Can. J. Math., Vol. XXV, No. 6, 1973, pp. 1165–1169

ON THE COMMUTANT OF CERTAIN AUTOMORPHISM GROUPS

P. K. TAM

1. Introduction. Let $\mathscr{A}$ be a W^*-algebra, $A(\mathscr{A})$ the group of all automorphisms of $\mathscr{A}$. In this paper we have determined the commutant G' of a subgroup G of $A(\mathscr{A})$ for certain classes of G and $\mathscr{A}$. The main results are as follows.

THEOREM 1. *If G is a locally compact abelian group acting by translation on the W^*-algebra $L^\infty(G)$, then the commutant of a dense subgroup of G is G itself.*

THEOREM 2. *Consider a W^*-algebra $\mathscr{A}$, a topological group G with a dense subgroup D and a continuous faithful representation $g \mapsto \alpha_g$ of G as an ergodic group of automorphisms of $\mathscr{A}$* (the topology on $A(\mathscr{A})$ being pointwise convergence in the strong topology). *Suppose that:*

(1) *K is a topological group;*

(2) *$k \mapsto U(k)$ is a strongly continuous representation of K as a unitary group generating $\mathscr{A}$;*

(3) *For each $g \in G$ and $k \in K$ there is a constant $c(g, k)$ such that*

$$\alpha_g(U(k)) = c(g, k)U(k);$$

and

(4) *If χ is a continuous character of K then there exists $g \in G$ such that*

$$\chi(k) = c(g, k) \quad \textit{for all } k \in K$$

or

(4′) *If $\beta \in A(\mathscr{A})$ and*

$$\beta(U(k)) = \chi(k)U(k) \quad \textit{for all } k \in K$$

then there is a $g \in G$ such that $\chi(k) = c(g, k)$ for all $k \in K$.

Then we can conclude that

$$\{\alpha_d : d \in D\}' = \{\alpha_g : g \in C\}$$

where C is the centralizer of G.

THEOREM 3. *If G_i is an ergodic group of automorphisms of the abelian W^*-algebra $\mathscr{M}_i$ $(i = 1, 2)$, then $(G_1 \otimes G_2)' = G_1' \otimes G_2'$ as groups on $\mathscr{M}_1 \otimes \mathscr{M}_2$.*

Received May 15, 1972.

https://doi.org/10.4153/CJM-1973-125-8 Published online by Cambridge University Press

THEOREM 4. (For notations, cf. [**3**].) *Suppose that for each $i \in I, \mathcal{M}_i$ is an abelian W^*-algebra, ω_i a normal state on $\mathcal{M}_i$ with $\omega_i(1) = 1$, and G_i an ergodic group of automorphisms of $\mathcal{M}_i$. Then*

$$\left(\coprod_{i \in I} (G_i, \omega_i)\right)' = \left(\bigotimes_{i \in I} (G_i, \omega_i)\right)' = \bigotimes_{i \in I} (G_i', \omega_i)$$

as groups on $\otimes_{i \in I} (\mathcal{M}_i, \omega_i)$, where

$$\bigotimes_{i \in I} (G_i, \omega_i) = \left\{\alpha \in A\left(\bigotimes_{i \in I} (\mathcal{M}_i, \omega_i)\right) : \alpha = \bigotimes_{i \in I} g_i \text{ for some } (g_i)_{i \in I} \in \prod_{i \in I} G_i\right\},$$

and

$$\coprod_{i \in I} (G_i, \omega_i) = \left\{\alpha \in A\left(\bigotimes_{i \in I} (\mathcal{M}_i, \omega_i)\right) : \alpha = \bigotimes_{i \in I} g_i \text{ for some } (g_i)_{i \in I} \in \coprod_{i \in I} G_i\right\}.$$

A few words about the organisation of this paper are in order. In § 2 we give the proofs of a preparational lemma and Theorem 2, which are independent of Theorem 1. In § 3 we apply Theorem 2 to establish Theorem 1. In § 4 we present some applications of Theorems 1 and 2. In § 5 we prove Theorem 3. Finally in § 6, as an addendum, we indicate a proof of Theorem 4.

Acknowledgement. The results of this paper are partly contained in, and partly inspired by, the author's doctoral thesis written under the very helpful supervision of Professor Donald Bures at the University of British Columbia. The author is also grateful to the referee for his many valuable suggestions; in particular, the statements and the proofs of Theorems 1 and 2 are suggested by him.

2. Proof of Theorem 2. We first establish the following preparational

LEMMA. *Given a W^*-algebra $\mathcal{A}$, a group G and a representation $g \mapsto \alpha_g$ of G as an ergodic group of automorphisms of $\mathcal{A}$, suppose that a unitary U in $\mathcal{A}$ satisfies*

$$\alpha_g(U) = c(g)U \quad \text{for all } g \in G$$

for some function $c : G \to \mathbf{C}$. If $A \in \mathcal{A}$ satisfies

$$\alpha_g(A) = c(g)A \quad \text{for all } g \in G,$$

then $A = \lambda U$ for some $\lambda \in \mathbf{C}$.

Proof. By a direct calculation (and the fact that $\alpha_g(1) = 1$),

$$\alpha_g(U^{-1}A) = U^{-1}A \quad \text{for all } g \in G.$$

Proof of Theorem 2. Suppose that $\beta \in A(\mathcal{A})$ commutes with all α_d for $d \in D$. Then by continuity β commutes with α_g for $g \in G$ so that

$$\alpha_g[\beta(U(k)) = c(g, k)[\beta(U(k))].$$

https://doi.org/10.4153/CJM-1973-125-8 Published online by Cambridge University Press

The lemma then implies that

$$\beta(U(k)) = \chi(k)U(k) \quad \text{for all } k \in K.$$

By condition (2), χ is a continuous character of K. Hence condition (4) or (4′) proves the existence of $g \in G$ such that

$$\beta(U(k)) = c(g, k)U(k) \quad \text{for all } k \in K;$$

hence

$$\beta(U(k)) = \alpha_g(U(k)).$$

As $\{U(k) : k \in K\}$ generates $\mathscr{A}$, $\beta = \alpha_g$. As the representation of G is one-to-one, g belongs to the centralizer of G. This completes the proof.

3. Proof of Theorem 1. Let $\mathscr{A} = L^\infty(G)$ acting on $L^2(G)$ by multiplication. Define $\alpha : G \rightarrow A(\mathscr{A})$ by

$$\alpha_g(M_F) = M_{gF},$$

where $M_F \in \mathscr{A}$ is the multiplication by $F \in L^\infty(G)$, and g_F is the translate of F by g (i.e., $g_F(x) = F(x - g)$, $x \in G$). Then α is a continuous faithful representation of G as an ergodic group of automorphisms of $\mathscr{A}$ as required in Theorem 2. To meet the other conditions of Theorem 2, let K be $\hat{G}$, the dual group of G, and let $U(k) \in \mathscr{A}$ be the multiplication by the continuous character k. Then by well-known theorems in harmonic analysis it is not difficult to check that all conditions (1)–(4) of Theorem 2 are satisfied. The proof is thus completed by an application of Theorem 2.

4. Applications. The following applications of Theorems 1 and 2 are not only interesting on their own right, but also useful in the problem of unitary equivalence of operators [**7**].

Let X be $[0, 1)$ with addition mod 1 (i.e. the circle group), or $\mathbf{R}$ with usual addition (and in both cases, with usual topology and Lebesque measure), and let D be a dense subgroup of X. For each $x \in X$, denote by T_x the automorphism of $\mathscr{M}$ ($= L_\infty(X)$ acting by multiplication on $L_2(X)$) induced by translation by x. Then by Theorem 1, $\{T_d : d \in D\}' = \{T_x : x \in X\}$.

PROPOSITION 1. *Let $\mathscr{M}$ be $L_\infty(\mathbf{R})$ acting by multiplication on $L_2(\mathbf{R})$. For each non-zero real number r, let s_r be the automorphism of $\mathscr{M}$ given by:*

$$(s_r f)(x) = f(r^{-1}x), \quad x \in \mathbf{R},$$

for any $f \in L_\infty(\mathbf{R})$. Then for any set D of strictly positive real numbers with $\ln(D)$ *a dense subgroup of* $\mathbf{R}$, *we have:*

$$\{s_r : |r| \in D\}' = \{s_r : r \text{ non-zero real}\}.$$

Proof. Suppose $\alpha \in \{s_r : |r| \in D\}'$. By ergodicity it is easy to see that either: for every measurable subset A of $\mathbf{R}_+$ (the positive reals), $\alpha(T_A) = T_B$ for some measurable subset B of $\mathbf{R}_+$, or: for every measurable subset A of $\mathbf{R}_+$,

https://doi.org/10.4153/CJM-1973-125-8 Published online by Cambridge University Press

$\alpha(T_A) = T_D$ for some measurable subset D of $\mathbf{R}_-$ (the negative reals), where T_A denotes the multiplication by the characteristic function 1_A on A. It then follows that α can be identified to an automorphism $\bar{\alpha}$ of $\mathscr{M}$, which commutes with automorphisms of $\mathscr{M}$ induced by translations by a certain dense subgroup of $\mathbf{R}$. Theorem 1 implies that $\bar{\alpha}$ is induced by translation, and the identification shows that $\alpha = s_r$ for some non-zero real number r. The proposition then follows immediately.

Let Z_2 be the additive group of two elements 0 and 1, S_0 the ring of all subsets of Z_2, μ_0 the measure on (Z_2, S_0) assigning q to 1 and $1 - q$ to 0 where $q \in [\frac{1}{2}, 1]$. For each $n \in Z$, let $X_n = Z_2$, $S_n = S_0$, and $\mu_n = \mu_0$. Let $X = \times_{n\in Z} X_n$, $S' = \times_{n\in Z} S_n$, and let (X, S, μ_q) be the completion of $\times_{n\in Z}\mu_n$ on (X, S'). Let $\Delta = \coprod_{n\in Z} X_n$. Let $\mathscr{M}$ be $L_\infty(X, S, \mu_q)$ acting by multiplication on $L_2(X, S, \mu_q)$. For each $\delta \in \Delta$ the translation in X by δ induces an automorphism α_δ of $\mathscr{M}$ [**5**; **9**]. When $q = \frac{1}{2}$, Kakutani's theorem [**4**] implies that translation by each $x \in X$ induces an automorphism α_x of $\mathscr{M}$. For simplicity we write α_n instead of α_{δ_n}, where $\delta_n \in \Delta$ $(n \in Z)$ is such that $\delta_n(m) = 0$ if $m \neq n$, $= 1$ if $m = n$.

PROPOSITION 2. *With the above notations we have:*
(i) *When* $q > \frac{1}{2}$, $\{\alpha_n : n \in Z\}' = \{\alpha_\delta : \delta \in \Delta\}$.
(ii) *When* $q = \frac{1}{2}$, $\{\alpha_n : n \in Z\}' = \{\alpha_x : x \in X\}$.

Proof. Let X_n have the discrete topology, X the product topology, and Δ the relative topology. Let $G = \Delta$ in case (i), and $G = X$ in case (ii). Let $D = K = \Delta$ in both cases. For $g \in G$, let α_g be the automorphism of $\mathscr{M}$ induced by translation by g. For $n \in Z$ let U_n be the multiplication by the function $U_n(\cdot)$, where $U_n(x) = -1$ when $x(n) = 1$, and $= 1$ when $x(n) = 0$ $(x \in X)$. For $\delta \in \Delta$, let U_δ be the multiplication by the function $U_\delta(\cdot)$, where $U_\delta(x) = \prod_{\delta_n=1} U_n(x)$, $x \in X$. Condition (4) of Theorem 2 is verified in case (ii) by direct calculation. Condition (4′) of Theorem 2 is verified in case (i) by similar calculation and Kakutani's theorem [**4**]. Other conditions of Theorem 2 are also satisfied by well-known theorems [**5**; **9**] or by simple calculations. The proposition then follows from Theorem 2.

5. Proof of Theorem 3. We shall need the following result of [**2**]; for the sake of completeness we include an indication of a proof.

LEMMA [**2**]. *Let* $\mathscr{M}_1, \mathscr{M}_2$ *be abelian* W*-*algebras, and* H *an ergodic group of automorphisms of* $\mathscr{M}_2$. *Then*

$$\{M \in \mathscr{M}_1 \otimes \mathscr{M}_2 : (1 \otimes h)(M) = M \text{ for all } h \in H\} = \mathscr{M}_1 \otimes \mathbf{C}.$$

Proof. Represent $\mathscr{M}_2$ as maximal abelian on a hilbert space $\mathscr{H}$. Then each automorphism h of $\mathscr{M}_2$ is induced by a unitary operator on $\mathscr{H}$. The lemma then follows from the commutant theorem.

https://doi.org/10.4153/CJM-1973-125-8 Published online by Cambridge University Press

Proof of Theorem 3. Suppose $\alpha \in (G_1 \otimes G_2)'$. For any $M_1 \in \mathscr{M}_1$, $M_2 \in \mathscr{M}_2$, $g_1 \in G_1$, $g_2 \in G_2$ we have

$$(1 \otimes g_2)[\alpha(M_1 \otimes 1)] = \alpha(M_1 \otimes 1),$$
$$(g_1 \otimes 1)[\alpha(1 \otimes M_2)] = \alpha(1 \otimes M_2).$$

By the preceding lemma there are $g_1' \in G_1'$ and $g_2' \in G_2'$ such that

$$\alpha = g_1' \otimes g_2'.$$

The theorem then follows immediately.

6. Addendum. As a generalization of Theorem 3 to the infinite tensor product algebra, we have Theorem 4. For technical reasons we shall only sketch the proof briefly. First it is obvious that

$$\bigotimes_{i \in I} (G_i', \omega_i) \subset \left(\bigotimes_{i \in I} (G_i, \omega_i) \right)' \subset \left(\coprod_{i \in I} (G_i, \omega_i) \right)'.$$

Let $\alpha \in (\coprod_{i \in I}(G_i, \omega_i))'$. Then by the associativity of the tensor product [**5**] and the ergodicity of $\coprod_{i \in I}(G_i, \omega_i)$ (cf. [**1**]), α induces an automorphism $g_i' \in G_i'$ for each $i \in I$ such that

$$\alpha = \bigotimes_{i \in I} g_i' \in \bigotimes_{i \in I} (G_i', \omega_i).$$

References

1. D. Bures, *Certain factors constructed as infinite tensor products*, Compositio Math. *15* (1963), 169–191.

2. ——— *Abelian subalgebras of von Neumann algebras* (to appear in Mem. Amer. Math. Soc.).

3. ——— *An extension of Kautani's theorem on infinite product measures to the tensor product of semi-finite W*-algebras*, Trans. Amer. Math. Soc. *135* (1969), 199–212.

4. S. Kakutani, *On equivalence of infinite product measures*, Ann. of Math. *49* (1948), 214–226.

5. J. von Neumann, *On infinite direct products*, Compositio Math. *6* (1938), 1–77.

6. L. Pukánszky, *Some examples of factors*, Publ. Math. Debrecen *4* (1956), 135–156.

7. P. K. Tam, *On unitary equivalence of certain classes of non-normal operators*, Can. J. Math. *23* (1971), 849–856.

The Chinese University of Hong Kong,
Shatin, N.T., Hong Kong.

https://doi.org/10.4153/CJM-1973-125-8 Published online by Cambridge University Press