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ERGODIC PROPERTIES OF LAMPERTI OPERATORS

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1. Introduction. We shall assume throughout this paper, unless otherwise specified, that p is a fixed number, $1 < p < \infty$.

It is well known that to prove the pointwise ergodic convergence of a contraction T on an L_p-space it is enough to prove a Dominated Ergodic Estimate (DEE) for T (see e.g. [**11**]). The earliest and simplest nontrivial DEE was proved by Hardy–Littlewood [**10**, Theorem 8], where T is induced by the (right) shift on nonnegative integers equipped with the counting measure. The DEE for general positive L_p contractions, for long an open problem, was finally proved by Akcoglu [**1**] in 1974. The proof involves several steps, the most difficult being a dilation that reduces it to the case of positive invertible isometries, first proved by A. Ionescu Tulcea [**11**]. Recently A. de la Torre [**8**] proved a DEE for a cyclic group of positive, uniformly norm-bounded L_p operators, using a technique developed by Calderón [**4**] and extended by Coifman and Weiss [**7**], which brings the Hardy–Littlewood theorem into play. This result generalizes [**11**], and its proof is considerably simpler, thereby in effect simplifying the proof of Akcoglu's theorem. Our first aim in this paper is to show, in §2, that Calderón's technique works for positive, not necessarily invertible L_p isometries. In §3 we introduce the concept of Lamperti operators, which include positive L_p isometries, and give sufficient conditions for L_p operators to be Lamperti, showing, in particular, that operators considered in [**8**] are so. In §4 we prove some structural theorems for Lamperti operators, which we use in §5 to prove our main results, the DEE for Lamperti contractions (Theorem 5.1) and the DEE for a class of Lamperti operators (Theorem 5.2), which generalizes and improves that of [**8**]. Finally in §6 we show how the dilation method in [**1**] can be simplified, using the results that we have proved in §§2, 5.

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2. DEE for positive isometries. Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space and $L_p = L_p(X, \mathscr{F}, \mu)$, $1 \leqq p \leqq \infty$, the usual (real or complex) Banach spaces. Statements concerning measurable functions and sets shall be read modulo μ-null sets. The indicator function of a set E is denoted 1_E. The *support* of a function f is the set $\operatorname{supp} f = \{x : f(x) \neq 0\}$. The *maximal operator* $M(T) \equiv M$ of an L_p operator T is defined by $Mf = \sup_{n \geqq 1} |T_n f|$, where

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$T_n = n^{-1}\sum_{i=0}^{n-1} T^i$. The *truncated maximal operator* M_N, N a positive integer, is defined similarly with the sup taken over $n = 1, \ldots, N$. T is said to have a *Dominated Ergodic Estimate* (DEE) with (finite) constant C if

(2.1) $\quad ||Mf|| \leqq C||f|| \quad$ for all $f \in L_p$.

This will be the case if (2.1) holds for all M_N with the same C.

THEOREM 2.1. *Suppose T is a positive isometry on L_p, $1 < p < \infty$. Then* (2.1) *holds with $C = p/(p-1)$.*

Proof. First note that T maps functions with disjoint supports to functions with disjoint supports. This follows from the fact that for $f, g \in L_p^+$, $||f+g||^p = ||f||^p + ||g||^p$ if and only if f and g have disjoint supports.

It suffices to prove (2.1) for all M_N and $f \in L_p^+$. Now $M_N f = \sum_{n=1}^{N} 1_{E_n} T_n f$ for a family of disjoint subsets $E_1, \ldots, E_N$. Since T and so T^k, $k = 1, 2, \ldots$, preserve disjointness of supports, $T^k(1_{E_n} T_n f)$, $n = 1, \ldots, N$, k fixed, have disjoint supports D_n. Hence

$$(2.2) \qquad T^k M_N f = \sum_{n=1}^{N} 1_{D_n} T^k(1_{E_n} T_n f) \leqq \sum_{n=1}^{N} 1_{D_n} T^k T_n f \leqq M_N(T^k f),$$

and so

$$(2.3) \qquad ||M_N f|| = ||T^k M_N f|| \leqq ||M_N(T^k f)||, \quad k = 0, 1, 2, \ldots .$$

Taking power p and averaging between $k = 0$ and $k = L-1$, $L \geqq 1$, we have

$$(2.4) \qquad ||M_N f||^p \leqq \frac{1}{L} \int \sum_{k=0}^{L-1} (M_N T^k f)^p d\mu.$$

Now the Hardy–Littlewood DEE says that for a finite sequence of nonnegative numbers $F(0), \ldots, F(N+L-2)$,

$$(2.5) \qquad \sum_{k=0}^{L-1} (M_N F(k))^p \leqq C^p \sum_{k=0}^{N+L-2} (F(k))^p$$

where

$$M_N F(k) = \max_{N \geqq n \geqq 1} \frac{1}{n} \sum_{i=0}^{n-1} F(k+i).$$

Applying (2.5) to $F_x(k) = T^k f(x)$ for each x fixed, and observing that $M_N T^k f(x) = M_N F_x(k)$, we have from (2.4)

$$||M_N f||^p \leqq C^p \frac{1}{L} \int \sum_{k=0}^{N+L-2} (T^k f)^p d\mu = C^p \frac{N+L-1}{L} ||f||^p,$$

which gives us (2.1) for M_N by letting $L \to \infty$.

Remark 2.1. (2.3) is crucial in the proof of Theorem 2.1. More generally if

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there exist positive numbers H, K so that $||T^n|| \leqq K$, $n = 0, 1, \ldots$, and

(2.3′) $\quad ||M_N f|| \leqq H||M_N T^k f|| \quad$ for all $k \geqq 0$, $N \geqq 1$,

then T has a DEE with constant $HKp/(p-1)$.

3. Lamperti operators.

Definition 3.1. A linear operator on a Banach space of functions is said to *separate supports* if it maps functions with disjoint supports to the same.

Definition 3.2. A bounded linear operator on an L_p-space, $1 \leqq p < \infty$, separating supports is called a *Lamperti operator.*

Lamperti operators include L_p isometries, $1 \leqq p < \infty$, $p \neq 2$, and positive L_2 isometries [**3**; **11**; **12**]. Their general structure, in the context of L_p isometries, $p \neq 2$, was investigated by J. Lamperti [**12**], but the idea goes back to Banach. It is interesting to note that the operators considered in [**8**] also fall into this category, by the following.

PROPOSITION 3.1. *Every positive linear operator on* L_p, $1 \leqq p \leqq \infty$, *that has a positive inverse separates supports.*

Proof. We need only show that such an operator T maps every pair $f, g \in L_p^+$ with $\min(f, g) = 0$ to Tf, Tg with $\min(Tf, Tg) = 0$. Call this minimum h. So $T^{-1}h \leqq f$, $T^{-1}h \leqq g$, implying supp $T^{-1}h \subset$ supp $f \cap$ supp g. Thus $T^{-1}h = 0$ and $h = 0$.

Remarks 3.1. However, as noted by de la Torre himself (oral communication), the result in [**8**] extends to an invertible L_p operator T such that only T(or T^{-1}) is positive and $||T^n|| \leqq K < \infty$, $n = 0, \pm 1, \pm 2, \ldots$. In fact the inequalities $M_N f \leqq T^k M_N(T^{-k}f)$, $f \in L_p$, $N \geqq 1$, still hold for positive (negative) k. This yields a DEE with constant $K^2p/(p-1)$ (cf. Remark 2.1). Such a T is not in general Lamperti (see Example 3.1 below) but it is so in the finite dimensional case.

PROPOSITION 3.2. *Let T be an invertible, nonnegative $n \times n$ matrix such that T^k, $k = 0, \pm 1, \ldots$, are uniformly bounded in any (equivalent) matrix norm. Then T is periodic and separates supports.*

Proof. The spectral radius formula shows that $r(T)$, $r(T^{-1}) \leqq 1$. This is possible only if the spectrum $\sigma(T) \subset$ unit circle. If T is irreducible, then by Frobenius' Theorem [**9**, Ch. 13], its elements a_{ij}, after a congruent change of rows and columns, are all 0 except when $j = i + 1$, and its characteristic polynomial $\lambda^n - a_{12} \ldots a_{n-1,n}a_{n1}$ is equal to $\lambda^n - 1$. It follows that T separates supports and is n-periodic. If T is reducible, then T splits, after a congruent change of rows and columns, into blocks T_{ij}, $i, j = 1, \ldots, m$, such that T_{ij} is a zero matrix for $j > i$, and each T_{ii} is an irreducible square matrix. Since $\sigma(T_{ii}) \subset \sigma(T)$, each T_{ii} separates supports and is periodic, by the first case.

Let N be the least common multiple of the periods. Then $T^N = I + P, P \geqq 0$. $T^{Nk} \geqq kP$, $k = 1, 2 \ldots$. The norm condition then implies $P = 0$. Now $T = D + Q$, $D = \operatorname{diag}(T_{11}, \ldots, T_{mm})$, $Q \geqq 0$. So

$$D^N = I = T^N = (D + Q)^N \geqq D^N + D^{N-1}Q,$$

implying $Q = 0$. Thus $T = D$, is periodic, and separates supports.

Example 3.1. Let l_p, $1 \leqq p \leqq \infty$, be the L_p-space on the set of integers with counting measure. Define operators U, E_{ij}, i, j any two integers, on l_p by $U\{x_n\} = \{y_n\}$, $y_n = x_{n+1}$, and $E_{ij}\{x_n\} = \{z_n\}$, $z_n = 0$ if $n \neq i$, $z_i = x_j$. Thus U is a positive invertible isometry. Let $0 < t < 1$ and $T = U + tE_{-1,1}$. Then T is positive and does not separate supports. $T^{-1} = U^{-1} - tE_{00}$. Routine calculations show that

$$|T^n x| \leqq U^n|x| + tU^{n+1}|x|, \quad |T^{-n}x| \leqq \sum_{i=0}^{n} t^{n-i}U^{-i}|x|,$$

for $x \in l_p$, $n \geqq 0$. Hence $||T^n|| \leqq 1 + t$, and $||T^{-n}|| \leqq (1 - t)^{-1}$.

Next we give a characterization of Lamperti operators. $|T|$ in Theorem 3.1 below is clearly the linear modulus of T (see [**5**]). This theorem is of interest since the linear modulus of an L_p operator, $1 < p < \infty$, that is not positive may not exist.

THEOREM 3.1. *A bounded linear operator T on an L_p-space, $1 \leqq p \leqq \infty$, separates supports if and only if there exists a positive linear operator $|T|$ on L_p such that*

(3.1) $\quad |Tf| = |T|\,|f| \quad$ *for every $f \in L_p$.*

Proof. Suppose (3.1) holds. Then for every pair f, $g \in L_p$ with disjoint supports, $|Tf + tTg|$ is the same for $t = \pm 1$ (real L_p), or $t = \pm 1, i$ (complex L_p). It follows that Tf and Tg have disjoint supports. Conversely suppose T separates supports. Then $|Tf| = |T|f||$, and $|T|$ defined on L_p^+ as $|T|f = |Tf|$ is linear. These are easy for simple functions. The general case follows from a routine approximation process. Then $|T|$ extends to a linear operator on L_p and (3.1) is true.

4. Structural theorems.

Definition 4.1. A σ-endormorphism Φ of the measure algebra $(X, \mathscr{F}, \mu)$ is an endomorphism of $\mathscr{F}$ modulo μ-null sets as a Boolean σ-algebra. This means:

(4.1) $\quad \Phi\left(\bigcup_{n=1}^{\infty} E_n\right) = \bigcup_{n=1}^{\infty} \Phi E_n, \quad$ for disjoint $E_n \in \mathscr{F}$,

(4.2) $\quad \Phi(X - E) = \Phi X - \Phi E, \quad$ for all $E \in \mathscr{F}$, and

(4.3) $\quad E \in \mathscr{F}, \mu E = 0 \Rightarrow \mu \Phi E = 0.$

Φ induces a unique positive linear operator, also denoted by Φ, on the space of (finite-valued or extended) measurable functions such that $\Phi 1_E = 1_{\Phi E}$(cf. [**12**]). We list here some properties of this operator which we will use later. Each of the last three is equivalent to positivity in the definition of the operator Φ. Let f, g be any measurable functions, and p any positive number. Then

(4.4) Φf is $\Phi\mathscr{F}$-measurable;

(4.5) $\operatorname{supp} \Phi f = \Phi \operatorname{supp} f$;

(4.6) $|\Phi f|^p = \Phi |f|^p$;

(4.7) $\Phi f . \Phi g = \Phi (f.g)$;

(4.8) Φ preserves a.e. convergence, i.e., $f_n \to f$ a.e. implies $\Phi f_n \to \Phi f$ a.e.

THEOREM 4.1. *Every Lamperti operator T on $L_p(X, \mathscr{F}, \mu)$, $1 \leqq p < \infty$, is induced by a σ-endomorphism Φ and a measurable function h. Specifically one such Φ (called the associated σ-endomorphism of T) is defined by $\Phi E = \operatorname{supp} T1_E$ for $E \in \mathscr{F}$, $\mu E < \infty$. There is then a unique $h = \sum_{i=1}^{\infty} T1_{X_i}$, where $\{X_i : i \geqq 1\}$ is a countable decomposition of X into subsets of finite measure, with* $\operatorname{supp} h = \Phi X$, *such that*

$$(4.9) \qquad \nu E = \int_{\Phi E} |h|^p d\mu$$

defines a measure on $(X, \mathscr{F})$, $\nu \leqq ||T||^p \mu$; and

$$(4.10) \qquad Tf(x) = h(x)\Phi f(x) \quad \textit{for all } f \in L_p.$$

Proof. Similar to that of [**12**, Theorem 3.1].

Remarks 4.1. There is a parallel structural theorem for bounded support-separating L_∞ operators, with σ-endomorphisms replaced by endomorphisms, a.e. convergence in (4.8) by L_∞ convergence, and (4.9) by $||h||_\infty = ||T||$.

Remark 4.2. In many cases of interest, Φ is induced by a non-singular point transformation ϕ, so that $\Phi f = f \circ \phi$; and it is always so in discrete measure spaces, and, by a theorem of Sikorski, in Borel spaces. (4.10) for non-isometric Lamperti operators was observed by several authors before; see e.g. [**11**].

With ν given by (4.9) and μ, p, Φ fixed, we denote $d\nu/d\mu$ by $D(h)$, i.e.,

$$\nu E = \int_E D(h) d\mu \quad \text{for all } E \in \mathscr{F}.$$

It is not difficult to prove the following.

PROPOSITION 4.1. *Let T be as in Theorem* 4.1.

(a) *If T is one-to-one, then so is Φ and $D(h) > 0$ a.e.*

(b) *If T is onto, then so is Φ and $h \neq 0$ a.e.*

(c) *The dual T^* of T separates supports if and only if Φ maps $\mathscr{F}$ onto $\mathscr{F} \cap \Phi X$.*

THEOREM 4.2. *Let T be as in Theorem* 4.1. *Then*

$$\int \Phi f \cdot |h|^p d\mu = \int f \cdot D(h) d\mu \tag{4.11}$$

for all nonnegative measurable functions f. Hence

(4.12) *T acts isometrically on $L_p(\{D(h) = 1\})$ and vanishes on $L_p(\{D(h) = 0\})$*,

and

$$||T||^p = ||D(h)||_\infty. \tag{4.13}$$

Proof. By definition of $D(h)$ and (4.9), (4.11) is true for indicator functions. The general case then follows. (4.12) is then obvious. For (4.13), observe that

$$||Tf||^p = \int |\Phi f|^p |h|^p d\mu = \int |f|^p D(h) d\mu,$$

by (4.11). Hence

$$||T||^p = \sup_{||f|| \leqq 1} ||Tf||^p = \sup_{||f|| \leqq 1} \int |f|^p D(h) d\mu = ||D(h)||_\infty.$$

THEOREM 4.3. *Let T be as in Theorem* 4.1. *Then*

$$T^n = \theta_n \cdot S^n \quad \text{and} \quad \theta_n = \theta \ldots \Phi^{n-1}\theta,\ n = 1, 2, \ldots, \tag{4.14}$$

where

(i) *S is a positive Lamperti contraction for which there is a decomposition of X into subsets X_1 and X_2, such that S acts isometrically on $L_p(X_1)$ and vanishes on $L_p(X_2)$;*

(ii) *θ is an L_∞ function whose support* $\operatorname{supp} \theta = \Phi X$ *and whose modulus $|\theta|$ is $\Phi\mathscr{F}$-measurable.*

Further,

$$||T^n|| \leqq ||\theta_n||_\infty, \tag{4.15}$$

where equality holds for $n = 1$ always, and for $n \geqq 2$ when T^* separates supports.

Proof. X decomposes into disjoint subsets X_1, X_2 such that ΦX_2 is null and Φ is one-to-one on $\mathscr{F} \cap X_1$. By (4.9), $D(h) = 0$ on X_2, and since $|h| > 0$ on ΦX, $D(h) > 0$ on X_1. Let S be the Lamperti operator induced by Φ and g, where $g = |h|. (\Phi D(h))^{-1/p}$ on ΦX, and 0 on $X - \Phi X$. Clearly $D(g) = 0$ on X_2. For each $E \in \mathscr{F} \cap X_1$,

$$\int_{\Phi E} |g|^p d\mu = \int \Phi(1_E D(h)^{-1}) |h|^p d\mu = \int_E D(h)^{-1} \cdot D(h) d\mu = \mu E,$$

by (4.11), and so $D(g) = 1$ on X_1. Hence by (4.12) and (4.13), S has the properties described in (4.14). Put $\theta = \Phi D(h)^{1/p}$. sgn h, where sgn $h = h/|h|$

on $\{h \neq 0\}$, and 0 on $\{h = 0\}$. Then the equality in (4.14) holds for $n = 1$, and hence also for $n \geqq 2$, by property (4.7) of Φ.

Obviously from (4.14), inequality (4.15) holds. Conversely, since $|\theta|$ is $\Phi\mathscr{F}$-measurable, given any $A < ||\theta||_\infty$, there is $E \in \mathscr{F} \cap X_1$, $0 < \mu E < \infty$, such that $|\theta| \geqq A$ on ΦE. Then supp $S1_E = \Phi E$ and $|T1_E| = |\theta|S1_E \geqq AS1_E$. It follows that $||T|| \geqq A$ and therefore equality holds in (4.15) for $n = 1$. If T^* separates supports, then Φ maps $\mathscr{F}$ onto $\mathscr{F} \cap \Phi X$, by Prop. 4.1(c), and hence Φ^n maps $\mathscr{F}$ onto $\mathscr{F} \cap \Phi^n X$. Since

$$\text{supp } \theta_n = \text{supp } \theta \cap \ldots \cap \text{supp } \Phi^{n-1}\theta = \Phi X \cap \ldots \cap \Phi^n X = \Phi^n X,$$

it is then clear that $|\theta_n|$ is $\Phi^n\mathscr{F}$-measurable for $n \geqq 2$. The same argument above shows that equality now holds in (4.15) for $n \geqq 2$.

COROLLARY 4.1. *Let T be a Lamperti operator on $L_p(X, \mathscr{F}, \mu)$, $1 \leqq p < \infty$, with $||T^n|| \leqq K < \infty$, $n = 0, 1, 2, \ldots$, such that*

(a) *T^* separates supports, or equivalently,*

(b) *the associated σ-endomorphism Φ maps $\mathscr{F}$ onto $\mathscr{F} \cap \Phi X$.*

Then there exists a positive Lamperti contraction S on L_p such that

(4.16) $|T^n f| \leqq KS^n|f|$ *for each* $f \in L_p$, $n = 0, 1, \ldots$.

In [**2**], it is shown that an L_p contraction, $1 \leqq p < \infty$, has a geometric dilation (as defined in [**2**]) to an L_p isometry, positive when $p = 2$, only if it separates supports. Conversely, we have the following.

THEOREM 4.4. *Every support-separating L_p contraction, $1 \leqq p \leqq \infty$, has a geometric dilation to a support-separating L_p isometry, which can be chosen positive if so is the contraction.*

Proof. Consider first the case $1 \leqq p < \infty$. With notation as in Theorem 4.1 and 4.2, X decomposes into $U = \{D(h) = 1\}$ and $V = \{D(h) < 1\}$. Define $X_1 = X$, and X_n, $n \geqq 2$, as disjoint copies of V, equipped with inherited σ-algebra and measure. Then the L_p-direct sum $\oplus_{n=1}^\infty L_p(X_n)$ defines an L_p-space $L_p(Y, \mathscr{G}, \lambda)$ such that $Y \supset X$, $\mathscr{G} \supset \mathscr{F}$ and λ extends μ. R, defined as

$$R(f_1, f_2, \ldots) = (Tf_1, (1 - D(h))^{1/p} \cdot 1_V f_1, f_2, f_3, \ldots),$$

is a Lamperti isometry on $L_p(Y)$, positive if so is T. It is easy to check that $T^n f = 1_X R^n f$, $n = 0, 1, \ldots$, for all $f \in L_p(X)$.

The same proof works through for the case $p = \infty$ if we read 1 for $(1 - D(h))^{1/p}$, sup $\{E: ||h \cdot 1_{\Phi E}||_\infty < 1\}$ for V and inf $\{E: ||h \cdot 1_{\Phi E}||_\infty = 1\}$ for U.

5. Ergodic properties.

THEOREM 5.1. *Let T be a Lamperti contraction on L_p, $1 < p < \infty$. Then T has a DEE with constant $p/(p - 1)$.*

Proof. From Theorem 3.1 or 4.1, T has a linear modulus $|T|$ which is a positive Lamperti contraction. For all $f \in L_p$, $M(T)f \leqq M(|T|)|f|$. From Theorem 4.4, the latter equals $1_X M(R)|f|$, where R is a positive isometry on a larger L_p-space. Theorem 2.1 then completes the proof.

THEOREM 5.2. *Let T be a Lamperti L_p operator, $1 < p < \infty$, with $||T^n|| \leqq K < \infty$, $n = 0, 1, 2, \ldots$, such that*

(a) *T^* separates supports, or equivalently,*

(b) *the associated σ-endomorphism Φ maps $\mathscr{F}$ onto $\mathscr{F} \cap \Phi X$.*

Then T has a DEE with constant $Kp/(p-1)$.

Proof. This follows from Corollary 4.1 and Theorem 5.1.

COROLLARY 5.1. *For T in Theorem 5.1 or 5.2, the individual ergodic theorem holds i.e. $T_n f$ converges a.e. for all $f \in L_p$.*

Remarks 5.1. By Propositions 3.1 and 4.1, a cyclic group of positive, uniformly bounded L_p operators, $1 < p < \infty$, is generated by a Lamperti operator satisfying conditions (a) equivalently (b) of Corollary 4.1. Thus Theorem 5.2 generalizes and improves the result of [**8**], giving a sharper constant $Kp/(p-1)$ instead of $K^2p/(p-1)$.

If T^* does not separate supports, we have the following weaker theorem whose proof is similar to that of Theorem 2.1. (See Remark 2.1).

THEOREM 5.3. *Suppose T is a Lamperti L_p operator, $1 < p < \infty$, with $||T^n|| \leqq K < \infty$, $n \geqq 0$,* such that for all $f \in L_p$ *with norm* 1,

$$(*) \qquad \limsup_{n\to\infty} n^{-1}(||f||^p + \ldots + ||T^{n-1}f||^p) \geqq H^p > 0.$$

Then the DEE holds for T with constant $Kp/H(p-1)$.

6. Akcoglu's theorem. A crucial step in the proof of Akcoglu's theorem [**1**] is the dilation of an n-dimensional $L_p(X, \mathscr{A}, m)$ operator T satisfying $||T|| = 1$ and $T_{ij} > 0$ to a positive *invertible* isometry. The same proof shows that by virtue of Theorem 2.1, it is enough to dilate T to a positive isometry. More generally, because of Theorem 5.1, it is enough to "super-dilate" T to a positive Lamperti *contraction* Q on an $L_p(X, \mathscr{B}, m)$ with $\mathscr{B} \supset \mathscr{A} : T^k f \leqq EQ^k f$, $k \geqq 1, f \in L_p^+(\mathscr{A})$, where E is the conditional expectation with respect to $\mathscr{A}$. This will follow if we can prove $TEf \leqq EQf, f \in L_p^+(\mathscr{B})$. The existence of Q comes up in a very natural way by a simpler adaptation of Akcoglu's original construction.

In fact, we can regard X as the union of disjoint intervals $I_1, \ldots, I_n$, of lengths $m_1, \ldots, m_n$, $\mathscr{A}$ as generated by $I_1, \ldots, I_n$, and m as the Lebesgue measure. Take $\mathscr{B} = \{$Borel sub-sets of $X\}$. Then

$$Ef(x) = \frac{1}{m_i} \int_{I_i} f dm \quad \text{for } x \in I_i.$$

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By dividing each I_i into sub-intervals I_{ij}, $j = 1, \ldots, n$, and mapping I_{ij} linearly onto I_j, we get a transformation ϕ of X and a σ-endomorphism $\Phi = \phi^{-1}$ on $\mathscr{B}$ such that

$$m(\Phi F \cap I_i) = m(F)\xi_{ij}m_i/m_j,$$

for all $F \in I_j \cap \mathscr{B}$, where $\xi_{ij} = m(I_{ij})/m_i$. Let Q be the Lamperti operator induced by Φ and $h \geqq 0$, where $h(x) = h_{ij} =$ constant, for $x \in I_{ij}$. Simple calculations show that $EQ = TE$ if and only if $h_{ij} = T_{ij}/\xi_{ij}$ and that

$$D(h)(x) = m_j^{-1} \sum_i h_{ij}^p \xi_{ij} m_i \quad \text{for } x \in I_j .$$

By Theorem 4.2, Q is a contraction if and only if $\sum_i h_{ij}^p \xi_{ij} m_i \leqq m_j$ (equalities for isometric Q), and if in addition $EQ = TE$, $\sum_i T_{ij}^p \xi_{ij}^{1-p} m_i \leqq m_j$. A natural choice for ξ_{ij} is $T_{ij}u_j/(Tu)_i$ where $u = (u_1, \ldots, u_n)$ and Tu are positive vectors. The last relations then become $\sum_i T_{ij}(Tu)_i^{p-1} m_i \leqq m_j u_j^{p-1}$, i.e. $T^*(Tu)^{p-1} \leqq u^{p-1}$. The existence of such a u is easy, and in fact it satisfies the equality so that Q is an isometry [**1**, Lemma 2.4].

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