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RESEARCH ARTICLE

Optimal allocation of a coherent system with statistical dependent subsystems

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Abstract
This paper studies the optimal allocation policy of a coherent system with independent heterogeneous components and dependent subsystems, the systems are assumed to consist of two groups of components whose lifetimes follow proportional hazard (PH) or proportional reversed hazard (PRH) models. We investigate the optimal allocation strategy by finding out the number k of components coming from Group A in the up-series system. First, some sufficient conditions are provided in the sense of the usual stochastic order to compare the lifetimes of two-parallel–series systems with dependent subsystems, and we obtain the hazard rate and reversed hazard rate orders when two subsystems have independent lifetimes. Second, similar results are also obtained for two-series–parallel systems under certain conditions. Finally, we generalize the corresponding results to parallel–series and series–parallel systems with multiple subsystems in the viewpoint of the minimal path and the minimal cut sets, respectively. Some numerical examples are presented to illustrate the theoretical findings.

1. Introduction

In reliability theory and survival analysis, it is of eternal interest to explore the stochastic properties of the system. Since Barlow and Proschan [1] has been devoted to investigating and enhancing the reliability of the system, a great deal of scholars and engineers have paid their attentions on reliability engineering in the past decades. As two important system structures in reliability theory and engineering, the parallel and series systems have been studied comprehensively, interested readers may refer to Boland *et al.* [6], Singh and Misra [41], Hu and Wang [25], Da *et al.* [11], Navarro and Spizzichino [38], Yan *et al.* [47], Yan and Luo [45], Chen *et al.* [7], Fang and Wang [20], Kundu *et al.* [29], Yan and Wang [46], Majumder *et al.* [36] and the reference therein. Some researchers also focus on the optimal allocation of components for k-out-of-n systems and the general coherent systems, refer to Bhattacharya and Samaniego [5], Li and Ding [32], Ding and Li [15], You *et al.* [50], Zhang [51], Ding *et al.* [16], Ling and Wei [34], Guo *et al.* [23], Torrado [44] and Yan *et al.* [49].

A parallel–series system may be regarded as a number of series subsystems connected in parallel. Similarly, a series–parallel system may be considered as a number of parallel subsystems arranged by series. Many scholars and engineers devoted themselves to improving the reliability of the parallel–series or series–parallel systems, the main topic of this line is how to allocate the components to different positions of the systems can construct the parallel–series or series–parallel systems with high-level reliability performance? El-Neweihi *et al.* [19] studied the optimal allocation of components to parallel–series and series–parallel systems by majorization order and Schur-convex function. Laniado and Lillo [30] developed the optimal allocation policy of components in two-parallel–series and two-series–parallel systems with two types independent components and independent subsystems, under the assumption that there are components in one subsystem whose lifetimes are independently and identically distributed random variables, and the lifetimes of the components in the other subsystem are independently and identically distributed random variables but with different distribution functions. Ling *et al.* [35] established the stochastic comparison of the independent heterogeneous components grouping in series–parallel and parallel–series systems. Fang *et al.* [21] investigated the optimal allocation policy of the dependent heterogeneous components of series–parallel and parallel–series systems under Archimedean copula dependence. For more relevant studies in the parallel–series and series–parallel systems, one can refer to Tavakkoli-Moghaddam *et al.* [43], Dao *et al.* [12], Feizabadi and Jahromi [22] and Sun *et al.* [42].

To the best of our knowledge, the optimal allocation policy for two-parallel–series and two-series–parallel systems with dependent subsystems has not been touched yet. In practice, due to the difference of production process and production technology, the lifetimes of different components from the same type are almost heterogeneous, and subsystems usually operate in the same environment or share the same load, thus, the operation of subsystems are dependent. In this paper, we assume that all components are independent and heterogeneous, and the lifetimes of two subsystems are dependent, consider the problem of allocation of components to the different positions in the system so as to improve the system's reliability, and have a further discussion on the two-parallel–series and two-series–parallel systems studied in Laniado and Lillo [30]. Also, we generalize the corresponding results to parallel–series and series–parallel systems with multiple subsystems in the viewpoint of the minimal path and the minimal cut sets.

Some of our results established here can be applied in guiding system assembly policy for engineers. Consider a production system having n machines whose lifetimes are independent and nonidentically distributed random variables, and suppose that the final product is obtained through a process of n successive stages (or steps), and the full process is finished when a product is treated by the n machines. Due to an increase in demand, it is necessary to open another production line with n different machines, whose lifetimes are not necessarily identically distributed with the original ones. Note that the total production system stops when at least one machine in each line has failed. One of subsystems fails will increase the load of the other subsystem. Thus, the lifetimes of the two subsystems are dependent, and the question is how to allocate the machine to each position to improve the reliability of the total production system? In Section 3, we give an answer to this question.

The remainder of this article is rolled out as follows. Section 2 recalls some pertinent notions and definitions used in the sequel. In Sections 3 and 4, we establish the optimal allocation policies of two-parallel–series and two-series–parallel systems, respectively. Section 5 generalizes the corresponding results obtained in Sections 3 and 4 to parallel–series and series–parallel systems with multiple subsystems in viewpoint of the minimal path and cut sets, respectively. Some conclusions and future directions are made in Section 6.

2. Preliminaries

Before proceeding to the main results, let us first review some basic concepts that will be used in the sequel. For simplicity of the discussion, we denote $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{R}^+ = (0, +\infty)$.

2.1. Stochastic orders

We first give the definitions of some stochastic orders between two random variables. For a random variable X, let us denote $F_X, \bar{F}_X, h_X$ and $\tilde{r}_X$ the distribution function, the survival function, the hazard rate function and the reversed hazard rate function, respectively.

Definition 1. *A random variable X is said to be smaller than Y in the*

(i) usual stochastic order (denoted by $X \leq_{st} Y$) if $\overline{F}_X(x) \leq \overline{F}_Y(x)$ for all $x \in \mathbb{R}$;

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(ii) hazard rate order (denoted by $X \le_{hr} Y$) if $h_X(x) \ge h_Y(x)$ for all $x \in \mathbb{R}$, or equivalently, if $\overline{F}_Y(x)/\overline{F}_X(x)$ is increasing in $x \in \mathbb{R}$;

(iii) reversed hazard rate order (denoted by $X \le_{rh} Y$) if $\tilde{r}_X(x) \le \tilde{r}_Y(x)$ for all $x \in \mathbb{R}$, or equivalently, if $F_Y(x)/F_X(x)$ is increasing in $x \in \mathbb{R}$.

It is well-known that the (reversed) hazard rate order implies the usual stochastic order, while the reversed statement is not true in general. For more comprehensive discussions on various stochastic orders and their applications, one may refer to the monographs by Shaked and Shanthikumar [40] and Belzunce *et al.* [3].

Definition 2. *The random vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ is said to follow*

(i) the proportional hazards (denoted by $\boldsymbol{X} \sim PH(\boldsymbol{\alpha}, \bar{F})$) model with baseline survival function $\overline{F}(x)$ and frailty parameter vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, if X_i has the survival function

$$\bar{F}_{X_i}(x) = \bar{F}^{\alpha_i}(x), \quad \text{for } \alpha_i > 0,\ i = 1, 2, \ldots, n.$$

(ii) the proportional reversed hazards (denoted by $\boldsymbol{X} \sim PRH(\boldsymbol{\beta}, F)$) model with baseline distribution function $F(x)$ and resilience parameter vector $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$, if X_i has the distribution function

$$F_{X_i}(x) = F^{\beta_i}(x), \quad \text{for } \beta_i > 0,\ i = 1, 2, \ldots, n.$$

It is well-known that the Exponential, Weibull, Lomax and Pareto distributions are special cases of the PH model, while Fréchet distribution is an example of the PRH model. For more detailed applications of the PH model, we refer the readers to Cox [9], Collett [8], Li *et al.* [33], Yan *et al.* [48] and Jarrahiferiz *et al.* [26], and for more comprehensive discussions on the PRH model, one may refer to Kalbfleisch and Lawless [27], Gupta and Gupta [24] and Belzunce and Martínez-Riquelme [2].

2.2. Majorization order

The notion of majorization order is a key tool in establishing various inequalities arising from many research areas. For any real two vectors $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$, denote $x_{1:n} \le x_{2:n} \le \cdots \le x_{n:n}$ and $y_{1:n} \le y_{2:n} \le \cdots \le y_{n:n}$ be the increasing arrangements of the components of $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively.

Definition 3. *A vector $\boldsymbol{x}$ is said to majorize $\boldsymbol{y}$ (denoted by $\boldsymbol{x} \overset{m}{\succeq} \boldsymbol{y}$), if $\sum_{i=1}^{j} x_{i:n} \le \sum_{i=1}^{j} y_{i:n}$ for $j = 1, 2, \ldots, n-1$ and $\sum_{i=1}^{n} x_{i:n} = \sum_{i=1}^{n} y_{i:n}$.*

For an elaborate discussion on the theory of majorization orders and their applications, one may refer to Marshall *et al.* [37] and Kundu *et al.* [28].

Definition 4 ([37])**.** *A real-valued function φ defined on a set $\mathbb{A} \subseteq \mathbb{R}^n$ is said to be Schur-convex [Schur-concave] on $\mathbb{A}$ if and only if $\boldsymbol{x} \overset{m}{\succeq} \boldsymbol{y}$ implies $\varphi(\boldsymbol{x}) \ge [\le] \varphi(\boldsymbol{y})$, for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{A}$.*

Lemma 1 ([37])**.** *Let φ be a continuously differentiable function on $\mathbb{A} \subseteq \mathbb{R}^n$. Then, φ is said to be Schur-convex [Schur-concave] on $\mathbb{A}$ if and only if φ is symmetric on $\mathbb{A}$, and for all $\boldsymbol{z} \in \mathbb{A}$ and $1 \le i \ne j \le n$,*

$$(z_i - z_j)\left(\frac{\partial \varphi(\boldsymbol{z})}{\partial z_i} - \frac{\partial \varphi(\boldsymbol{z})}{\partial z_j}\right) \ge [\le] 0.$$

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2.3. Copula

For a random vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ with joint cumulative distribution function H and univariate marginal distribution functions $F_1, F_2, \ldots, F_n$, its *copula* is a distribution function $C : [0,1]^n \to [0,1]$, such that

$$H(\boldsymbol{x}) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)).$$

Similarly, a *survival copula* is a joint cumulative survival function $\hat{C} : [0,1]^n \to [0,1]$, such that

$$\bar{H}(\boldsymbol{x}) = \mathbb{P}(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) = \hat{C}(\bar{F}_1(x_1), \bar{F}_2(x_2), \ldots, \bar{F}_n(x_n)),$$

where $\bar{F}_i = 1 - F_i$ $(i = 1, 2, \ldots, n)$ is the marginal survival function and $\bar{H}$ is the joint survival function.

When C is Schur-concave, C is said to be the Schur-concave copula. The family of the Schur-concave copula includes some important copulas, such as Archimedean copula and Farlie–Gumble–Morgenstern (FGM). For a decreasing and continuous function $\psi : \mathbb{R}^+ \to [0,1]$ such that $\psi(0) = 1$, $\psi(+\infty) = 0$, $(-1)^k \psi^{(k)}(x) \geq 0$, $k = 0, 1, \ldots, n-2$, and $(-1)^{n-2}\psi^{(n-2)}(x)$ is decreasing and convex. Then,

$$C_\psi(u_1, u_2, \cdots, u_n) = \psi\left(\sum_{i=1}^{n} \psi^{-1}(u_i)\right), \quad u_i \in [0,1]$$

is called Archimedean copula. If we take $\psi_\theta^{-1}(t) = t^\theta - 1$ as a generator, then we get the Clayton copula (cf. [10])

$$C_\theta(u_1, u_2, \cdots, u_n) = \left(\sum_{i=1}^{n} u_i^{-\theta} - n + 1\right)^{-1/\theta}, \quad u_i \in [0,1] \text{ and } \theta \in (0, \infty). \tag{1}$$

In fact, Archimedean copula includes Clayton family, Ali-Mikhail-Had (AMH) family and Gumbel family, etc.

As pointed in Durante and Sempi [18], FGM family defined as

$$C_\theta(u_1, u_2, \ldots, u_n) = \prod_{i=1}^{n} u_i + \theta \prod_{i=1}^{n} u_i(1 - u_i), \quad u_i \in [0,1] \text{ and } \theta \in [-1,1], \tag{2}$$

which must be Schur-concave but not be Archimedean copula, and every Archimedean copula must be Schur-concave, but the Schur-concave copula is not necessarily an Archimedean copula. Hence, the Schur-concave copula characterize more extensive dependency to some extent. For detailed discussions on copulas and its applications, one may refer to Durante and Sempi [18], Nelsen [39] and Durante and Papini [17].

Next, we recall the definition of the concordance order.

Definition 5. *Suppose that C_1 and C_2 are two copulas. C_1 is said to be smaller than C_2 in the concordance order (denoted by $C_1 \prec C_2$) if $C_1(u,v) \leq C_2(u,v)$, for all $(u,v) \in [0,1]^2$.*

The concordance order is referred to as the positive quadrant dependent order. For an elaborate discussion of the positive quadrant dependent and concordance order, one may refer to Lehmann [31], Dhaene and Goovaerts [13,14] , and Nelsen [39].

The following lemmas play a vital role in establishing the main results.

Lemma 2. *If $C(u,v)$ is a Schur-concave function, then $C(1-\alpha^x, 1-\alpha^y)$ is also a Schur-concave function in (x,y), for any $x, y > 0$ and $\alpha \in (0,1)$.*

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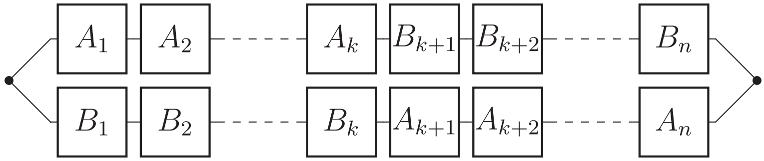

Figure 1. *Two-parallel–series system under allocation policy a_k.*

Proof. Suppose $u = 1 - \alpha^x$, $v = 1 - \alpha^y$, without loss of generality, assume $x \geq y$, which implies $u \geq v$. From the Schur-concavity of $C(u, v)$, we have

$$\frac{\partial C(u,v)}{\partial u} - \frac{\partial C(u,v)}{\partial v} \leq 0.$$

Note that $\partial C(u, v)/\partial u$ or $\partial C(u, v)/\partial v$ is non-negative, it holds that,

$$\begin{aligned}\frac{\partial C(u,v)}{\partial x} - \frac{\partial C(u,v)}{\partial y} &= \frac{\partial C(u,v)}{\partial u}(-\alpha^x \ln \alpha) - \frac{\partial C(u,v)}{\partial v}(-\alpha^y \ln \alpha) \\ &= \left[\frac{\partial C(u,v)}{\partial v}(\alpha^y - \alpha^x) + \left(\frac{\partial C(u,v)}{\partial v} - \frac{\partial C(u,v)}{\partial u}\right)\alpha^x\right] \ln \alpha \leq 0.\end{aligned}$$

And thus

$$(x - y)\left(\frac{\partial C(u,v)}{\partial x} - \frac{\partial C(u,v)}{\partial y}\right) \leq 0.$$

It follows from Lemma 1 that $C(1 - \alpha^x, 1 - \alpha^y)$ is also Schur-concave. The proof is completed. □

Lemma 3 ([30])**.** *The function $R(p, \delta, \lambda) = (p^\delta + p^{1-\delta} - p)/(p^\lambda + p^{1-\lambda} - p)$ is decreasing in $p \in (0, 1]$, for all $0 < \delta \leq \lambda \leq \frac{1}{2}$.*

Combining Proposition C.1 in Part I in [37] p. 92 with Lemma 1 in Laniado and Lillo [30], we obtain immediately the following Lemma 4.

Lemma 4. *The function $\psi(x_1, x_2) = \sum_{i=1}^{2} x_i p^{x_i}/(1 - p^{x_i})$ is Schur-convex in $(x_1, x_2) \in \mathbb{R}^{+2}$, for any $p \in [0, 1)$.*

Throughout this article, all random variables are assumed to be positive and absolutely continuous, the terms increasing and decreasing are used instead of monotone nondecreasing and monotone nonincreasing, respectively.

3. Optimal allocation of the two-parallel–series system

In this section, we investigate the allocation policies of components in the two-parallel–series system consisting of two dependent or independent subsystems with respective independent components $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$. Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ and $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n)$ be the lifetimes vectors of components $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$, respectively. Denote a_k $(k = 1, 2, \ldots, n)$ the allocation policy with consecutive allocation of components A_1 to A_k and components B_{k+1} to B_n to the up-series system. The resulting two-parallel–series system under allocation policy a_k is illustrated in Figure 1. It is obvious that the possible different allocations policies of the two-parallel–series system are n types. A natural question is, what is the optimal strategy among these allocation policies in order to improve the reliability of the system? We will explore the optimal value of allocation policies for the two-parallel–series system.

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Denote the lifetimes of the two-parallel–series, up-series and down-series systems under allocation policy a_k by S_k, S_{uk} and S_{dk}, $k = 1, 2, \ldots, n$, respectively. Then,

$$\begin{aligned} S_k &= \max\{\min(X_1, \ldots, X_k, Y_{k+1}, \ldots, Y_n), \min(Y_1, \ldots, Y_k, X_{k+1}, \ldots, X_n)\} \\ &= \max\{S_{uk}, S_{dk}\}, \end{aligned}$$

where $S_{uk} = \min(X_1, \ldots, X_k, Y_{k+1}, \ldots, Y_n)$ and $S_{dk} = \min(Y_1, \ldots, Y_k, X_{k+1}, \ldots, X_n)$. We assume that for each k, the system has different copula C_k, hence, the reliability function of the two-parallel–series system can be expressed as

$$\bar{H}_S(k; x) = \mathbb{P}(S_k > x) = 1 - \mathbb{P}(S_{uk} \le x, S_{dk} \le x) = 1 - C_k(F_{S_{u_k}}(x), F_{S_{d_k}}(x)),$$

where C_k is the joint copula of S_{uk} and S_{dk}, $F_{S_{uk}}(x)$ and $F_{S_{dk}}(x)$ are the distribution functions of S_{uk} and S_{dk}, respectively. When $\boldsymbol{X} \sim \mathrm{PH}(\boldsymbol{\alpha}, \bar{F})$ and $\boldsymbol{Y} \sim \mathrm{PH}(\boldsymbol{\beta}, \bar{F})$, we have

$$F_{S_{uk}}(x) = 1 - [\bar{F}(x)]^{\sum_{i=1}^{k} \alpha_i + \sum_{i=k+1}^{n} \beta_i} \quad \text{and} \quad F_{S_{dk}}(x) = 1 - [\bar{F}(x)]^{\sum_{i=1}^{k} \beta_i + \sum_{i=k+1}^{n} \alpha_i}.$$

3.1. Dependence case

In this subsection, we consider the case of two subsystems are dependent. Denote the reliability function of the resulting two-parallel–series system under allocation policy a_k by $\bar{H}_S(k; x)$, $k = 1, 2, \ldots, n$. Then,

$$\bar{H}_S(k; x) = 1 - C_k(1 - \bar{F}^{\sum_{i=1}^{k} \alpha_i + \sum_{i=k+1}^{n} \beta_i}, 1 - \bar{F}^{\sum_{i=1}^{k} \beta_i + \sum_{i=k+1}^{n} \alpha_i}).$$

Theorem 1 presents some sufficient conditions for the lifetime of the two-parallel–series system under different allocation policies in terms of the usual stochastic order.

Theorem 1. *For $\boldsymbol{X} \sim PH(\boldsymbol{\alpha}, \bar{F})$ and $\boldsymbol{Y} \sim PH(\boldsymbol{\beta}, \bar{F})$, suppose that C_{k_1} or C_{k_2} is Schur-concave and $C_{k_1} \prec C_{k_2}$, $k_1 < k_2$. If $\boldsymbol{\alpha} \overset{m}{\succeq} \boldsymbol{\beta}$ and $\sum_{i=k_1+1}^{k_2} \alpha_{i:n} \ge \sum_{i=k_1+1}^{k_2} \beta_{i:n}$, then $S_{k_1} \ge_{st} S_{k_2}$.*

Proof. The survival function of the resulting two-parallel–series system under allocation policy a_{k_j} can be written as

$$\begin{aligned} \bar{H}_S(k_j; x) &= 1 - C_{k_j}(1 - \bar{F}^{\sum_{i=1}^{k_j} \alpha_{i:n} + \sum_{i=k_j+1}^{n} \beta_{i:n}}, 1 - \bar{F}^{\sum_{i=1}^{k_j} \beta_{i:n} + \sum_{i=k_j+1}^{n} \alpha_{i:n}}) \\ &= 1 - C_{k_j}(1 - \bar{F}^{s_{j1}}, 1 - \bar{F}^{s_{j2}}), \end{aligned}$$

where $s_{j1} = \sum_{i=1}^{k_j} \alpha_{i:n} + \sum_{i=k_j+1}^{n} \beta_{i:n}$ and $s_{j2} = \sum_{i=1}^{k_j} \beta_{i:n} + \sum_{i=k_j+1}^{n} \alpha_{i:n}$, $j = 1, 2$. From Lemma 2, the Schur-concavity of C_{k_1} implies that $C_{k_j}(1 - \bar{F}^{s_{j1}}, 1 - \bar{F}^{s_{j2}})$ is also Schur-concave. According to $\boldsymbol{\alpha} \overset{m}{\succeq} \boldsymbol{\beta}$, it holds that

$$\sum_{i=1}^{k_j} \alpha_{i:n} \le \sum_{i=1}^{k_j} \beta_{i:n}, \quad \sum_{i=k_j+1}^{n} \beta_{i:n} \le \sum_{i=k_j+1}^{n} \alpha_{i:n}, \quad j = 1, 2,$$

thus,

$$\sum_{i=1}^{k_j} \alpha_{i:n} + \sum_{i=k_j+1}^{n} \beta_{i:n} \le \sum_{i=1}^{k_j} \beta_{i:n} + \sum_{i=k_j+1}^{n} \alpha_{i:n}, \quad j = 1, 2. \tag{3}$$

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Based on conditions $k_1 < k_2$ and $\sum_{i=k_1+1}^{k_2} \alpha_{i:n} \geq \sum_{i=k_1+1}^{k_2} \beta_{i:n}$, we have

$$\sum_{i=1}^{k_1} \alpha_{i:n} + \sum_{i=k_1+1}^{k_2} \beta_{i:n} + \sum_{i=k_2+1}^{n} \beta_{i:n} \leq \sum_{i=1}^{k_1} \alpha_{i:n} + \sum_{i=k_1+1}^{k_2} \alpha_{i:n} + \sum_{i=k_2+1}^{n} \beta_{i:n},$$

which is equivalent to

$$\sum_{i=1}^{k_1} \alpha_{i:n} + \sum_{i=k_1+1}^{n} \beta_{i:n} \leq \sum_{i=1}^{k_2} \alpha_{i:n} + \sum_{i=k_2+1}^{n} \beta_{i:n}. \tag{4}$$

According to (3) and (4), we obtain the following majorization order

$$\begin{aligned} &\left(\sum_{i=1}^{k_1} \alpha_{i:n} + \sum_{i=k_1+1}^{n} \beta_{i:n}, \sum_{i=1}^{k_1} \beta_{i:n} + \sum_{i=k_1+1}^{n} \alpha_{i:n}\right) \\ &\overset{\text{m}}{\geq} \left(\sum_{i=1}^{k_2} \alpha_{i:n} + \sum_{i=k_2+1}^{n} \beta_{i:n}, \sum_{i=1}^{k_2} \beta_{i:n} + \sum_{i=k_2+1}^{n} \alpha_{i:n}\right), \end{aligned} \tag{5}$$

hence, we have

$$(s_{11}, s_{12}) \overset{\text{m}}{\geq} (s_{21}, s_{22}),$$

combining Schur-concavity of $C_{k_j}(1 - \bar{F}^{s_{j1}}, 1 - \bar{F}^{s_{j2}})$ with the majorization order, it holds that

$$\begin{aligned} \bar{H}_S(k_1; x) &= 1 - C_{k_1}(1 - \bar{F}^{s_{11}}, 1 - \bar{F}^{s_{12}}) \\ &\geq 1 - C_{k_1}(1 - \bar{F}^{s_{21}}, 1 - \bar{F}^{s_{22}}) \\ &\geq 1 - C_{k_2}(1 - \bar{F}^{s_{21}}, 1 - \bar{F}^{s_{22}}) = \bar{H}_S(k_2; x), \end{aligned}$$

the last inequality follows from the assumption $C_{k_1} \prec C_{k_2}$. The proof is completed. □

Remark 1. *Theorem 1 shows that the reliability of the two-parallel–series system can be improved when the components are allocated by unbalancing as much as possible in two subsystems in the sense of the usual stochastic order, that means fewer components from group A are in the first series subsystem, meanwhile, more components from group A are in the second series subsystem. Note that Theorem 1 generalizes Theorem 2.1 of El-Neweihi et al. [19] to the case of dependent subsystems.*

The following Example 1(i) illustrates the result of Theorem 1, and Example 1(ii) shows that the condition $\sum_{i=k_1+1}^{k_2} \alpha_{i:n} \geq \sum_{i=k_1+1}^{k_2} \beta_{i:n}$ can not be dropped.

Example 1. *Under the assumptions of Theorem 1, let $\bar{F}(x) = e^{-\lambda x}$, suppose subsystems S_1 and S_2 have FGM copula as (2)*

$$C_{\theta_j}(u_1, u_2) = u_1 u_2 (1 + \theta_j (1 - u_1)(1 - u_2)), \quad u_1, u_2 \in [0, 1] \text{ and } \theta_j \in [-1, 1],$$

$j = 1, 2, 3, 4$. Assume $\lambda = 0.008, \theta_1 = -0.9, \theta_2 = -0.3, \theta_3 = 0.2$, and $\theta_4 = 0.9$. For convenience, let $k_1 = 1, k_2 = 2, k_3 = 3, k_4 = 4$ in all examples of this paper. To plot the entire survival curves of S_{k_1} and S_{k_2} on $[0, \infty)$, we perform the transformation $e^{-x} : [0, \infty) \to (0, 1]$. Then, it is obvious that $S_{k_1} \leq_{st} S_{k_2}$ is equivalent to $e^{-S_{k_1}} \geq_{st} e^{-S_{k_2}}$.

(i) Seting $\boldsymbol{\alpha} = (1, 8, 11, 23) \overset{m}{\geq} (6, 7, 10, 20) = \boldsymbol{\beta}$, from Figure 2(a) we can see that the difference function $\bar{H}_S(k_i; x) - \bar{H}_S(k_{i+1}; x)$ is always non-negative, $i = 1, 2, 3$, for all $x = -\ln u$ and $u \in (0, 1]$. Hence, $\bar{H}_S(k_1; x) \geq \bar{H}_S(k_2; x) \geq \bar{H}_S(k_3; x) \geq \bar{H}_S(k_4; x)$, which is in accordance with Theorem 1.

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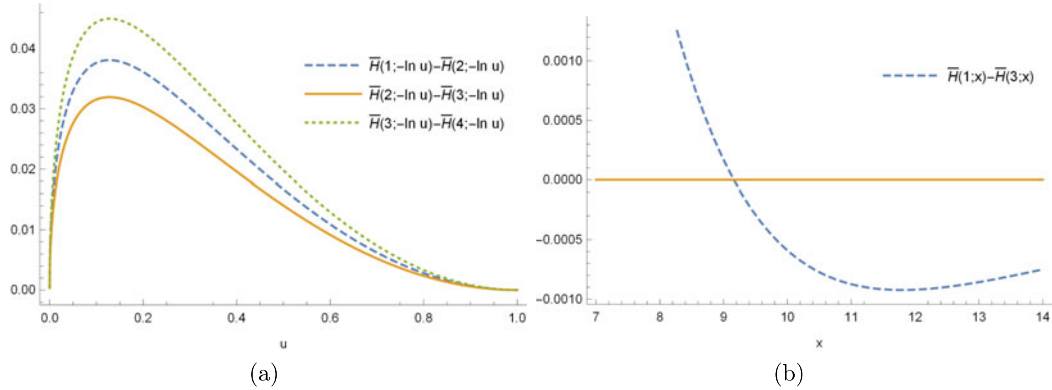

Figure 2. *Plots of the difference functions $\bar{H}_S(k_i; -\ln u) - \bar{H}_S(k_{i+1}; -\ln u)$, $i = 1, 2, 3$, for all $u \in (0, 1]$.*

(ii) *Taking $\boldsymbol{\alpha} = (1, 8, 11, 23) \overset{m}{\succeq} (2, 9, 12, 20) = \boldsymbol{\beta}$ with remaining conditions unchanged, notice that $\sum_{i=2}^{3} \alpha_{i:n} < \sum_{i=2}^{3} \beta_{i:n}$. Figure 2(b) displays the curves of $\bar{H}_U(k_1; x) - \bar{H}_U(k_3; x)$, which is crossing with the line $y = 0$, this shows that the usual stochastic order does not hold between S_{k_1} and S_{k_2}. Therefore, the condition $\sum_{i=k_1+1}^{k_2} \alpha_{i:n} \geq \sum_{i=k_1+1}^{k_2} \beta_{i:n}$ can not be dropped.*

Combining Theorem 1 with the transitivity of the usual stochastic order, we have the following corollary.

Corollary 1. *For $\boldsymbol{X} \sim PH(\boldsymbol{\alpha}, \bar{F})$ and $\boldsymbol{Y} \sim PH(\boldsymbol{\beta}, \bar{F})$, suppose that $n-1$ copulas of C_j $(j = 1, 2, \ldots, n)$ are Schur-concave, and $C_k \leq C_{k+1} (k = 1, 2, \ldots, n-1)$. If $\boldsymbol{\alpha} \overset{m}{\succeq} \boldsymbol{\beta}$ and $\alpha_{i:n} \geq \beta_{i:n}$, $i = 2, 3, \ldots, n$, then*

$$S_1 \geq_{st} S_2 \geq_{st} \cdots \geq_{st} S_n.$$

Corollary 1 shows that a_1 is the optimal allocation policy and a_n is the worst allocation policy within all possible allocations.

3.2. Independence case

In this part, we consider that the case of two subsystems are independent. The reliability function of the resulting two-parallel–series system can be written as

$$\bar{H}_S(k; x) = 1 - (1 - \bar{F}^{\sum_{i=1}^{k} \alpha_i + \sum_{i=k+1}^{n} \beta_i})(1 - \bar{F}^{\sum_{i=1}^{k} \beta_i + \sum_{i=k+1}^{n} \alpha_i}).$$

Theorem 2 provides some sufficient conditions to compare the lifetime of two-parallel–series system with respect to the hazard rate and the reversed hazard rate orders.

Theorem 2. *For $\boldsymbol{X} \sim PH(\boldsymbol{\alpha}, \bar{F})$ and $\boldsymbol{Y} \sim PH(\boldsymbol{\beta}, \bar{F})$. If $\boldsymbol{\alpha} \overset{m}{\succeq} \boldsymbol{\beta}$ and $\sum_{i=k_1+1}^{k_2} \alpha_{i:n} \geq \sum_{i=k_1+1}^{k_2} \beta_{i:n}$, then $k_1 < k_2$ implies*

(i) *$S_{k_1} \geq_{hr} S_{k_2}$ and*
(ii) *$S_{k_1} \geq_{rh} S_{k_2}$.*

Proof. (i) Note that the survival function of S_{k_j} $(j = 1, 2)$ can be expressed by

$$\bar{H}_S(k_j; x) = \bar{F}^{\sum_{i=1}^{k_j} \alpha_{i:n} + \sum_{i=k_j+1}^{n} \beta_{i:n}} + \bar{F}^{\sum_{i=1}^{k_j} \beta_{i:n} + \sum_{i=k_j+1}^{n} \alpha_{i:n}} - \bar{F}^{\sum_{i=1}^{n} \alpha_{i:n} + \sum_{i=1}^{n} \beta_{i:n}}.$$

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To reach the desired result, it suffices to show that

$$\begin{aligned} R_S(k_1,k_2;x) &= \frac{\bar{F}^{\sum_{i=1}^{k_1}\alpha_{i:n}+\sum_{i=k_1+1}^{n}\beta_{i:n}} + \bar{F}^{\sum_{i=1}^{k_1}\beta_{i:n}+\sum_{i=k_1+1}^{n}\alpha_{i:n}} - \bar{F}^{\sum_{i=1}^{n}\alpha_{i:n}+\sum_{i=1}^{n}\beta_{i:n}}}{\bar{F}^{\sum_{i=1}^{k_2}\alpha_{i:n}+\sum_{i=k_2+1}^{n}\beta_{i:n}} + \bar{F}^{\sum_{i=1}^{k_2}\beta_{i:n}+\sum_{i=k_2+1}^{n}\alpha_{i:n}} - \bar{F}^{\sum_{i=1}^{n}\alpha_{i:n}+\sum_{i=1}^{n}\beta_{i:n}}} \\ &= \frac{p(x)^{t_\delta}+p(x)^{1-t_\delta}-p(x)}{p(x)^{t_\lambda}+p(x)^{1-t_\lambda}-p(x)} \end{aligned}$$

is increasing in $x \in \mathbb{R}^+$, where

$$p(x) = \bar{F}^{\sum_{i=1}^{n}\alpha_{i:n}+\sum_{i=1}^{n}\beta_{i:n}}, \quad t_\delta = \frac{\sum_{i=1}^{k_1}\alpha_{i:n}+\sum_{i=k_1+1}^{n}\beta_{i:n}}{\sum_{i=1}^{n}\alpha_{i:n}+\sum_{i=1}^{n}\beta_{i:n}}$$

and

$$t_\lambda = \frac{\sum_{i=1}^{k_2}\alpha_{i:n}+\sum_{i=k_2+1}^{n}\beta_{i:n}}{\sum_{i=1}^{n}\alpha_{i:n}+\sum_{i=1}^{n}\beta_{i:n}}.$$

Note that (3) and (4) imply $t_\delta \le t_\lambda \le 1/2$, it holds from Lemma 3 that $R_S(k_1,k_2;x)$ is decreasing in $p(x)$, and consider that $p(x)$ is decreasing in $x \in \mathbb{R}^+$, hence, $R_S(k_1,k_2;x)$ is increasing in $x \in \mathbb{R}^+$, which completes the proof of (i) .

(ii) Denote the distribution function and the reversed hazard rate function of S under allocation policy a_{k_j} by $H_S(k_j;x)$ and $\tilde{r}_S(k_j;x)$, respectively. Then,

$$H_S(k_j;x) = (1-\bar{F}^{\sum_{i=1}^{k_j}\alpha_{i:n}+\sum_{i=k_j+1}^{n}\beta_{i:n}})(1-\bar{F}^{\sum_{i=1}^{k_j}\beta_{i:n}+\sum_{i=k_j+1}^{n}\alpha_{i:n}})$$

and

$$\begin{aligned} \tilde{r}_S(k_j;x) &= \frac{\mathrm{d}}{\mathrm{d}x}[\ln(1-\bar{F}^{\sum_{i=1}^{k_j}\alpha_{i:n}+\sum_{i=k_j+1}^{n}\beta_{i:n}}) + \ln(1-\bar{F}^{\sum_{i=1}^{k_j}\beta_{i:n}+\sum_{i=k_j+1}^{n}\alpha_{i:n}})] \\ &= \frac{f(x)}{\bar{F}(x)}\left(\frac{(\sum_{i=1}^{k_j}\alpha_{i:n}+\sum_{i=k_j+1}^{n}\beta_{i:n})\bar{F}^{\sum_{i=1}^{k_j}\alpha_{i:n}+\sum_{i=k_j+1}^{n}\beta_{i:n}}}{1-\bar{F}^{\sum_{i=1}^{k_j}\alpha_{i:n}+\sum_{i=k_j+1}^{n}\beta_{i:n}}}\right) \\ &\quad + \frac{f(x)}{\bar{F}(x)}\left(\frac{(\sum_{i=1}^{k_j}\beta_{i:n}+\sum_{i=k_j+1}^{n}\alpha_{i:n})\bar{F}^{\sum_{i=1}^{k_j}\beta_{i:n}+\sum_{i=k_j+1}^{n}\alpha_{i:n}}}{1-\bar{F}^{\sum_{i=1}^{k_j}\beta_{i:n}+\sum_{i=k_j+1}^{n}\alpha_{i:n}}}\right) \\ &= r(x)\sum_{m=1}^{2}\frac{t_{jm}\bar{F}^{t_{jm}}}{1-\bar{F}^{t_{jm}}}, \end{aligned}$$

where $t_{j1} = \sum_{i=1}^{k_j}\alpha_{i:n}+\sum_{i=k_j+1}^{n}\beta_{i:n}$ and $t_{j2} = \sum_{i=1}^{k_j}\beta_{i:n}+\sum_{i=k_j+1}^{n}\alpha_{i:n},\ j=1,2.$

For any fixed $x \in \mathbb{R}^+$, it follows from Lemma 4 that $\sum_{m=1}^{2} t_{jm}\bar{F}^{t_{jm}}/(1-\bar{F}^{t_{jm}})$ is Schur-convex in (t_{j1},t_{j2}), and thus $\tilde{r}_S(k_j;x)$ is also Schur-convex. Note that (5) implies $(t_{11},t_{12}) \overset{\mathrm{m}}{\succeq} (t_{21},t_{22})$, we have

$$\tilde{r}_S(k_1;x) = r(x)\sum_{m=1}^{2}\frac{t_{1m}\bar{F}^{t_{1m}}}{1-\bar{F}^{t_{1m}}} \ge r(x)\sum_{m=1}^{2}\frac{t_{2m}\bar{F}^{t_{2m}}}{1-\bar{F}^{t_{2m}}} = \tilde{r}_S(k_2;x),$$

for all $x \in \mathbb{R}^+$. Then, the theorem is proved. □

Remark 2. *Theorem 2 indicates that the performance of the two-parallel–series system can be improved in the sense of the (reversed) hazard rate order when two-series subsystems are as unbalancing as*

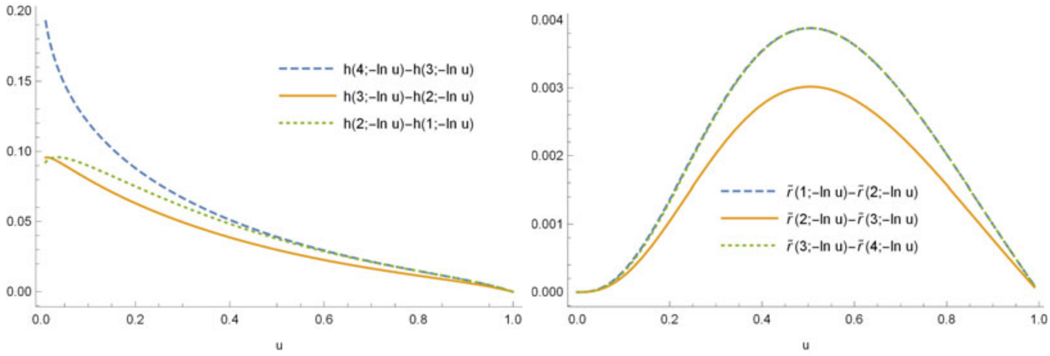

Figure 3. *Plots of the difference functions* $h_S(k_{i+1};x) - h_S(k_i;x)$ *and* $\tilde{r}_S(k_i;x) - \tilde{r}_S(k_{i+1};x)$, $i = 1,2,3,$ *for all* $x = -\ln u$ *and* $u \in (0,1]$.

possible. Theorems IV.1 and III.1 of Laniado and Lillo [30] are the special cases of Theorem 2 under some certain conditions.

The next example illustrates the theoretical result of Theorem 2.

Example 2. *Under the setups of Theorem 2, assume* $\bar{F}(x) = e^{-\lambda x}$, *set* $\boldsymbol{\alpha} = (0.1, 0.8, 1.1, 2.3) \overset{m}{\geq} (0.6, 0.7, 1, 2) = \boldsymbol{\beta}$, $\lambda = 0.8$. *Figure 3(a) and 3(b) plots the difference functions* $h_S(k_{i+1};x) - h_S(k_i;x)$ *and* $\tilde{r}_S(k_i;x) - \tilde{r}_S(k_{i+1};x)$, *respectively,* $i = 1,2,3,$ *for all* $x = -\ln u$ *and* $u \in (0,1]$. *They are always non-negative, which demonstrates the theoretical results of Theorem 2.*

The following corollary can be obtained from Theorem 2 and the transitivity of the hazard rate order and the reversed hazard rate order.

Corollary 2. *For* $\boldsymbol{X} \sim PH(\boldsymbol{\alpha}, \bar{F})$ *and* $\boldsymbol{Y} \sim PH(\boldsymbol{\beta}, \bar{F})$. *If* $\boldsymbol{\alpha} \overset{m}{\geq} \boldsymbol{\beta}$ *and* $\alpha_{i:n} \geq \beta_{i:n}$, *i = 2,3,. . . ,n, then*

(i) $S_1 \geq_{hr} S_2 \geq_{hr} \cdots \geq_{hr} S_n$ *and*
(ii) $S_1 \geq_{rh} S_2 \geq_{rh} \cdots \geq_{rh} S_n$.

In accordance with Corollary 2, a_1 is the optimal allocation policy and a_n is the worst allocation policy within all possible allocations.

4. Optimal allocation of the two-series–parallel system

In this section, we are interested in the optimal allocation policy in the two-series–parallel system. Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ and $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n)$ be the lifetime vectors of components $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$, respectively. Denote the allocation policy with consecutive allocation of components A_1 to A_k and components B_{k+1} to B_n being allocated to the left-parallel system by b_k, $k = 1,2,\ldots,n$. The resulting two-series–parallel system under allocation policy b_k is illustrated in Figure 4.

Denote the lifetimes of the two-series–parallel, left-parallel and right-parallel systems under allocation b_k by U_k, U_{lk} and U_{rk}, $k = 1,2,\ldots,n$, respectively. Then, the reliability function of the two-series–parallel system can be expressed as

$$\bar{H}_U(k;x) = \mathbb{P}(U_k > x) = \mathbb{P}(U_{lk} > x, U_{rk} > x) = \hat{C}_k(\bar{F}_{U_{lk}}(x), \bar{F}_{U_{rk}}(x)),$$

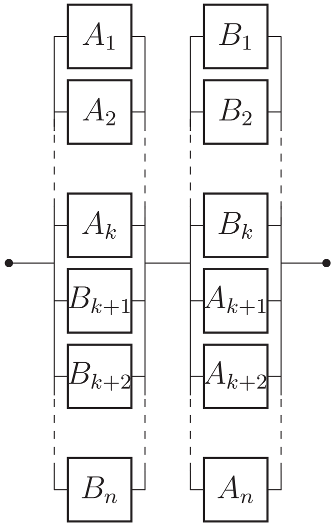

Figure 4. *Two-series–parallel system under allocation policy b_k.*

where $\hat{C}_k$ is the joint survival copula of U_{lk} and U_{rk}, $\bar{F}_{U_{lk}}(x)$ and $\bar{F}_{U_{rk}}(x)$ are the survival functions of U_{lk} and U_{rk}, respectively. When $\boldsymbol{X} \sim \text{PRH}(\boldsymbol{\alpha}, F)$ and $\boldsymbol{Y} \sim \text{PRH}(\boldsymbol{\beta}, F)$, we have

$$\bar{F}_{U_{lk}}(x) = 1 - [F(x)]^{\sum_{i=1}^{k} \alpha_i + \sum_{i=k+1}^{n} \beta_i} \quad \text{and} \quad \bar{F}_{U_{rk}}(x) = 1 - [F(x)]^{\sum_{i=1}^{k} \beta_i + \sum_{i=k+1}^{n} \alpha_i}.$$

4.1. Dependent case

In this subsection, we focus that the case of two subsystems are dependent. Denote the reliability function of the resulting two-series–parallel system under the allocation policy b_k by $\bar{H}_U(k;x)$, $k = 1, 2, \ldots, n$. Then,

$$\bar{H}_U(k;x) = \hat{C}_k(1 - F^{\sum_{i=1}^{k} \alpha_i + \sum_{i=k+1}^{n} \beta_i}, 1 - F^{\sum_{i=1}^{k} \beta_i + \sum_{i=k+1}^{n} \alpha_i}).$$

The following result establishes the usual stochastic order for the two-series–parallel system. The proof can be obtained along the same way with that of Theorem 1, and thus omitted here.

Theorem 3. *For $\boldsymbol{X} \sim PRH(\boldsymbol{\alpha}, F)$ and $\boldsymbol{Y} \sim PRH(\boldsymbol{\beta}, F)$, suppose that $\hat{C}_{k_1}$ or $\hat{C}_{k_2}$ is Schur-concave and $\hat{C}_{k_1} \prec \hat{C}_{k_2}$, $k_1 < k_2$. If $\boldsymbol{\alpha} \overset{m}{\succeq} \boldsymbol{\beta}$ and $\sum_{i=k_1+1}^{k_2} \alpha_{i:n} \geq \sum_{i=k_1+1}^{k_2} \beta_{i:n}$, then $U_{k_1} \leq_{st} U_{k_2}$.*

Remark 3. *Theorem 3 manifests that a two-series–parallel system is more reliable with respect to the usual stochastic order for a bigger k. It should be noted that Theorem 3 generalizes Theorem 3.1 in El-Neweihi et al. [19] to the case of the dependent subsystems.*

By the transitivity of the usual stochastic orders, the following corollary follows immediately from Theorem 3.

Corollary 3. *For $\boldsymbol{X} \sim PRH(\boldsymbol{\alpha}, F)$ and $\boldsymbol{Y} \sim PRH(\boldsymbol{\beta}, F)$, suppose that $n-1$ copulas of $\hat{C}_j$ $(j = 1, \ldots, n)$ are Schur-concave, and $C_k \leq C_{k+1}$ $(k = 1, 2, \ldots, n-1)$. If $\boldsymbol{\alpha} \overset{m}{\succeq} \boldsymbol{\beta}$ and $\alpha_{i:n} \geq \beta_{i:n}$, i = 2,3,. . . ,n, then*

$$U_1 \leq_{st} U_2 \leq_{st} \cdots \leq_{st} U_n.$$

In accordance with Corollary 3, b_1 is the worst allocation policy and b_n is the optimal allocation policy within all possible allocations.

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4.2. Independence case

When the lifetimes of two subsystems are independent, the reliability function of the two-series–parallel system can be expressed as

$$\bar{H}_U(k;x) = (1 - F^{\sum_{i=1}^{k}\alpha_i + \sum_{i=k+1}^{n}\beta_i})(1 - F^{\sum_{i=1}^{k}\beta_i + \sum_{i=k+1}^{n}\alpha_i}).$$

Now, we develop some sufficient conditions for the reversed hazard rate and the hazard rate orders of the two-series–parallel system.

Theorem 4. *For $\boldsymbol{X} \sim PRH(\boldsymbol{\alpha}, F)$ and $\boldsymbol{Y} \sim PRH(\boldsymbol{\beta}, F)$. If $\boldsymbol{\alpha} \overset{m}{\geq} \boldsymbol{\beta}$ and $\sum_{i=k_1+1}^{k_2}\alpha_{i:n} \geq \sum_{i=k_1+1}^{k_2}\beta_{i:n}$, then $k_1 < k_2$ implies*

(i) $U_{k_1} \leq_{rh} U_{k_2}$ and
(ii) $U_{k_1} \leq_{hr} U_{k_2}$.

Proof. (i) Note that the distribution functions of U_{lk} and U_{rk} are given by

$$F_{U_l}(k;x) = [F(x)]^{\sum_{i=1}^{k}\alpha_{i:n} + \sum_{i=k+1}^{n}\beta_{i:n}} \quad \text{and} \quad F_{U_r}(k;x) = [F(x)]^{\sum_{i=1}^{k}\beta_{i:n} + \sum_{i=k+1}^{n}\alpha_{i:n}},$$

respectively. In order to obtain the desired result, let $\tilde{U}_{lk} = 1/U_{lk}$ and $\tilde{U}_{rk} = 1/U_{rk}$, then

$$\begin{aligned}\bar{F}_{\tilde{U}_l}(k;x) &= \mathbb{P}(1/U_{lk} > x) = \mathbb{P}\left(U_{lk} < \frac{1}{x}\right)\\ &= F_{U_l}\left(k; \frac{1}{x}\right) = \left[F\left(\frac{1}{x}\right)\right]^{\sum_{i=1}^{k}\alpha_{i:n} + \sum_{i=k+1}^{n}\beta_{i:n}}\end{aligned}$$

and

$$F\left(\frac{1}{x}\right) = \mathbb{P}\left(X \leq \frac{1}{x}\right) = \mathbb{P}(1/X \geq x) = \bar{F}_0(x).$$

Hence, the survival functions of $\tilde{U}_{lk}$ and $\tilde{U}_{rk}$ can be expressed by

$$\bar{F}_{\tilde{U}_l}(k;x) = [\bar{F}_0(x)]^{\sum_{i=1}^{k}\alpha_{i:n} + \sum_{i=k+1}^{n}\beta_{i:n}}$$

and

$$\bar{F}_{\tilde{U}_r}(k;x) = [\bar{F}_0(x)]^{\sum_{i=1}^{k}\beta_{i:n} + \sum_{i=k+1}^{n}\alpha_{i:n}},$$

respectively. Obviously, $\tilde{U}_{lk}$ and $\tilde{U}_{rk}$ follow the PH model. Let $\tilde{U} = \max\{\tilde{U}_{lk}, \tilde{U}_{rk}\}$, according to (i) of Theorem 2, we have $\tilde{U}_{k_1} \geq_{\rm hr} \tilde{U}_{k_2}$, which is equivalent to

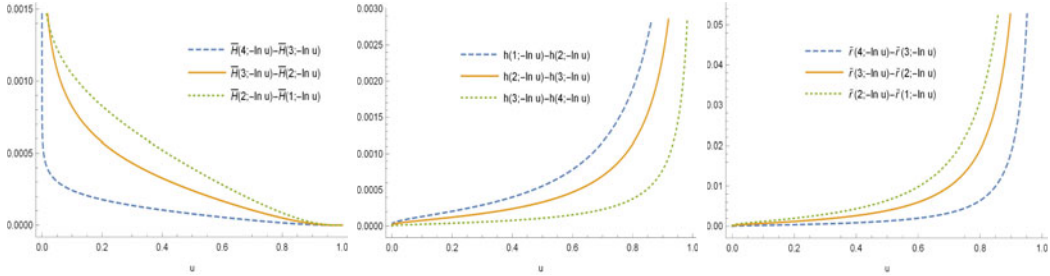
$$R_{\tilde{U}}(k_1, k_2; x) = \frac{\bar{H}_{\tilde{U}}(k_1;x)}{\bar{H}_{\tilde{U}}(k_2;x)} = \frac{H_U\left(k_1; \frac{1}{x}\right)}{H_U\left(k_2; \frac{1}{x}\right)} = R_U\left(k_1, k_2; \frac{1}{x}\right)$$

is increasing in $x \in \mathbb{R}^+$, and thus

$$R_U(k_1, k_2; x) = \frac{H_U(k_1;x)}{H_U(k_2;x)}$$

is decreasing in $x \in \mathbb{R}^+$, that is, $U_{k_1} \leq_{\rm rh} U_{k_2}$. The desired result (i) is proved.

(ii) The proof of (ii) is similar to that of (i), and thus is omitted here. □



(a) (b) (c)

Figure 5. *Plots of the difference functions* $\bar{H}_U(k_{i+1};x) - \bar{H}_U(k_i;x)$, $h_U(k_i;x) - h_U(k_{i+1};x)$ *and* $\tilde{r}_U(k_{i+1};x) - \tilde{r}_U(k_i;x)$, $i = 1, 2, 3$, *for all* $x = -\ln u$ *and* $u \in (0, 1]$.

Remark 4. *Theorem 4 indicates that the two-series–parallel system is more unreliable in terms of the reversed hazard rate order and the hazard rate order when two subsystems are as unbalancing as possible. It should be pointed out that Theorem 4 extends Theorem VI.1 of Laniado and Lillo [30] to the case of the heterogeneous components.*

Example 3. *Under the assumptions of Theorems 3 and 4, let* $F(x)$ *be a Fréchet distribution, that is,* $F(x) = e^{-(x/\lambda)^{-\gamma}}$. *Set* $\boldsymbol{\alpha} = (0.1, 0.9, 1.3, 1.7) \overset{m}{\geq} (0.4, 0.8, 1.2, 1.6) = \boldsymbol{\beta}$.

(i) Assume that subsystems U_1 *and* U_2 *have the Clayton survival copula as (1)*

$$\hat{C}_{\theta_j}(u_1, u_2) = (u_1^{-\theta_j} + u_2^{-\theta_j} - 1)^{-1/\theta_j}, \quad u_1, u_2 \in [0, 1] \text{ and } \theta_j \in (0, +\infty).$$

Setting $\gamma = 0.3$, $\lambda = 2$, $\theta_1 = 0.1$, $\theta_2 = 0.2$, $\theta_3 = 0.7$, *and* $\theta_4 = 0.8$. *From Figure 5(a), we can see that all the difference functions* $\bar{H}_U(k_{i+1};x) - \bar{H}_U(k_i;x)$ *are always non-negative,* $i = 1, 2, 3$, *which justifies the validity of the result in Theorem 3.*

(ii) Taking $\gamma_1 = 0.2$, $\lambda_1 = 0.1$ *with remaining conditions unchanged. Figure 5(b) and 5(c) presents the plots of the difference functions* $h_U(k_i;x) - h_U(k_{i+1};x)$ *and* $\tilde{r}_U(k_{i+1};x) - \tilde{r}_U(k_i;x)$, *respectively, which are always non-negative,* $i = 1, 2, 3$. *Thus, the result of Theorem 4 is verified.*

Similar to Corollary 2, from Theorem 4, we have the following Corollary.

Corollary 4. *For* $\boldsymbol{X} \sim PRH(\boldsymbol{\alpha}, F)$ *and* $\boldsymbol{Y} \sim PRH(\boldsymbol{\beta}, F)$. *If* $\boldsymbol{\alpha} \overset{m}{\geq} \boldsymbol{\beta}$ *and* $\alpha_{i:n} \geq \beta_{i:n}$, $i = 2, 3, \ldots, n$, *then*

(i) $U_1 \leq_{rh} U_2 \leq_{rh} \cdots \leq_{rh} U_n$ *and*
(ii) $U_1 \leq_{hr} U_2 \leq_{hr} \cdots \leq_{hr} U_n$.

According to Corollary 4, b_1 is the worst allocation policy and b_n is the optimal allocation policy within all possible allocations.

5. Optimal allocations of the coherent system

In this section, we try to generalize the results obtained in the previous sections to parallel–series and series–parallel systems with multiple subsystems in view of minimal path sets and minimal cut sets, respectively. To provide these extensions, we need to introduce some notions of the classical work of Barlow and Proschan [1]. A system is called coherent if the system structure function τ is increasing in each component and each component is relevant. In other words, any improvement of a component cannot decrease the lifetime of the system. The minimal path sets and the minimal cut sets play a vital role in characterizing the lifetime of the coherent system. A minimal path set is a collection of

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components such that the system works if all the components in the collection work and it fails if one of them fails along with all components outside the collection. Suppose that there are l minimal path sets $P_1, P_2, \ldots, P_l$ in a coherent system, then the lifetime of the coherent system with component lifetimes $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ can be given by

$$\tau(\boldsymbol{X}) = \max_{1\le j\le l} \min_{i\in P_j} X_i, \quad 1 \le l \le n.$$

A collection of components is a cut if their failures cause the system failure, and the one such that any its subset is not a cut any more is called a minimal cut. Suppose that there are s minimal cut sets $C_1, C_2, \ldots, C_s$, then the lifetime of a coherent system can be represented as

$$\tau(\boldsymbol{X}) = \min_{1\le j\le s} \max_{i\in C_j} X_i, \quad 1 \le s \le n.$$

5.1. Viewpoint of the minimal path sets

In this subsection, we investigate the allocation policies of a parallel–series system with multiple subsystems in the viewpoint of the minimal path sets. Consider a parallel–series system having structure function τ, and minimal path sets $P_1, P_2, \ldots, P_l$. For convenience, denote the lifetime of minimal set P_j by W_j, where $W_j = \min_{i\in P_j} X_i$, $1 \le j \le l$. Suppose that there are two classes of independent components $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ with lifetimes vectors $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ and $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n)$ to be allocated, respectively. For two disjoint minimal path sets P_p and P_q $(1 \le p < q \le l)$, denote $a_{\tau k}$ the policy that allocate k components $A_i (i = 1, 2, \ldots, k)$ and $n-k$ components $B_i (i = k+1, k+2, \ldots, n)$ to minimal path sets P_p, and denote the corresponding copula of allocation policy $a_{\tau k}$ by C_k, $k = 1, 2, \ldots, n$, then the lifetime of the resulting system under allocation policy $a_{\tau k}$ can be expressed by

$$\begin{aligned} W_{\tau k} &= \max\left\{\max_{1\le m\le l, m\ne p,q}\{W_m\}, \max\{W_p, W_q\}\right\} \\ &= \max\left\{\max_{1\le m\le l, m\ne p,q}\{W_m\}, S_k\right\}, \end{aligned} \tag{6}$$

where $S_k = \max\{W_p, W_q\}$.

In fact, there are situations where two disjoint minimal paths, two crossing minimal paths and a minimal path contains or is contained by another minimal path. Hence, owing to the complexity of modeling, we are ready to develop the main results in the following context:

A1: A1: $P_1, P_2, \ldots, P_l$ are disjoint minimal path sets, that is, for $1 \le i \ne j \le l$, $P_i \bigcap P_j = \emptyset$;
A2: A2: There exist two minimal path sets P_p and P_q having the same numbers of components;
A3: A3: The component lifetime vectors in the minimal path sets P_p and P_q both follow the PH model, that is, $\boldsymbol{X} \sim \mathrm{PH}(\boldsymbol{\alpha}_{P_p}, \bar{F})$ and $\boldsymbol{Y} \sim \mathrm{PH}(\boldsymbol{\beta}_{P_q}, \bar{F})$.

Next, we present some sufficient conditions in the sense of the usual stochastic order for two disjoint minimal path sets.

Theorem 5. *Under the context of A1, A2 and A3. Suppose that C_{k_1} or C_{k_2} is Schur-concave, and $C_{k_1} \prec C_{k_2}$, $k_1 < k_2$. If $\boldsymbol{\alpha}_{P_p} \overset{m}{\succeq} \boldsymbol{\beta}_{P_q}$ and $\sum_{i=k_1+1}^{k_2} \alpha_{i:n} \ge \sum_{i=k_1+1}^{k_2} \beta_{i:n}$, then $W_{\tau k_1} \ge_{st} W_{\tau k_2}$.*

Proof. The idea of the proof is borrowed from Theorem 1 of Belzunce *et al.* [4]. According to (6), the lifetime of the parallel–series system under two different allocation policies $a_{\tau k_1}$ and $a_{\tau k_2}$ can be written as

$$W_{\tau k_j} = \max\left\{\max_{1\le m\le l, m\ne p,q}\{W_m\}, S_{k_j}\right\}, \quad j = 1, 2.$$

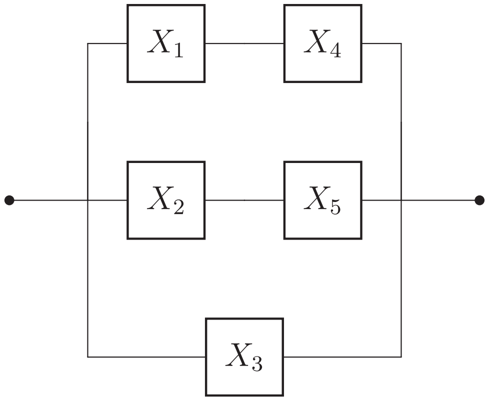

Figure 6. *Parallel–series system.*

Since the structure function τ is an increasing function, thus, according to Theorem 1 and the usual stochastic order with respect to the preservation property of increasing function, it follows that

$$\max\left\{\max_{1\le m\le l, m\neq p,q}\{W_m\}, S_{k_1}\right\} \ge_{\mathrm{st}} \max\left\{\max_{1\le m\le l, m\neq p,q}\{W_m\}, S_{k_2}\right\},$$

which implies $W_{\tau k_1} \ge_{\mathrm{st}} W_{\tau k_2}$, yielding the desired result. □

In combination with the result of Theorem 2, it is easy to obtain the following Theorem 6 in terms of the reversed hazard rate order, the proof is similar to that of Theorem 5, thus is omitted here.

Theorem 6. *Under the context of A1, A2 and A3. If* $\boldsymbol{\alpha}_{P_p} \overset{m}{\ge} \boldsymbol{\beta}_{P_q}$ *and* $\sum_{i=k_1+1}^{k_2} \alpha_{i:n} \ge \sum_{i=k_1+1}^{k_2} \beta_{i:n}$, *then* $k_1 < k_2$ *implies*

$$W_{\tau k_1} \ge_{rh} W_{\tau k_2}.$$

The following example is presented to illustrate Theorems 5 and 6.

Example 4. *Consider a parallel–series system as shown in Figure 6, its lifetime is*

$$W = \max\{\min\{X_1, X_4\}, X_3, \min\{X_2, X_5\}\}.$$

The minimal path sets of the system are $P_1 = \{1,4\}$, $P_2 = \{3\}$ *and* $P_3 = \{2,5\}$. *Let* $\bar{F}_i(x) = [\bar{F}(x)]^{a_i}$ *be the survival function of* X_i, *where* $\bar{F}(x) = e^{-\lambda x}$, $i = 1,2,3,4,5$. *Set* $\lambda = 0.02$, $a_1 = 3, a_2 = 4, a_3 = 2, a_4 = 6, a_5 = 5$. *For two minimal path sets* P_1 *and* P_3, *we can check that* $\boldsymbol{\alpha}_{P_1} = (3,6) \overset{m}{\ge} (4,5) = \boldsymbol{\beta}_{P_3}$.

(i) Suppose that P_1, P_2 *and* P_3 *has FGM copula*

$$C_{\theta_j}(u_1,u_2,u_3) = \prod_{i=1}^{3} u_i + \theta_j \prod_{i=1}^{3} u_i(1-u_i), \quad u_i \in [0,1] \text{ and } \theta_j \in [-1,1].$$

Set $\theta_1 = -0.8$, $\theta_2 = 0.9$. *As Figure 7(a) displays,* $W_{\tau k_1} \ge_{st} W_{\tau k_2}$.

(ii) As shown in Figure 7(b), $\tilde{r}_W(1; -\ln u) - \tilde{r}_W(2; -\ln u)$ *is always non-negative.*

5.2. Viewpoint of the minimal cut sets

In parallel to the previous subsection, here, we study the allocation policies of a series–parallel system in the viewpoint of the minimal cut sets. Consider a series–parallel system having structure function τ, and

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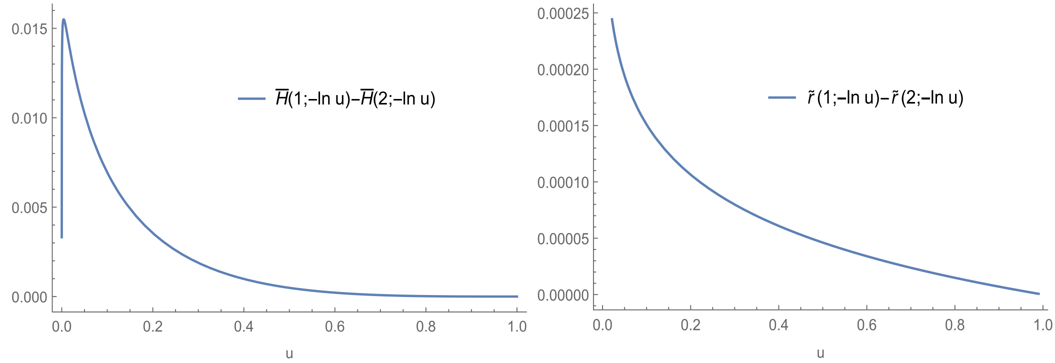

(a) (b)

Figure 7. *Plots of the difference functions (a)* $\bar{H}_W(1;-\ln u) - \bar{H}_W(2;-\ln u)$ *and (b)* $\tilde{r}_W(1;-\ln u) - \tilde{r}_W(2;-\ln u)$*, for all* $x = -\ln u$ *and* $u \in (0,1]$.

minimal cut sets $C_1, C_2, \ldots, C_s$. For converience, we denote the lifetime of minimal set C_j by Z_j, where $Z_j = \max_{i\in C_j} X_i$, $1 \le j \le s$. For two disjoint minimal cut sets C_p and C_q $(1 \le p < q \le s)$, denote the allocation policy with k components A_i $(i = 1, 2, \ldots, k)$ and $n-k$ components B_i $(i = k+1, k+2, \ldots, n)$ being allocated to minimal cut sets C_p by $b_{\tau k}$, and denote the corresponding survival copula of allocation policy $b_{\tau k}$ by $\hat{C}_k$, $k = 1, 2, \ldots, n$, then lifetime of the resulting system under allocation policy $b_{\tau k}$ can be written as

$$\begin{aligned} Z_{\tau k} &= \min\left\{\min_{1\le m\le s, m\ne p,q}\{Z_m\}, \min\{Z_p, Z_q\}\right\} \\ &= \min\left\{\min_{1\le m\le s, m\ne p,q}\{Z_m\}, U_k\right\}, \end{aligned} \tag{7}$$

where $U_k = \min\{Z_p, Z_q\}$.

In general, for two minimal cuts, they may be disjoint, crossing or a minimal cut contains or is contained another minimal cut. Hence, owing to the complexity of modeling, we investigate the allocation policies of the coherent system in the following context:

B1: B1: $C_1, C_2, \ldots, C_s$ are disjoint minimal cut sets, that is, for $1 \le i \ne j \le l$, $C_i \cap C_j = \emptyset$;
B2: B2: There exist two minimal cut sets C_p and C_q having the same numbers of components;
B3: B3: The component lifetime vectors in the minimal cut sets C_p and C_q both follow the PRH model, that is, $\boldsymbol{X} \sim \mathrm{PRH}(\boldsymbol{\alpha}_{C_p}, F)$ and $\boldsymbol{Y} \sim \mathrm{PRH}(\boldsymbol{\beta}_{C_q}, F)$.

Theorem 7. *Under the context of B1, B2 and B3. Suppose that* $\hat{C}_{k_1}$ *or* $\hat{C}_{k_2}$ *is Schur-concave and* $\hat{C}_{k_1} \prec \hat{C}_{k_2}$, $k_1 < k_2$. *If* $\boldsymbol{\alpha}_{C_p} \overset{m}{\ge} \boldsymbol{\beta}_{C_q}$ *and* $\sum_{i=k_1+1}^{k_2} \alpha_{i:n} \ge \sum_{i=k_1+1}^{k_2} \beta_{i:n}$, *then* $Z_{\tau k_1} \le_{st} Z_{\tau k_2}$.

Proof. By means of (7), the lifetimes of the series–parallel system under two different allocation policies $b_{\tau k_1}$ and $b_{\tau k_2}$ can be expressed by

$$Z_{k_j} = \min\left\{\min_{1\le m\le s, m\ne p,q}\{Z_m\}, U_{k_j}\right\}, \quad j = 1, 2.$$

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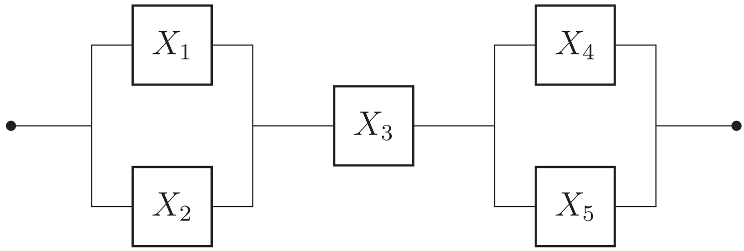

Figure 8. *Series–parallel system.*

Owing to the structure function, τ is an increasing function, hence, combining Theorem 4 with the usual stochastic order with respect to the preservation property of increasing function, it holds that

$$\min\left\{\min_{1\le m\le s, m\neq p,q}\{Z_m\}, U_{k_1}\right\} \le_{\text{st}} \min\left\{\min_{1\le m\le s, m\neq p,q}\{Z_m\}, U_{k_2}\right\},$$

which means that $Z_{\tau k_1} \le_{\text{st}} Z_{\tau k_2}$. Hence, the result is proved. □

The following theorem provides the results with respect to the hazard rate order. The proof can be obtained along the same line as in Theorem 7 and is hence omitted.

Theorem 8. *Under the context of B1, B2 and B3. If* $\boldsymbol{\alpha}_{C_p} \overset{m}{\succeq} \boldsymbol{\beta}_{C_q}$ *and* $\sum_{i=k_1+1}^{k_2} \alpha_{i:n} \ge \sum_{i=k_1+1}^{k_2} \beta_{i:n}$, *then* $k_1 < k_2$ *implies*

$$Z_{\tau k_1} \le_{hr} Z_{\tau k_2}.$$

The next example gives some illustrations of Theorems 7 and 8.

Example 5. *Consider a series–parallel system as shown in Figure 8, its lifetime is*

$$Z = \min\{\max\{X_1, X_2\}, X_3, \max\{X_4, X_5\}\}.$$

The minimal cut sets are $C_1 = \{1,2\}$, $C_2 = \{3\}$ *and* $C_3 = \{4,5\}$. *Let* $F_i(x) = [F(x)]^{b_i}$ *be the distribution function of* X_i, *where* $F(x) = e^{-(x/\lambda)^{-\gamma}}$, $i = 1,2,3,4,5$. *Set* $\lambda = 1$, $\gamma = 0.1$, $b_1 = 3$, $b_2 = 6$, $b_3 = 2$, $b_4 = 4$, $b_5 = 5$. *For two minimal cut sets* C_1 *and* C_3, *it is easily checked that* $\boldsymbol{\alpha}_{C_1} = (3,6) \overset{m}{\succeq} (4,5) = \boldsymbol{\beta}_{C_3}$.

(i) Suppose that C_1, C_2 *and* C_3 *have Clayton copula*

$$\hat{C}_{\theta_j}(u_1,u_2,u_3) = \left(\sum_{i=1}^{3} u_i^{-\theta_j} - 2\right)^{-1/\theta_j}, \quad u_i \in [0,1],\ \theta_j \in (0,\infty),\ j = 1,2.$$

Set $\theta_1 = 6$, $\theta_2 = 8$. *As displayed in Figure 9(a),* $\bar{H}_Z(2;x) - \bar{H}_Z(1;x)$ *is always non-negative.*

(ii) As Figure 9(b) shows, $h_Z(1;x) - h_Z(2;x)$ *is always non-negative.*

6. Conclusion

In this paper, we developed the optimal allocation policy for two-parallel–series system and two-series–parallel system consisting of dependent subsystems with independent and heterogeneous components with respect to the usual stochastic order, the hazard rate order and the reversed hazard rate order under some certain conditions. The obtained results indicate that the reliability of the two-parallel–series system will be increased by unbalancing the components as much as possible, and the performance of the two-series–parallel system will be improved when both subsystems are allocated to

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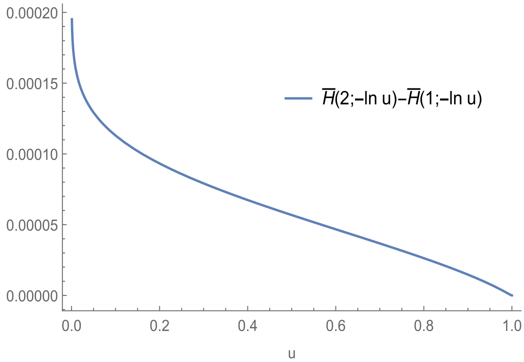
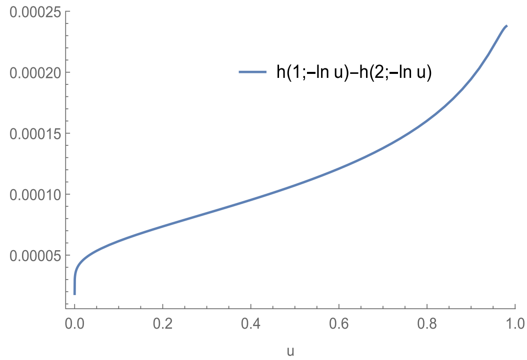


(a)



(b)

Figure 9. *Plots of the difference functions (a)* $\bar{H}_Z(2;x) - \bar{H}_Z(1;x)$ *and (b)* $h_Z(1;x) - h_Z(2;x)$*, for all* $x = -\ln u$ *and* $u \in (0, 1]$.

be more similar. Finally, we popularize the corresponding stochastic comparisons to parallel–series and series–parallel systems with multiple subsystems in the viewpoint of the minimal path and the minimal cut sets, respectively.

As described by [21], the components of the subsystem may be interdependent. In fact, it will be of great interest to investigate the optimal allocation policy of a coherent system with dependent components. However, owing to the complexity of modeling for the interdependent systems with dependent components, these problems remain open and merit further discussion. It is also of interest to investigate whether the results similar to the ones obtained in this paper hold for other stochastic orders, such as the (reversed) hazard rate order and the likelihood ratio order.

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Conflict of interest. The authors declare that they have no conflict of interest.

References

[1] Barlow, E. & Proschan, F. (1975). *Statistical theory of reliability and life testing: Probability models*. New York: Holt, Rinehart & Winston.
[2] Belzunce, F. & Martínez-Riquelme, C. (2019). Bounds for the hazard rate and the reversed hazard rate of the convolution of dependent random lifetimes. *Journal of Applied Probability* 56(4): 1033–1043.
[3] Belzunce, F., Riquelme, C.M., & Mulero, J. (2016). *An introduction to stochastic orders*. New York: Academic Press.
[4] Belzunce, F., Martínez-Riquelme, C., & Ruiz, J.M. (2018). Allocation of a relevation in redundancy problems. *Applied Stochastic Models in Business and Industry* 35(3): 492–503.
[5] Bhattacharya, D. & Samaniego, F.J. (2008). On the optimal allocation of components within coherent systems. *Statistics and Probability Letters* 78(7): 938–943.
[6] Boland, P.J., El-Neweihi, E., & Proschan, F. (1992). Stochastic order for redundancy allocations in series and parallel systems. *Advances in Applied Probability* 24(1): 161–171.
[7] Chen, J., Zhang, Y., & Zhao, P. (2019). Comparisons of order statistics from heterogeneous negative binomial variables with applications. *Statistics* 53(5): 990–1011.
[8] Collett, D. (1994). *Modelling survival data in medical research*. London: Chapman & Hall.
[9] Cox, D.R. (1992). *Regression models and life-tables*. Springer Series in Statistics. New York: Springer, pp. 527–541.
[10] Cuvelier, E. & Noirhomme-Fraiture, M. (2005). Clayton copula and mixture decomposition. *ASMDA 2005*, pp. 699–708.
[11] Da, G., Ding, W., & Li, X. (2010). On hazard rate ordering of parallel systems with two independent components. *Journal of Statistical Planning and Inference* 140(7): 2148–2154.
[12] Dao, C.D., Zuo, M.J., & Pandey, M. (2014). Selective maintenance for multi-state series–parallel systems under economic dependence. *Reliability Engineering & System Safety* 121: 240–249.
[13] Dhaene, J. & Goovaerts, M.J. (1996). Dependency of risks and stop-loss order. *ASTIN Bulletin: The Journal of the IAA* 26(2): 201–212.

[14] Dhaene, J. & Goovaerts, M.J. (1997). On the dependency of risks in the individual life model. *Insurance: Mathematics and Economics* 19(3): 243–253.
[15] Ding, W. & Li, X. (2012). The optimal allocation of active redundancies to k-out-of-n systems with respect to hazard rate ordering. *Journal of Statistical Planning and Inference* 142(7): 1878–1887.
[16] Ding, W., Fang, R., & Zhao, P. (2019). Reliability analysis of k-out-of-n systems based on grouping of components. *Advances in Applied Probability* 51(2): 339–357.
[17] Durante, F. & Papini, P.L. (2007). A weakening of Schur-concavity for copulas. *Fuzzy Sets and Systems* 158(12): 1378–1383.
[18] Durante, F. & Sempi, C. (2003). Copulae and Schur-concavity. *International Mathematical Journal* 3: 893–905.
[19] El-Neweihi, E., Proschan, F., & Sethuraman, J. (1986). Optimal allocation of components in parallel–series and series–parallel systems. *Journal of Applied Probability* 23(3): 770–777.
[20] Fang, R. & Wang, B. (2020). Stochastic comparisons on sample extremes from independent or dependent gamma samples. *Statistics* 54(4): 841–855.
[21] Fang, L., Balakrishnan, N., & Jin, Q. (2020). Optimal grouping of heterogeneous components in series-parallel and parallel-series systems under Archimedean copula dependence. *Journal of Computational and Applied Mathematics* 377: 112916.
[22] Feizabadi, M. & Jahromi, A.E. (2017). A new model for reliability optimization of series-parallel systems with non-homogeneous components. *Reliability Engineering & System Safety* 157: 101–112.
[23] Guo, Z., Zhang, J., & Yan, R. (2021). On inactivity times of failed components of coherent system under double monitoring. *Probability in the Engineering and Informational Sciences*, 1–18. doi:10.1017/S0269964821000152.
[24] Gupta, R.C. & Gupta, R.D. (2007). Proportional reversed hazard rate model and its applications. *Journal of Statistical Planning and Inference* 137(11): 3525–3536.
[25] Hu, T. & Wang, Y. (2009). Optimal allocation of active redundancies in k-out-of-n systems. *Journal of Statistical Planning and Inference* 139(10): 3733–3737.
[26] Jarrahiferiz, J., Kayid, M., & Izadkhah, S. (2019). Stochastic properties of a weighted frailty model. *Statistical Papers* 60(1): 53–72.
[27] Kalbfleisch, J.D. & Lawless, J.F. (1989). Inference based on retrospective ascertainment: An analysis of the data on transfusion-related AIDS. *Journal of the American Statistical Association* 84(406): 360–372.
[28] Kundu, A., Chowdhury, S., Nanda, A.K., & Hazra, N.K. (2016). Some results on majorization and their applications. *Journal of Computational and Applied Mathematics* 301: 161–177.
[29] Kundu, P., Hazra, N.K., & Nanda, A.K. (2020). Reliability study of series and parallel systems of heterogeneous component lifetimes following proportional odds model. *Statistics* 54(2): 375–401.
[30] Laniado, H. & Lillo, R.E. (2014). Allocation policies of redundancies in two-parallel-series and two-series-parallel systems. *IEEE Transactions on Reliability* 63(1): 223–229.
[31] Lehmann, E.L. (1966). Some concepts of dependence. *The Annals of Mathematical Statistics*: 1137–1153.
[32] Li, X. & Ding, W. (2010). Optimal allocation of active redundancies to k-out-of-n systems with heterogeneous components. *Journal of Applied Probability* 47(1): 254–263.
[33] Li, X., Yan, R., & Zuo, M.J. (2009). Evaluating a warm standby system with components having proportional hazard rates. *Operations Research Letters* 37(1): 56–60.
[34] Ling, X. & Wei, Y. (2020). Optimal allocation of randomly selected redundancies to k-out-of-n system with independent but nonidentical components. *IEEE Access* 8: 88464–88473.
[35] Ling, X., Wei, Y., & Li, P. (2018). On optimal heterogeneous components grouping in series-parallel and parallel-series systems. *Probability in the Engineering and Informational Sciences* 33(4): 564–578.
[36] Majumder, P., Ghosh, S., & Mitra, M. (2020). Ordering results of extreme order statistics from heterogeneous Gompertz–Makeham random variables. *Statistics* 54(3): 595–617.
[37] Marshall, A.W., Olkin, I., & Arnold, B.C. (2011). *Inequalities: Theory of majorization and its applications*. New York: Springer.
[38] Navarro, J. & Spizzichino, F. (2010). Comparisons of series and parallel systems with components sharing the same copula. *Applied Stochastic Models in Business and Industry* 26(6): 775–791.
[39] Nelsen, R.B. (2006). *An introduction to copulas*. Springer Series in Statistics, Vol. 47. Berlin, Heidelberg: Springer-Verlag.
[40] Shaked, M. & Shanthikumar, G. (2007). *Stochastic orders*. New York: Springer.
[41] Singh, H. & Misra, N. (1994). On redundancy allocations in systems. *Journal of Applied Probability* 31(4): 1004–1014.
[42] Sun, M.-X., Li, Y.-F., & Zio, E. (2019). On the optimal redundancy allocation for multi-state series–parallel systems under epistemic uncertainty. *Reliability Engineering & System Safety* 192: 106019.
[43] Tavakkoli-Moghaddam, R., Safari, J., & Sassani, F. (2008). Reliability optimization of series-parallel systems with a choice of redundancy strategies using a genetic algorithm. *Reliability Engineering and System Safety* 93(4): 550–556.
[44] Torrado, N. (2021). On allocation policies in systems with dependence structure and random selection of components. *Journal of Computational and Applied Mathematics* 388: 113274.
[45] Yan, R. & Luo, T. (2018). On the optimal allocation of active redundancies in series system. *Communications in Statistics – Theory and Methods* 47(10): 2379–2388.

[46] Yan, R. & Wang, J. (2020). Component level versus system level at active redundancies for coherent systems with dependent heterogeneous components. *Communications in Statistics – Theory and Methods*, 1–22. doi:10.1080/03610926.2020.1767140.
[47] Yan, R., Da, G., & Zhao, P. (2013). Further results for parallel systems with two heterogeneous exponential components. *Statistics* 47(5): 1128–1140.
[48] Yan, R., Lu, B., & Li, X. (2018). On redundancy allocation to series and parallel systems of two components. *Communications in Statistics – Theory and Methods* 48(18): 4690–4701.
[49] Yan, R., Zhang, J., & Zhang, Y. (2021). Optimal allocation of relevations in coherent systems. *Journal of Applied Probability* 58(4): 1–22. In press.
[50] You, Y., Fang, R., & Li, X. (2016). Allocating active redundancies to k-out-of-n reliability systems with permutation monotone component lifetimes. *Applied Stochastic Models in Business and Industry* 32(5): 607–620.
[51] Zhang, Y. (2018). Optimal allocation of active redundancies in weighted k-out-of-n systems. *Statistics and Probability Letters* 135: 110–117.

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