log\left|\frac{1-\overline{\zeta}z}{z-\zeta}\right| \frac{d\sigma(\zeta)}{1-|\zeta|^2} + P_\nu(z) =: S_\sigma(z) + P_\nu(z), \tag{1.2}$$

for a unique finite positive Borel measure σ on $\mathbb{D}$ and a unique finite positive Borel measure ν on the unit circle $\mathbb{T} := \partial\mathbb{D}$ (see [2]). Here,

$$P_\nu(z) := \int_{\mathbb{T}} \frac{1-|z|^2}{|1-\overline{\xi}z|^2} d\nu(\xi)$$

is the Poisson transform of the measure ν on $\mathbb{T}$. Let ω be a weight of the form $\omega(z) = (1-|z|^2)^\alpha u(z), \alpha > -1$. The space $\mathcal{D}_\omega$ is called the perturbed superharmonically weighted Dirichlet space. We adopt the notation $\mathcal{D}_\sigma$ (resp. $\mathcal{D}_\nu$) instead of $\mathcal{D}_{S_\sigma}$ (resp. $\mathcal{D}_{P_\nu}$). The generalized Nevanlinna counting function of φ associated with a weight ω is defined by

$$N_{\varphi,\omega}(w) = \sum_{w=\varphi(z)} \omega(z) \ \text{ if } w \in \varphi(\mathbb{D}); \ \ N_{\varphi,\omega}(w) = 0 \ \text{ if } w \notin \varphi(\mathbb{D}).$$

We write $N_{\varphi,\nu}$ instead of N_{φ,P_ν}. In [24], Sarason and Silva characterized boundedness and compactness of operators $C_\varphi : \mathcal{D}_\nu \to \mathcal{D}_\nu$ in terms of $N_{\varphi,\nu}$. They proved that C_φ is bounded (resp. compact) on $\mathcal{D}_\nu$ if and only if

$$\int_{\Delta_w} \frac{N_{\varphi,\nu}(z)}{P_\nu(z)} dA(z) = O(|\Delta_w|) \qquad (\text{resp. } o(|\Delta_w|) \text{ as } |w| \to 1^-),$$

where $\Delta_w = \Delta(w, \frac{1}{2})$. In [12], El-Fallah, Mahzouli, Marrhich, and Naqos proved that $C_\varphi \in \mathcal{S}_p(\mathcal{D}_\nu)$, for $p > 0$, if and only if

$$\sum_{n=0}^{+\infty} \sum_{j=0}^{2^n-1} \left(\frac{1}{|R_{n,j}|} \int_{R_{n,j}} \frac{N_{\varphi,\nu}(z)}{P_\nu(z)} dA(z) \right)^{\frac{p}{2}} < \infty,$$

where

$$R_{n,j} := \left\{ re^{it} : \frac{1}{2^{n+1}} < 1-r \le \frac{1}{2^n} \text{ and } \frac{2\pi j}{2^n} \le t < \frac{2\pi(j+1)}{2^n} \right\}, \qquad n \in \mathbb{N} \text{ and } 0 \le j \le 2^n - 1,$$

are the dyadic disks.

On the space $\mathcal{D}_\sigma$, Bao, Göğüş, and Pouliasis [5] proved that C_φ is bounded (resp. compact) if and only if

$$\check{N}_{\varphi,\sigma}(w) = O(U_\sigma(w)) \qquad (\text{resp. } o(U_\sigma(w)) \text{ as } (|w| \to 1^-)),$$

https://doi.org/10.4153/S0008439523000292 Published online by Cambridge University Press

with $U_\sigma(z) = \int_{\mathbb{D}} \frac{1-|z|^2}{|1-\overline{\zeta}z|^2} d\sigma(\zeta)$ and

$$\check{N}_{\varphi,\sigma}(w) = \sum_{w=\varphi(z)} U_\sigma(z) \ \text{ if } w \in \varphi(\mathbb{D}); \ \ \check{N}_{\varphi,\sigma}(w) = 0 \ \text{ if } w \notin \varphi(\mathbb{D}).$$

We will characterize boundedness, compactness, and Schatten classes membership of composition operators on perturbed superharmonically weighted Dirichlet spaces $\mathcal{D}_\omega$ with $\omega(z) = (1-|z|^2)^\alpha (S_\sigma(z) + P_\nu(z))$. In particular, we prove that $C_\varphi : \mathcal{D}_\sigma \to \mathcal{D}_\sigma$ belongs to $\mathcal{S}_p(\mathcal{D}_\sigma)$, for $p > 0$, if and only if

$$\int_{\mathbb{D}} \left(\frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)} \right)^{p/2} d\lambda(z) < \infty.$$

In this paper, we are interested also in composition operators on weighted Bergman spaces. In particular, we extend the result obtained by Constantin in [8] concerning the membership of C_φ to $\mathcal{S}_p(A^2_\omega)$, for $p \geq 2$ and in the Bekollé–Bonami weights setting, to all $p > 0$ and for $\omega \in \mathcal{C}_{p,t}$.

Throughout this paper, we decompose $\mathbb{D}$ by using the disks $\Delta(z,r)$, $0 < r < 1$. The sets $\Delta(z,r)$ give a (ρ,δ)–lattice of $\mathbb{D}$ for $\rho(z) = (1-|z|^2)/2$ and for some choice of δ. Let $(\Delta(z_n,\delta))_n$ be the corresponding (ρ,δ)–lattice of $\mathbb{D}$, and let $(\Delta_n)_n$ be an enumeration of $\Delta(z_n,\delta)$. Let $b > 1$ such that $b\Delta_n = \Delta(z_n, b\delta)$ is a covering of $\mathbb{D}$ of finite multiplicity (see [11, Proposition 3.1] and [20] for details and generalization).

2 Composition operators on weighted Dirichlet spaces

2.1 General results

Suppose that ω is a weight such that $\omega \in \mathcal{C}_{p_0,t}$ for some $p_0 > 1$ and $t \geq 0$. The weighted Bergman space associated with ω is defined by

$$A^2_\omega = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \ \|f\|_{A^2_\omega} := \left(\int_{\mathbb{D}} |f(z)|^2 \omega(z) dA(z) \right)^{1/2} < \infty \right\}.$$

Notice that A^2_ω is a reproducing kernel Hilbert space since each point evaluation $e_z : A^2_\omega \to \mathbb{C}$, which takes f to $f(z)$, is a bounded linear functional on A^2_ω (see [7]). The reproducing kernel of A^2_ω will be denoted by K^ω. The Toeplitz operator T_μ, associated with a positive Borel measure μ on $\mathbb{D}$, acting on A^2_ω is the transformation

$$T_\mu f(z) = \int_{\mathbb{D}} f(\zeta) K^\omega(z,\zeta)\omega(\zeta) d\mu(\zeta), \quad f \in A^2_\omega,\ z \in \mathbb{D}.$$

In the sequel, for a positive Borel measure μ on $\mathbb{D}$, we denote $d\mu_\omega = \omega d\mu$. The following results are proved in [7].

Theorem A *Let μ be a finite positive Borel measure on $\mathbb{D}$. The following assertions are equivalent.*

(1) *The Toeplitz operator T_μ is bounded (resp. compact) on A^2_ω.*
(2) $\mu_\omega(\Delta_n) = O(A_\omega(\Delta_n))$ *(resp.* $\mu_\omega(\Delta_n) = o(A_\omega(\Delta_n))$*)*, $n \to \infty$.

Theorem B *Let μ be a finite positive Borel measure on $\mathbb{D}$ such that T_μ is compact on A^2_ω, and let $p > 0$. Then T_μ belongs to $\mathcal{S}_p(A^2_\omega)$ if and only if*

$$\sum_{n=0}^{\infty}\left(\frac{\mu_\omega(\Delta_n)}{A_\omega(\Delta_n)}\right)^p < \infty.$$

We will apply Theorems A and B to characterize boundedness, compactness, and Schatten class composition operators on $\mathcal{D}_\omega$. The following lemma, which implies, in particular, that $(\mathcal{D}_\omega, \|.\|_{\mathcal{D}_\omega})$ is a Hilbert space, will be needed for the proof of next result.

Lemma 2.1 *Suppose that ω is a weight such that $\omega \in \mathcal{C}_p$ for some $p > 1$. Then, each point evaluation is bounded on $(\mathcal{D}_\omega, \|.\|_{\mathcal{D}_\omega})$.*

Proof Fix z in $\mathbb{D}$, and let $f \in \mathcal{D}_\omega$. We have $|f(z) - f(0)|^2 \leq \int_0^1 |f'(sz)|^2 ds$. Since $f' \in A^2_\omega$ and $\omega \in \mathcal{C}_p$, then

$$|f'(sz)|^2 \lesssim \frac{1}{(1-|sz|^2)^2\tilde{\omega}(sz)}\|f'\|^2_{A^2_\omega},\ s \in [0,1]$$

(see [3] or [7]). Let $r \in (|z|, 1)$. We have

$$\inf_{w\in[0,z]} \tilde{\omega}(w) \geq \inf_{w\in\Delta(0,r)} \tilde{\omega}(w) \gtrsim \tilde{\omega}(0) > 0,$$

since $\tilde{\omega}(w) \asymp \tilde{\omega}(0)$ when $w \in \Delta(0, r)$ (see Lemma 2.2 in [8]). We obtain

$$|f(z) - f(0)|^2 \lesssim \|f'\|^2_{A^2_\omega}\int_0^1 \frac{1}{(1-|sz|^2)^2\tilde{\omega}(sz)}ds \lesssim \|f'\|^2_{A^2_\omega} \leq \|f\|^2_{\mathcal{D}_\omega}.$$

Consequently, $|f(z)|^2 \lesssim |f(z) - f(0)|^2 + |f(0)|^2 \lesssim \|f\|^2_{\mathcal{D}_\omega}$. ∎

Theorem 2.2 *Suppose that ω is a weight such that $\omega \in \mathcal{C}_{p_0,t}$ for some $p_0 > 1$ and $t \geq 0$. Let φ be an analytic self-map of $\mathbb{D}$. Then:*

(1) *C_φ is bounded on $\mathcal{D}_\omega$ if and only if*

$$\int_{\Delta_n} N_{\varphi,\omega}(z)dA(z) \lesssim \int_{\Delta_n} \omega(z)dA(z), \quad n \in \mathbb{N}.$$

(2) *C_φ is compact on $\mathcal{D}_\omega$ if and only if*

$$\int_{\Delta_n} N_{\varphi,\omega}(z)dA(z) = o\left(\int_{\Delta_n} \omega(z)dA(z)\right), \quad (n \to \infty).$$

Proof Suppose that C_φ is bounded on $\mathcal{D}_\omega$. Let $V_\omega : \mathcal{D}_\omega \to A^2_\omega$ be the bounded operator defined by $V_\omega f = f'$, and let $D_{\varphi,\omega} : A^2_\omega \to A^2_\omega$ be the operator defined by

$$D_{\varphi,\omega} f := V_\omega C_\varphi V^*_\omega f.$$

https://doi.org/10.4153/S0008439523000292 Published online by Cambridge University Press

By a direct calculation, we have $D_{\varphi,\omega}f = \varphi'.f\circ\varphi$, $f\in A^2_\omega$. The operator $D^*_{\varphi,\omega}D_{\varphi,\omega}$ is then bounded on A^2_ω. For $f\in A^2_\omega$, the change of variable formula [1] gives

$$\begin{aligned} D^*_{\varphi,\omega}D_{\varphi,\omega}f(z) &= \langle D_{\varphi,\omega}f, D_{\varphi,\omega}K^\omega_z\rangle_{A^2_\omega} \\ &= \int_{\mathbb{D}} f(\xi)\overline{K^\omega_z(\xi)}\omega(\xi)d\omega_\varphi(\xi) \\ &= T_{\omega_\varphi}f(z), \end{aligned}$$

where ω_φ is the measure defined on $\mathbb{D}$ by $d\omega_\varphi = \frac{N_{\varphi,\omega}}{\omega}dA$. It follows that T_{ω_φ} is bounded on A^2_ω. By (1) of Theorem A, we deduce that

$$\int_{\Delta_n} N_{\varphi,\omega}(z)dA(z) \lesssim \int_{\Delta_n}\omega(z)dA(z), \quad n\in\mathbb{N}. \tag{2.1}$$

Conversely, assume that (2.1) holds, and let $f\in\mathcal{D}_\omega$. By using once again the change of variable formula, we get

$$\begin{aligned} \|C_\varphi f\|^2_{\mathcal{D}_\omega} &= |f(\varphi(0))|^2 + \left\langle T_{\omega_\varphi}f', f'\right\rangle_{A^2_\omega} \\ &\le |f(\varphi(0))|^2 + \left\|T_{\omega_\varphi}\right\| \|f'\|^2_{A^2_\omega} \\ &\lesssim |f(\varphi(0))|^2 + \|f\|^2_{\mathcal{D}_\omega}. \end{aligned}$$

By Lemma 2.1, we deduce that $\|C_\varphi f\|^2_{\mathcal{D}_\omega} \lesssim \|f\|^2_{\mathcal{D}_\omega}$, $f\in\mathcal{D}_\omega$. Therefore, C_φ is bounded on $\mathcal{D}_\omega$.

To prove the second assertion, we may assume that C_φ is bounded on $\mathcal{D}_\omega$. We have

$$C_\varphi f = V^*_\omega D_{\varphi,\omega}V_\omega f + Kf, \quad f\in\mathcal{D}_\omega, \tag{2.2}$$

where $Kf(z) = f(\varphi(0))$, $f\in\mathcal{D}_\omega$, and $z\in\mathbb{D}$. Since K is bounded on $\mathcal{D}_\omega$ by Lemma 2.1 then, by definition of $D_{\varphi,\omega}$ on the one hand and by (2.2) on the other hand, C_φ is compact on $\mathcal{D}_\omega$ if and only if $D_{\varphi,\omega}$ is compact on A^2_ω. In other words, C_φ is compact on $\mathcal{D}_\omega$ if and only if T_{ω_φ} is compact on A^2_ω. The result follows now by (2) of Theorem A. ∎

Theorem 2.3 *Suppose that ω is a weight such that $\omega\in\mathcal{C}_{p_0,t}$ for some $p_0>1$ and $t\ge 0$. Assume that φ is an analytic self-map of $\mathbb{D}$ such that C_φ is compact on $\mathcal{D}_\omega$. Then C_φ belongs to $\mathcal{S}_p(\mathcal{D}_\omega)$, for $p>0$, if and only if*

$$\sum_{n=0}^{\infty}\left(\frac{\int_{\Delta_n}N_{\varphi,\omega}(z)dA(z)}{\int_{\Delta_n}\omega(z)dA(z)}\right)^{p/2} < \infty.$$

Proof Note that C_φ belongs to $\mathcal{S}_p(\mathcal{D}_\omega)$ if and only if $D_{\varphi,\omega}$ belongs to $\mathcal{S}_p(A^2_\omega)$ by (2.2). Since

$$D^*_{\varphi,\omega}D_{\varphi,\omega}f = T_{\omega_\varphi}f, \quad f\in A^2_\omega,$$

then $C_\varphi\in\mathcal{S}_p(\mathcal{D}_\omega)$ if and only if $T_{\omega_\varphi}\in\mathcal{S}_{\frac{p}{2}}(A^2_\omega)$. The result follows by Theorem B. ∎

If ω is a weight such that for some (equivalently for all) $r \in (0,1)$, we have

$$\omega(z) \asymp \omega(w), \; w \in \Delta(z,r), \tag{2.3}$$

then $\omega \in \mathcal{C}_p$ for all $p > 1$. Moreover, under the condition (2.3), we have

$$\frac{1}{|\Delta(z,\delta)|}\int_{\Delta(z,\delta)} \omega(\zeta)dA(\zeta) \asymp \omega(z), \; z \in \mathbb{D}$$

for all $\delta \in (0,1)$. In this case, Theorems 2.2 and 2.3 can be reduced to the following result.

Corollary 2.4 *Let φ be an analytic self-map of $\mathbb{D}$. Suppose that ω is a weight satisfying (2.3) and such that $\omega_t \lesssim \omega$ for some $t \geq 0$. Then:*

(1) *C_φ is bounded on $\mathcal{D}_\omega$ if and only if*

$$\frac{1}{|\Delta_n|}\int_{\Delta_n} \frac{N_{\varphi,\omega}(z)}{\omega(z)} dA(z) = O(1), \quad \forall n \in \mathbb{N}.$$

(2) *C_φ is compact on $\mathcal{D}_\omega$ if and only if*

$$\frac{1}{|\Delta_n|}\int_{\Delta_n} \frac{N_{\varphi,\omega}(z)}{\omega(z)} dA(z) = o(1), \quad (n \to \infty).$$

(3) *C_φ belongs to $\mathcal{S}_p(\mathcal{D}_\omega)$, for $p > 0$, if and only if*

$$\sum_{n=0}^{\infty}\left(\frac{1}{|\Delta_n|}\int_{\Delta_n} \frac{N_{\varphi,\omega}(z)}{\omega(z)} dA(z)\right)^{p/2} < \infty.$$

2.2 Radial weights

A radial weight ω in $\mathcal{C}^2[0,1)$ is called admissible if: $(\mathcal{W}_1)\omega$ is nonincreasing. $(\mathcal{W}_2)$ $\omega(r)(1-r)^{-(1+\delta)}$ is nondecreasing for some $\delta > 0$. $(\mathcal{W}_3)$ $\lim_{r\to 1^-} \omega(r) = 0$. $(\mathcal{W}_4)$ One of the two properties of convexity is fulfilled

$$\begin{cases} (\mathcal{W}_4^{(I)}) : \omega \text{ is convex and } \lim_{r\to 1^-} \omega'(r) = 0, \\ (\mathcal{W}_4^{(II)}) : \omega \text{ is concave.} \end{cases}$$

If ω satisfies $(\mathcal{W}_1)$–$(\mathcal{W}_3)$ and $(\mathcal{W}_4^{(I)})$ (resp. $(\mathcal{W}_4^{(II)})$), then we say that ω is (I)-admissible (resp. (II)-admissible). Kellay and Lefèvre [15] proved the following result.

Theorem C (1) *Let ω be a (II)-admissible weight. Then C_φ is bounded on $\mathcal{D}_\omega$ if and only if $N_{\varphi,\omega}(z) = O(\omega(z)), z \in \mathbb{D}$.*

(2) *Let ω be an admissible weight. Then C_φ is compact on $\mathcal{D}_\omega$ if and only if $N_{\varphi,\omega}(z) = o(\omega(z)), |z| \to 1^-$.*

As noticed in [15], C_φ is always bounded on $\mathcal{D}_\omega$ if ω is an (I)-admissible weight. We describe in the following theorem the membership of C_φ in $\mathcal{S}_p(\mathcal{D}_\omega)$ for admissible weights.

Theorem 2.5 *Let φ be an analytic self-map of $\mathbb{D}$, and let ω be an admissible weight. Then C_φ belongs to $\mathcal{S}_p(\mathcal{D}_\omega)$, for $p>0$, if and only if*

$$\int_{\mathbb{D}} \left(\frac{N_{\varphi,\omega}(z)}{\omega(z)}\right)^{p/2} d\lambda(z) < \infty. \tag{2.4}$$

Proof Let z in $\mathbb{D}$, and let $w \in \Delta(z,r)$. Suppose that $|z| \le |w|$. By $(\mathcal{W}_1)$, we have $\omega(z) \ge \omega(w)$, and by $(\mathcal{W}_2)$, we have

$$\omega(z) = \frac{\omega(z)}{(1-|z|)^{\delta+1}}(1-|z|)^{\delta+1} \le \frac{\omega(w)}{(1-|w|)^{\delta+1}}(1-|z|)^{\delta+1} \asymp \omega(w),$$

where δ is the constant in $(\mathcal{W}_2)$. Similarly, we have $\omega(z) \asymp \omega(w)$ if $|w| \le |z|$. Therefore, ω satisfies the condition (2.3). On the other hand, the conditions $(\mathcal{W}_1)$ and $(\mathcal{W}_2)$ imply that $\omega_{2+2\delta} \lesssim \omega$ (see [15, Lemma 2.4]). By (3) of Corollary 2.4, it follows that C_φ belongs to $\mathcal{S}_p(\mathcal{D}_\omega)$, for $p>0$, if and only if

$$\sum_{n=0}^{\infty} \left(\frac{1}{|\Delta_n|}\int_{\Delta_n} \frac{N_{\varphi,\omega}(z)}{\omega(z)} dA(z)\right)^{p/2} < \infty. \tag{2.5}$$

Now, since $N_{\varphi,\omega}$ satisfies the sub-mean-value property, that is,

$$N_{\varphi,\omega}(z) \lesssim \frac{1}{|\Delta(z,r)|}\int_{\Delta(z,r)} N_{\varphi,\omega}(w)dA(w), \quad z \in \mathbb{D} \tag{2.6}$$

(see Lemmas 2.2 and 2.3 in [15]), then, similarly to the proof of Theorem 2.9, the condition (2.5) is equivalent to the condition (2.4). ∎

2.3 Remarks and examples

• A radial weight ω is called almost standard if ω satisfies $(\mathcal{W}_1)$–$(\mathcal{W}_3)$. In the recent paper [13], Esmaeili and Kellay studied the boundedness and the compactness of weighted composition operators on Bergman and Dirichlet spaces associated with almost standard weights. As noticed in the proof of Theorem 2.5, every almost standard weight satisfies (2.3) and $\omega_{2+2\delta} \lesssim \omega$. Therefore, Corollary 2.4 can be applied for any almost standard weight.

• Let ω be a weight on $\mathbb{D}$ such that there are constants $s \in (-1,0)$ and $\eta \ge 0$ for which

$$\omega_{s,\eta}(z) := \int_{\mathbb{D}} \frac{\omega(\xi)(1-|\xi|^2)^s(1-|z|^2)^\eta}{|1-\overline{\xi}z|^{2+s+\eta}} dA(\xi) \lesssim \omega(z), \quad z \in \mathbb{D}. \tag{2.7}$$

This condition is similar to the one that appears in [6]. Note that any standard weight $\omega_\alpha(z) = (1-|z|^2)^\alpha$, for $\alpha > -1$, satisfies the condition (2.7) since for $s \in (\max(-1,-1-\alpha),0)$ and $\eta > \max(0,\alpha)$, we have

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^{\alpha+s}}{|1-\overline{w}z|^{2+s+\eta}} dA(z) \asymp (1-|w|^2)^{\alpha-\eta}, \quad w \in \mathbb{D}.$$

The following lemma is stated in [7].

Lemma A *Let ω be a weight satisfying (2.7) for some constants $s \in (-1,0)$ and $\eta \geq 0$. Then ω satisfies (2.7) for all $s' > s$ and $\beta > \eta$.*

Let $1 < p, p' < \infty$ such that $\frac{1}{p} + \frac{1}{p'} = 1$, and let $\eta > -1$. The class of Bekollé–Bonami weights $B_p(\eta)$ consists of weights ω such that

$$\left(\int_{S(\theta,h)} \omega dA_\eta\right)\left(\int_{S(\theta,h)} \omega^{-p'/p} dA_\eta\right)^{p/p'} \lesssim \left(A_\eta(S(\theta,h))\right)^p$$

for any Carleson square

$$S(\theta,h) := \left\{re^{i\alpha} : 1-h < r < 1,\ |\theta-\alpha| < h/2\right\}, \quad \theta \in [0,2\pi],\ h \in (0,1).$$

Note that if $\frac{\omega}{(1-|z|^2)^\eta} \in B_p(\eta)$, for some $p > 1$ and $\eta > -1$, then $\omega \in \mathcal{C}_{p,t}$ for all $t \geq p(\eta+2)-2$ (see [3, Lemma 2.1]).

If ω is a weight on $\mathbb{D}$ that satisfies (2.7), then by Lemma A, $\omega_\eta \lesssim \omega$ for some $\eta \geq 0$. Using Corollary 4.4 from [4] (see the proof of $(c) \Rightarrow (b)$), we find that $(1-|z|^2)^{-\eta}\omega$ belongs to $B_p(\eta)$ for all $p > 1$. We conclude that if ω satisfies (2.7), then $\omega \in \mathcal{C}_{p,t}$ for all $p > 1$ and some $t \geq 0$. As examples, we consider here weights which appear in [6]. Let μ be a finite positive Borel mesure on $\mathbb{D}$ and $b \in \mathbb{R}$ such that $\int_{\mathbb{D}}(1-|w|^2)^b d\mu(w) < \infty$. Let ν be a finite positive Borel measure on $\mathbb{T}$. Let $a > -1$, and let $c < a+2$. Using Lemma 2.5 in [14], one can verify that the weight

$$\omega(z) = (1-|z|^2)^a\left(\int_{\mathbb{D}} \frac{(1-|w|^2)^b}{|1-z\overline{w}|^c} d\mu(w) + \int_{\mathbb{T}} \frac{d\nu(\zeta)}{|1-\overline{\zeta}z|^c}\right)$$

satisfies the condition (2.7) for all $\eta > a$ and $c-a-2 < s < 0$. Notice that the previous weight satisfies, in addition, the condition (2.3).

Remark 2.6 Let ω be a weight satisfying (2.7) with $s \in (-1,0)$ and $t \geq 0$. Then $(1-|z|^2)^\alpha\omega$ satisfies (2.7) for all $\alpha > s$. Indeed, if $\alpha > 0$, then for $\varepsilon \in (0,1)$ such that $\alpha - \varepsilon > 0$ and $\beta \geq t + \alpha - \varepsilon$ we have by Lemma A

$$\int_{\mathbb{D}} \frac{\omega(z)(1-|z|^2)^{\alpha-\varepsilon}(1-|w|^2)^{\beta-\alpha+\varepsilon}}{|1-\overline{w}z|^{2+\beta}} dA(z) \lesssim \omega(w).$$

If $\alpha \in (s,0)$, then for $s' = s-\alpha$ and $\beta = t+\alpha+1$ once again by Lemma A, we obtain

$$\begin{aligned}\int_{\mathbb{D}} \frac{\omega(z)(1-|z|^2)^{\alpha+s'}(1-|w|^2)^\beta}{|1-\overline{w}z|^{2+s'+\beta}} dA(z) &= \int_{\mathbb{D}} \frac{\omega(z)(1-|z|^2)^s(1-|w|^2)^{t+\alpha+1}}{|1-\overline{w}z|^{3+s+t}} dA(z) \\ &\lesssim (1-|w|^2)^\alpha\omega(w).\end{aligned}$$

2.4 Composition operators on Dirichlet spaces induced by perturbed superharmonic weights

In this subsection, we examine the case of perturbed superharmonic weights. We begin with the following proposition.

https://doi.org/10.4153/S0008439523000292 Published online by Cambridge University Press

Proposition 2.7 *Let $\omega \in \mathcal{C}^2(\mathbb{D})$ be a positive superharmonic function on $\mathbb{D}$. Then $(1-|z|^2)^\alpha \omega$ verifies (2.7) for all $\alpha > -1$.*

Proof Let σ and ν be the unique finite positive Borel measures on $\mathbb{D}$ and $\mathbb{T}$, respectively, such that $\omega = S_\sigma + P_\nu$. It is proved in [17] that P_ν satisfies (2.7) for all $s > -1$ and $t > 1$. On the other hand, note that for $s \in (-1, 0)$ and $t > 1$, we have

$$\begin{aligned}\int_{\mathbb{D}} \frac{(1-|z|^2)^s}{|1-\overline{w}z|^{2+s+t}} \log\left|\frac{1-\overline{\zeta}z}{z-\zeta}\right| dA(z) &\asymp \int_{\mathbb{D}} \frac{(1-|z|^2)^s}{|1-\overline{w}z|^{2+s+t}}\left(1-\left|\frac{z-\zeta}{1-\overline{\zeta}z}\right|^2\right) dA(z)\\ &= (1-|\zeta|^2)\int_{\mathbb{D}} \frac{(1-|z|^2)^{s+1}}{|1-\overline{w}z|^{2+s+t}|1-\overline{\zeta}z|^2} dA(z)\\ &\lesssim \frac{(1-|\zeta|^2)}{(1-|w|^2)^{t-1}|1-\overline{\zeta}w|^2}.\end{aligned}$$

Therefore, for $s \in (-1, 0)$ and $t > 1$, we have

$$\begin{aligned}\int_{\mathbb{D}} \frac{S_\sigma(z)(1-|z|^2)^s}{|1-\overline{w}z|^{2+s+t}} dA(w) &= \int_{\mathbb{D}}\left(\int_{\mathbb{D}} \log\left|\frac{1-\overline{\zeta}z}{z-\zeta}\right| \frac{d\sigma(\zeta)}{1-|\zeta|^2}\right)\frac{(1-|z|^2)^s}{|1-\overline{w}z|^{2+s+t}} dA(z)\\ &= \int_{\mathbb{D}}\left(\int_{\mathbb{D}} \frac{(1-|z|^2)^s}{|1-\overline{w}z|^{2+s+t}} \log\left|\frac{1-\overline{\zeta}z}{z-\zeta}\right| dA(z)\right)\frac{d\sigma(\zeta)}{1-|\zeta|^2}\\ &\lesssim \frac{1}{(1-|w|^2)^t}\int_{\mathbb{D}}\left(\frac{(1-|\zeta|^2)(1-|w|^2)}{|1-\overline{\zeta}w|^2}\right)\frac{d\sigma(\zeta)}{1-|\zeta|^2}\\ &\lesssim \frac{S_\sigma(w)}{(1-|w|^2)^t}.\end{aligned}$$

Thus, S_σ satisfies (2.7). It follows that $S_\sigma + P_\nu$ satisfies (2.7) for all $s > -1$ and $t > 1$. Therefore, by Remark 2.6, $(1-|z|^2)^\alpha \omega$ verifies (2.7) for all $\alpha > -1$. ∎

In the rest of this subsection, let $\omega(z) = (1-|z|^2)^\alpha (S_\sigma(z) + P_\nu(z))$, for a fixed $\alpha > -1$ and finite positive Borel measures σ and ν on $\mathbb{D}$ and $\mathbb{T}$, respectively. Let $\check{\omega}$ be the weight given by $\check{\omega}(z) = (1-|z|^2)^\alpha (U_\sigma(z) + P_\nu(z))$, $\quad z \in \mathbb{D}$.

Theorem 2.8 *Let ω and $\check{\omega}$ be as given above, and let φ be an analytic self-map of $\mathbb{D}$. The following assertions hold.*

(1) *C_φ is bounded (resp. compact) on $\mathcal{D}_\omega$ if and only if*

$$\frac{1}{|\Delta_n|}\int_{\Delta_n} \frac{N_{\varphi,\check{\omega}}(z)}{\check{\omega}(z)} dA(z) = O(1), \quad (\textit{resp. } o(1) \textit{ as } n \to \infty).$$

(2) *C_φ belongs to $\mathcal{S}_p(\mathcal{D}_\omega)$, $p > 0$, if and only if*

$$\sum_{n=0}^{+\infty}\left(\frac{1}{|\Delta_n|}\int_{\Delta_n} \frac{N_{\varphi,\check{\omega}}(z)}{\check{\omega}(z)} dA(z)\right)^{\frac{p}{2}} < \infty.$$

https://doi.org/10.4153/S0008439523000292 Published online by Cambridge University Press

Proof Let $f \in \mathcal{D}_\omega$. We have

$$\begin{aligned}
&\mathcal{D}_\omega(f) - \int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)^\alpha P_\nu(z)dA(z) \\
&= \int_{\mathbb{D}}\int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)^\alpha \log\left|\frac{1-\overline{\zeta}z}{z-\zeta}\right| dA(z)\frac{d\sigma(\zeta)}{1-|\zeta|^2} \\
&\asymp \int_{\mathbb{D}}\int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)^\alpha \left(1-\left|\frac{z-\zeta}{1-\overline{\zeta}z}\right|^2\right) dA(z)\frac{d\sigma(\zeta)}{1-|\zeta|^2} \\
&= \int_{\mathbb{D}}\int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)^\alpha \frac{1-|z|^2}{|1-\overline{\zeta}z|^2} dA(z)d\sigma(\zeta) \\
&= \int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)^\alpha U_\sigma(z)dA(z).
\end{aligned}$$

It follows that $\mathcal{D}_\omega = \mathcal{D}_{\check{\omega}}$ with equivalent norms. Therefore, C_φ is bounded (resp. compact) on $\mathcal{D}_\omega$ if and only if C_φ is bounded (resp. compact) on $\mathcal{D}_{\check{\omega}}$. Taking into account that $\check{\omega}(z) \asymp \check{\omega}(z_n)$, for $z \in \Delta_n$, the first assertion follows by combining Proposition 2.7, Lemma A, and (1) and (2) of Corollary 2.4.

For the proof of the second assertion, note that since $\mathcal{D}_\omega = \mathcal{D}_{\check{\omega}}$ with equivalent norms then the operator $I : \mathcal{D}_\omega \to \mathcal{D}_{\check{\omega}}$ which takes f to f is bounded and invertible. It follows that I^*, the adjoint of I defined by $\langle I^* f, g\rangle_{\mathcal{D}_\omega} = \langle f, Ig\rangle_{\mathcal{D}_{\check{\omega}}}$ is bounded and invertible on $\mathcal{D}_{\check{\omega}}$. This implies

$$I^*\left(C_{\varphi,\mathcal{D}_{\check{\omega}}}\right)^*\left(C_{\varphi,\mathcal{D}_{\check{\omega}}}\right)I \asymp \left(C_{\varphi,\mathcal{D}_\omega}\right)^*\left(C_{\varphi,\mathcal{D}_\omega}\right),$$

where $C_{\varphi,\mathcal{D}_\omega}$ is the operator $C_\varphi : \mathcal{D}_{\check{\omega}} \to \mathcal{D}_{\check{\omega}}$ and $C_{\varphi,\mathcal{D}_\omega}$ is the operator $C_\varphi : \mathcal{D}_\omega \to \mathcal{D}_\omega$. It follows that if $C_{\varphi,\mathcal{D}_\omega}$ is compact, then $C_{\varphi,\mathcal{D}_\omega}$ belongs to $\mathcal{S}_p(\mathcal{D}_\omega)$ if and only if $C_{\varphi,\mathcal{D}_{\check{\omega}}}$ belongs to $\mathcal{S}_p(\mathcal{D}_{\check{\omega}})$. Hence, using once again Proposition 2.7, Lemma A, and (3) of Corollary 2.4, we obtain the second assertion of the theorem. ■

The following theorem extend the result obtained by Pau and Pérez [21, Theorem 4.1] in standard Dirichlet spaces setting to the Green potential of the Riesz measure of any positive superharmonic function. Recall that

$$\check{N}_{\varphi,\sigma}(w) = \sum_{w=\varphi(z)} U_\sigma(z) \ \text{ if } w \in \varphi(\mathbb{D}) \ ; \ \check{N}_{\varphi,\sigma}(w) = 0 \ \text{ if } w \notin \varphi(\mathbb{D}).$$

Theorem 2.9 *Let $p > 0$, and let φ be an analytic self-map of $\mathbb{D}$. Let σ be a finite positive measure on $\mathbb{D}$. Then, C_φ belongs to $\mathcal{S}_p(\mathcal{D}_\sigma)$ if and only if*

$$\int_{\mathbb{D}}\left(\frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)}\right)^{p/2} d\lambda(z) < \infty.$$

Proof By Theorem 2.8, we have

$$C_\varphi \in \mathcal{S}_p(\mathcal{D}_\sigma) \Longleftrightarrow \sum_{n=0}^{+\infty}\left(\frac{1}{|\Delta_n|}\int_{\Delta_n}\frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)}dA(z)\right)^{p/2} < \infty.$$

https://doi.org/10.4153/S0008439523000292 Published online by Cambridge University Press

Therefore, it suffices to show that

$$\frac{1}{|\Delta_n|}\int_{\Delta_n}\frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)}dA(z)\in\ell^{p/2}\Longleftrightarrow\frac{\check{N}_{\varphi,\sigma}(w)}{U_\sigma}\in L^{p/2}(\mathbb{D},d\lambda).$$

We use for the proof some standard arguments. First, we prove that for all $p>0$, we have

$$\check{N}_{\varphi,\sigma}(z)^p\lesssim\frac{1}{|\Delta(z,r)|}\int_{\Delta(z,r)}\check{N}_{\varphi,\sigma}(w)^p dA(w),\quad z\in\mathbb{D}.\tag{2.8}$$

The function $\check{N}_{\varphi,\sigma}$ satisfies the sub-mean-value property, that is,

$$\check{N}_{\varphi,\sigma}(z)\lesssim\frac{1}{|\Delta(z,r)|}\int_{\Delta(z,r)}\check{N}_{\varphi,\sigma}(w)dA(w),\quad z\in\mathbb{D}$$

(see [5, Lemma 5.2]). Therefore, there exists a subharmonic function u on $\mathbb{D}$ such that $N_{\varphi,\sigma}\le u$ on $\mathbb{D}$ and $u=N_{\varphi,\sigma}$ almost everywhere on $\mathbb{D}$ (see [19]. Since

$$u(z)^p\lesssim\frac{1}{|\Delta(z,r)|}\int_{\Delta(z,r)}u(w)^p dA(w),\quad z\in\mathbb{D},$$

by [19, Lemma 3], we obtain (2.8). Now, we have

$$\int_{\mathbb{D}}\left(\frac{\check{N}_{\varphi,\sigma}(w)}{U_\sigma(w)}\right)^{p/2}d\lambda(w)\asymp\sum_{n=0}^{+\infty}\int_{\Delta_n}\left(\frac{\check{N}_{\varphi,\sigma}(w)}{U_\sigma(w)}\right)^{p/2}d\lambda(w).$$

Taking into account that $U_\sigma(w)\asymp U_\sigma(z)$ if $w\in\Delta_n$ and $z\in b\Delta_n$, the inequality (2.8) gives

$$\begin{aligned}\int_{\mathbb{D}}\left(\frac{\check{N}_{\varphi,\sigma}(w)}{U_\sigma(w)}\right)^{p/2}d\lambda(w)&\lesssim\sum_{n=0}^{+\infty}\int_{\Delta_n}\left(\frac{1}{|\Delta_n|}\int_{b\Delta_n}\frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)}dA(z)\right)^{p/2}d\lambda(w)\\&\asymp\sum_{n=0}^{+\infty}\left(\frac{1}{|\Delta_n|}\int_{b\Delta_n}\frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)}dA(z)\right)^{p/2}.\end{aligned}$$

Since $(b\Delta_n)_n$ is a covering of $\mathbb{D}$ of finite multiplicity, then for all n there exist $n_1,n_2,\dots,n_N$ such that $b\Delta_n\subset\cup_{k=1}^N\Delta_{n_k}$, for some $N\in\mathbb{N}^*$ not depending on n. Hence,

$$\int_{b\Delta_n}\frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)}dA(z)\lesssim\int_{\Delta_{m_n}}\frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)}dA(z),$$

where m_n is such that $\int_{\Delta_{m_n}}\frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)}dA(z)=\max_{1\le k\le N}\int_{\Delta_{n_k}}\frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)}dA(z)$. Therefore,

$$\begin{aligned}\int_{\mathbb{D}}\left(\frac{\check{N}_{\varphi,\sigma}(w)}{U_\sigma(w)}\right)^{p/2}d\lambda(w)&\lesssim\sum_{n=0}^{+\infty}\left(\frac{1}{|\Delta_n|}\int_{\Delta_{m_n}}\frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)}dA(z)\right)^{p/2}\\&\le\sum_{n=0}^{+\infty}\left(\frac{1}{|\Delta_n|}\int_{\Delta_n}\frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)}dA(z)\right)^{p/2}.\end{aligned}$$

On the other hand, let $\zeta_n \in \overline{\Delta}_n$ such that $\dfrac{\check{N}_{\varphi,\sigma}(\zeta_n)}{U_\sigma(\zeta_n)} = \sup_{z\in\Delta_n} \dfrac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)}$. We have

$$\begin{aligned}\sum_{n=0}^{+\infty}\left(\frac{1}{|\Delta_n|}\int_{\Delta_n}\frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)}dA(z)\right)^{p/2} &\lesssim \sum_{n=0}^{+\infty}\left(\frac{\check{N}_{\varphi,\sigma}(\zeta_n)}{U_\sigma(\zeta_n)}\right)^{p/2}\\ &\lesssim \sum_{n=0}^{+\infty}\frac{1}{|\Delta_n|}\int_{b\Delta_n}\left(\frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)}\right)^{p/2}dA(z)\\ &\asymp \sum_{n=0}^{+\infty}\int_{b\Delta_n}\left(\frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)}\right)^{p/2}d\lambda(z)\\ &\asymp \int_{\mathbb{D}}\left(\frac{\check{N}_{\varphi,\sigma}(z)}{U_\sigma(z)}\right)^{p/2}d\lambda(z).\end{aligned}$$

The proof is complete. ∎

3 Composition operators on weighted Bergman spaces

3.1 Radial weights

Let $\omega : [0,1) \to (0,\infty)$ be a continuous radial weight. We associate with ω, the weight ω_* defined by

$$\omega_*(r) = \int_r^1 (t-r)\omega(t)dt.$$

As pointed in [15], $A^2_\omega = \mathcal{D}_{\omega_*}$ with equivalent norms and ω_* always satisfies $(\mathcal{W}_1)$, $(\mathcal{W}_3)$, and $(\mathcal{W}_4^{(I)})$. Therefore, as a consequence of Theorem 2.5, we have the following result.

Theorem 3.1 *Let φ be an analytic self-map of $\mathbb{D}$, and let $p > 0$. Let ω be a continuous radial weight such that ω_* satisfies $(\mathcal{W}_2)$. Then*

$$C_\varphi \in \mathcal{S}_p(A^2_\omega) \Longleftrightarrow \int_{\mathbb{D}}\left(\frac{N_{\varphi,\omega_*}(z)}{\omega_*(z)}\right)^{p/2} d\lambda(z) < \infty.$$

Corollary 3.2 *Let φ be an analytic self-map of $\mathbb{D}$, and let $p > 0$. Let ω be a continuous radial weight such that ω_* satisfies $(\mathcal{W}_2)$ fore some $\delta > 0$. The following assertions hold.*

(1) *If C_φ belongs to $\mathcal{S}_p(H^2)$, then C_φ belongs to $\mathcal{S}_p(A^2_\omega)$.*
(2) *If C_φ belongs to $\mathcal{S}_p(A^2_\omega)$, then C_φ belongs to $\mathcal{S}_p(A^2_{\delta-1})$.*

Proof Since ω_* satisfies $(\mathcal{W}_2)$ and always satisfies $(\mathcal{W}_1)$ then, by [15, Lemma 2.1], each composition operator induced by the symbol $q_{\varphi(0)}(z) = \frac{\varphi(0)-z}{1-\overline{\varphi(0)}z}$ is bounded on $A^2_\omega = \mathcal{D}_{\omega_*}$. It is also known that each composition operator induced by $q_{\varphi(0)}$ is bounded on $\mathcal{H}_\alpha$. Hence, by standard arguments, we may assume without loss of

https://doi.org/10.4153/S0008439523000292 Published online by Cambridge University Press

generality that $\varphi(0)=0$. The condition $(\mathcal{W}_2)$ gives

$$\frac{\omega_*(r)}{\omega_*(t)} \leq \left(\frac{1-r}{1-t}\right)^{1+\delta}, \quad 0 \leq r \leq t < 1.$$

On the other hand, by a direct calculation, $s \in [0,1) \to \frac{\omega_*(s)}{1-s}$ is a nonincreasing function. It follows that

$$\frac{\omega_*(r)}{\omega_*(t)} \geq \frac{1-r}{1-t}, \quad 0 \leq r \leq t < 1.$$

Let $z \in \varphi(\mathbb{D})$ and $w \in \mathbb{D}$ such that $\varphi(w) = z$. By Schwarz's lemma and the above inequalities, we obtain

$$\left(\frac{1-|w|^2}{1-|z|^2}\right)^{\delta+1} \lesssim \frac{\omega_*(w)}{\omega_*(z)} \lesssim \frac{1-|w|^2}{1-|z|^2}.$$

It follows that

$$\frac{N_{\varphi,\delta+1}(z)}{(1-|z|^2)^{\delta+1}} \lesssim \frac{N_{\varphi,\omega_*}(z)}{\omega_*(z)} \lesssim \frac{N_{\varphi,1}(z)}{(1-|z|^2)}, \quad z \in \mathbb{D}. \tag{3.1}$$

The assertions of the corollary are obtained by combining Theorem 3.1 and (1.1). ∎

A radial weight ω belongs to the class $\hat{D}$ if $\int_r^1 \omega(s)ds \lesssim \int_{\frac{1+r}{2}}^1 \omega(s)ds, r \in [0,1)$. Peláez and Rättyä obtained in [23], a trace class criteria for Toeplitz operators on Dirichlet spaces associated with regular weights and they obtained that, for $\omega \in \hat{D}$, C_φ belongs to $\mathcal{S}_p(A^2_\omega)$, for $p > 0$, if and only if

$$\int_{\mathbb{D}} \left(\frac{N_{\varphi,\omega^*}(z)}{\omega^*(z)}\right)^{p/2} d\lambda(z) < \infty,$$

where $\omega^*(r) := \int_r^1 s\log\left(\frac{s}{r}\right)\omega(s)ds$, $r \in (0,1)$. We point out that Lemma 2.4 in [15] and Theorem 2.5 for (I)-admissible weights still hold if we replace the condition $(\mathcal{W}_2)$ by the following one:

$(\mathcal{W}_2')$ there is $\delta > 0$ such that $\omega(r)(1-r)^{-(1+\delta)} \lesssim \omega(t)(1-t)^{-(1+\delta)}, \quad 0 \leq r \leq t < 1.$

Theorem 3.1 can be applied to continuous weights belonging to $\hat{D}$ thanks to the following lemma.

Lemma 3.3 *If ω belongs to $\hat{D}$, then ω_* satisfies $(\mathcal{W}_2')$.*

Proof Assume that ω belongs to $\hat{D}$. We have

$$\omega_*(r) \geq \int_{\frac{1+r}{2}}^1 (t-r)\omega(t)dt \gtrsim (1-r)\int_{\frac{1+r}{2}}^1 \omega(t)dt \gtrsim (1-r)\int_r^1 \omega(t)dt.$$

https://doi.org/10.4153/S0008439523000292 Published online by Cambridge University Press

It follows that $\omega_*(r) \asymp (1-r)\int_r^1 \omega(t)dt$. On the other hand, since $\omega \in \hat{D}$, there exists a constant $\delta > 0$ such that

$$\int_r^1 \omega(s)ds \lesssim \left(\frac{1-r}{1-t}\right)^{\delta} \int_t^1 \omega(s)ds, \quad 0 \le r \le t < 1$$

(see [22]). We obtain

$$\frac{\omega_*(r)}{(1-r)^{\delta+1}} \asymp \frac{1}{(1-r)^{\delta}} \int_r^1 \omega(s)ds \lesssim \frac{1}{(1-t)^{\delta}} \int_t^1 \omega(s)ds \asymp \frac{\omega_*(t)}{(1-t)^{\delta+1}},$$

for $0 \le r \le t < 1$. ∎

3.2 General case

Let ω be a weight not necessarily radial and consider the composition operator $C_\varphi : A^2_\omega \to A^2_\omega$. For the weights ω such that $\frac{\omega}{(1-|z|^2)^\eta} \in B_{p_0}(\eta)$ for some $p_0 > 1$ and $\eta > -1$, Constantin [8] characterized boundedness, compactness, and membership of C_φ in $\mathcal{S}_p(A^2_\omega)$, for $p \ge 2$, in terms of the pullback measure of ωdA under φ.

If C_φ is bounded on A^2_ω then $C_\varphi^* C_\varphi = T_{\frac{1}{\omega}d\mu}$ with $\mu(E) = A_\omega\left(\varphi^{-1}(E)\right)$ for any Borel subset E of $\mathbb{D}$. Using Theorem B, we obtain the following result, which extend [8, Theorem 6.2].

Theorem 3.4 *Let ω be a weight in $\omega \in \mathcal{C}_{p_0,t}$ for some $p_0 > 1$ and $t \ge 0$. Assume that φ is an analytic self-map of $\mathbb{D}$ such that C_φ is compact on A^2_ω. Then C_φ belongs to $\mathcal{S}_p(A^2_\omega)$, for $p > 0$, if and only if*

$$\sum_{n=0}^{\infty} \left(\frac{\int_{\varphi^{-1}(\Delta_n)} \omega(z)dA(z)}{\int_{\Delta_n} \omega(z)dA(z)} \right)^{p/2} < \infty.$$

In particular, if ω is an almost standard weight, then C_φ belongs to $\mathcal{S}_p(A^2_\omega)$ if and only if

$$\sum_{n=0}^{\infty} \left(\frac{A_\omega\left(\varphi^{-1}(\Delta_n)\right)}{(1-|z_n|^2)^2\omega(z_n)} \right)^{p/2} < \infty.$$

Note that, when ω is an almost standard weight, C_φ is bounded (resp. compact) on A^2_ω if and only if $A_\omega\left(\varphi^{-1}(\Delta_z)\right) = O\left((1-|z|^2)^2\omega(z)\right)$ (resp. $o\left((1-|z|^2)^2\omega(z)\right)$), $|z| \to 1^-$.

Here, we characterize boundedness, compactness, and membership of C_φ in $\mathcal{S}_p(A^2_\omega)$, for ω in some class $\mathcal{C}_{p_0,t}$, in terms of Nevanlinna counting function. Denote $\omega_{[2]} = (1-|z|^2)^2\omega$.

Theorem 3.5 *Let φ be an analytic self-map of $\mathbb{D}$, and let $p > 0$. Suppose that ω is a weight such that $\omega \in \mathcal{C}_{p_0,t}$ for some $p_0 > 1$ and $t \ge 0$. Then:*

https://doi.org/10.4153/S0008439523000292 Published online by Cambridge University Press

(1) *C_φ is bounded on A^2_ω if and only if*

$$\frac{1}{|\Delta_n|}\int_{\Delta_n} N_{\varphi,\omega_{[2]}}(z)dA(z) \lesssim \int_{\Delta_n}\omega(z)dA(z), \quad \forall n\in\mathbb{N}.$$

(2) *C_φ is compact on A^2_ω if and only if*

$$\frac{1}{|\Delta_n|}\int_{\Delta_n} N_{\varphi,\omega_{[2]}}(z)dA(z) = o\left(\int_{\Delta_n}\omega(z)dA(z)\right), \quad (n\to\infty).$$

(3) *C_φ belongs to $\mathcal{S}_p(A^2_\omega)$ if and only if*

$$\sum_{n=0}^{\infty}\left(\frac{\frac{1}{|\Delta_n|}\int_{\Delta_n} N_{\varphi,\omega_{[2]}}(z)dA(z)}{\int_{\Delta_n}\omega(z)dA(z)}\right)^{p/2} < \infty.$$

Proof Note that, since $\omega\in\mathcal{C}_{p_0,t}$ for some $p_0>1$ and $t\geq 0$, we have the following Littlewood–Paley estimates:

$$\|f\|^2_{A^2_\omega} \asymp |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)^2\omega(z)dA(z), \quad f\in \mathrm{Hol}(\mathbb{D}) \tag{3.2}$$

(see [7]). Therefore,

$$\int_{\mathbb{D}} |C_\varphi f(z)|^2\omega(z)dA(z) \asymp |f(\varphi(0))|^2 + \int_{\mathbb{D}} |(C_\varphi f)'(z)|^2\omega_{[2]}(z)dA(z), \quad f\in \mathrm{Hol}(\mathbb{D}).$$

It follows that $C_\varphi : A^2_\omega \to A^2_\omega$ is bounded (resp. compact) if and only if $C_\varphi : \mathcal{D}_{\omega_{[2]}} \to \mathcal{D}_{\omega_{[2]}}$ is bounded (resp. compact). Also, note that if $\omega\in\mathcal{C}_{p_0,t}$, then $\omega_{[2]}\in\mathcal{C}_{p_0,t+2}$. By Theorem 2.2, we obtain the first and the second assertions of the theorem.

By (3.2), the operator $X : A^2_\omega \to \mathcal{D}_{\omega_{[2]}}$ defined by $Xf = f$ is bounded and invertible. Similarly to the proof of the second assertion of Theorem 2.8, it follows that $C_\varphi : A^2_\omega \to A^2_\omega$ belongs to $\mathcal{S}_p(A^2_\omega)$ if and only if $C_\varphi : \mathcal{D}_{\omega_{[2]}} \to \mathcal{D}_{\omega_{[2]}}$ belongs to $\mathcal{S}_p(\mathcal{D}_{\omega_{[2]}})$. Hence, by Theorem 2.3, we obtain the third assertion of the theorem. ∎

Remark 3.6 Let $\alpha>-1$. Luecking and Zhu proved in [19] that the condition

$$\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{2+\alpha} \in L^{p/2}(\mathbb{D}, d\lambda)$$

is necessary when $p\geq 2$ and sufficient when $p\leq 2$ for C_φ to be in $\mathcal{S}_p(A^2_\alpha)$. Suppose that ω is a weight in $\mathcal{C}_{p_0,t}$ for some $p_0>1$ and $t\geq 0$. It is proved in [7] that $A^2_\omega = A^2_{\tilde{\omega}}$ with $\|f\|_{A^2_\omega} \asymp \|f\|_{A^2_{\tilde{\omega}}}$ for all $f\in Hol(\mathbb{D})$. It is proved also in [7] that $A^2_{\tilde{\omega}}$ is a reproducing kernel space with kernel $K^{\tilde{\omega}}$ satisfying

$$\|K^{\tilde{\omega}}_z\|^2_{A^2_{\tilde{\omega}}} \asymp \frac{1}{(1-|z|^2)^2\tilde{\omega}(z)}, \quad z\in\mathbb{D}.$$

https://doi.org/10.4153/S0008439523000292 Published online by Cambridge University Press

Since $C_\varphi \in \mathcal{S}_p(A^2_\omega)$ if and only if $C_\varphi \in \mathcal{S}_p(A^2_{\tilde{\omega}})$, using the same argument given in [19], we obtain that the condition

$$\frac{\int_{\Delta(z,(1-|z|^2)/2)} \omega(\zeta)dA(\zeta)}{\int_{\Delta(\varphi(z),(1-|\varphi(z)|^2)/2)} \omega(\zeta)dA(\zeta)} \in L^{p/2}(\mathbb{D}, d\lambda) \tag{3.3}$$

is necessary when $p \geq 2$ and sufficient when $p \leq 2$ for C_φ to be in $\mathcal{S}_p(A^2_\omega)$. Note that if in addition ω verifies (2.3), then the condition (3.3) is equivalent to

$$\frac{(1-|z|^2)^2\omega(z)}{(1-|\varphi(z)|^2)^2\omega(\varphi(z))} \in L^{p/2}(\mathbb{D}, d\lambda).$$

Acknowledgment The authors would like to thank Professor Omar El-Fallah for helpful discussions.

References

[1] A. Aleman, *Hilbert spaces of analytic functions between the Hardy and the Dirichlet space*. Proc. Amer. Math. Soc. **115**(1992), 97–104.
[2] A. Aleman, *The multiplication operator on Hilbert spaces of analytic functions*, Habilitationsschrift, Fern Universitat, Hagen, 1993.
[3] A. Aleman and O. Constantin, *Spectra of integration operators on weighted Bergman spaces*. J. Anal. Math. **109**(2009), 199–231.
[4] A. Aleman, S. Pott, and M. C. Reguera, *Characterizations of a limiting class B_∞ of Békollé–Bonami weights*. Rev. Mat. Iberoam. **35**(2019), 1766–1692.
[5] G. Bao, N. G. Göğüş, and S. Pouliasis, *On Dirichlet spaces with a class of superharmonic weights*. Canad. J. Math. **70**(2018), 721–741.
[6] G. Bao, H. Wulan, and K. Zhu, *A Hardy–Littlewood theorem for Bergman spaces*. Ann. Acad. Sci. Fenn. Math. **43**(2018), 807–821.
[7] M. Bourass and I. Marrhich, Littlewood–Paley estimates with applications to Toeplitz and integration operators on weighted Bergman spaces. Banach J. Math. Anal. **17**(2023), 10.
[8] O. Constantin, *Carleson embeddings and some classes of operators on weighted Bergman spaces*. J. Math. Anal. Appl. **365**(2010), 668–682.
[9] C. C. Cowen and B. MacCluer, *Composition operators on spaces of analytic functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, 1995.
[10] O. El-Fallah and M. ElIbbaoui, *Trace estimates of Toeplitz operators on Bergman spaces and applications to composition operators*. Rev. Mat. Iberoam. **38**(2022), 1723–1762.
[11] O. El-Fallah, H. Mahzouli, I. Marrhich, and H. Naqos, *Asymptotic behavior of eigenvalues of Toeplitz operators on the weighted analytic spaces*. J. Funct. Anal. **270**(2016), 4614–4630.
[12] O. El-Fallah, H. Mahzouli, I. Marrhich, and H. Naqos, *Toeplitz operators on harmonically weighted Bergman spaces and applications to composition operators on Dirichlet spaces*. J. Math. Anal. Appl. **466**(2018), 471–489.
[13] K. Esmaeili and K. Kellay, *Weighted composition operators on weighted Bergman and Dirichlet spaces*. Canad. Math. Bull. **66**(2023), 286–302.
[14] J. Fàbrega and J. M. Ortega, *Pointwise multipliers and corona type decomposition in BMOA*. Ann. Inst. Fourier (Grenoble) **46**(1996), 111–137.
[15] K. Kellay and P. Lefèvre, *Compact composition operators on weighted Hilbert spaces of analytic functions*. J. Math. Anal. Appl. **386**(2012), 718–727.
[16] P. Lefèvre, D. Li, H. Queffélec, and L. Rodríguez-Piazza, *Approximation numbers of composition operators on the Dirichlet space*. Ark. Mat. **53**(2015), 155–175.
[17] X. Liu, G. Chacón, and Z. Lou, *Characterizations of the Dirichlet-type space*. Complex Anal. Oper. Theory **9**(2015), 1269–1286.
[18] D. H. Luecking, *Representation and duality in weighted spaces of analytic functions*. Indiana Univ. Math. J. **34**(1985), 319–336.
[19] D. H. Luecking and K. Zhu, *Composition operators belonging to the Schatten ideals*. Amer. J. Math. **114**(1992), 1127–1145.

https://doi.org/10.4153/S0008439523000292 Published online by Cambridge University Press

[20] V. L. Oleinik, *Embedding theorems for weighted classes of harmonic and analytic functions*. J. Soviet Math. **9**(1978), 228–243.
[21] J. Pau and P. A. Pérez, *Composition operators acting on weighted Dirichlet spaces*. J. Math. Anal. Appl. **401**(2013), 682–694.
[22] J. A. Peláez, *Small weighted Bergman spaces*. In: Proceedings of the summer school in complex and harmonic analysis, and related topics, University of Eastern Finland, Joensuu (2016), 29–98.
[23] J. A. Peláez and J. Rättyä, *Trace class criteria for Toeplitz and composition operators on small Bergman spaces*. Adv. Math. **293**(2016), 606–643.
[24] D. Sarason and O. Silva, *Composition operators on a local Dirichlet space*. J. Anal. Math. **87**(2002), 433–450.
[25] J. H. Shapiro, *The essential norm of a composition operator*. Ann. Math. **125**(1987), 375–404.
[26] J. H. Shapiro, *Composition operators and classical function theory*, Springer, New York, 1993.
[27] N. Zorboska, *Composition operators on weighted Dirichlet spaces*. Proc. Amer. Math. Soc. **126**(1998), 2013–2023.

Faculty of Sciences, Mohammed V University in Rabat, CeReMAR, LAMA, B.P. 1014, Rabat, Morocco
e-mail: marouane.bourass@gmail.com fouzia.mkadmii@gmail.com

Laboratoire Mathématiques Fondamentales et Appliquées, Faculty of Sciences Ain-Chock, Hassan II University of Casablanca, B.P. 5366 Maarif, Casablanca, Morocco
e-mail: ibrahim.marrhich@univh2c.ma brahim.marrhich@gmail.com

https://doi.org/10.4153/S0008439523000292 Published online by Cambridge University Press