# A LINEAR DIOPHANTINE PROBLEM

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**1. Introduction.** Let $a_1, a_2, \ldots, a_t$ be a set of groupwise relatively prime positive integers. Several authors, **(2; 3; 5; 6)**, have determined bounds for the function $F(a_1, \ldots, a_t)$ defined by the property that the equation

$$n = a_1x_1 + a_2x_2 + \ldots + a_tx_t \tag{1}$$

has a solution in positive integers $x_1, \ldots, x_t$ for $n > F(a_1, \ldots, a_t)$. If $F(a_1, \ldots, a_t)$ is a function of this type, it is easy to see that

$$G(a_1, \ldots, a_t) = F(a_1, \ldots, a_t) - a_1 - a_2 - \ldots - a_t \tag{2}$$

is the corresponding function for the solvability of (1) in non-negative $x$'s.

It is well known that $a_1a_2$ is the best bound for $F(a_1, a_2)$ and $a_1a_2 - a_1 - a_2$ for $G(a_1, a_2)$. Otherwise only in very special cases have the best bounds been found, even for $t = 3$.

In the present paper a symmetric expression is developed for the best bound for $F(a_1, a_2, a_3)$ which solves that problem and gives insight on the general problem for larger values of $t$. In addition, some relations are developed which may be of interest in themselves.

**2. A General Property.** For $t \geqslant 2$, let $B(a_1, a_2, \ldots, a_t)$ be the best bound for $F(a_1, a_2, \ldots, a_t)$, that is, $B$ is the maximum number $N$ where

$$N \neq \sum_{i=1}^{t} x_ia_i \qquad \text{for any } x_i > 0. \tag{3}$$

Then note that $B$ is the maximum $N$ from a restricted set of numbers $N$ satisfying both (3) and

$$N + a_i = \sum_{j=1}^{t} y_{ij}a_j, \qquad y_{ij} > 0 \text{ for each } i. \tag{4}$$

since the definition of $B$ implies $B$ satisfies (4). Thus, in particular,

$$N = (y_{11} - 1)a_1 + y_{12}a_2 + \ldots + y_{1t}a_t, \; y_{1j} > 0.$$

But by (3), $y_{11} - 1 \leqslant 0$ so that $y_{11} = 1$ since $y_{11} > 0$. By symmetry we have

THEOREM 1. *For every $N$ satisfying* (3) *and* (4) *there are representations of $N$* for each $i = 1, 2, \ldots, t$ *of the form*

$$N = \sum_{\substack{j=1 \\ j \neq i}}^{t} y_{ij}a_j, \qquad y_{ij} > 0, \tag{5}$$

*and $B$ is the maximum such $N$.*

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**3. The Case $t = 3$. A reduction formula.** We seek an expression for $B = B(a_1, a_2, a_3)$ having the property that (1) is satisfied for $n > B$ but is not satisfied for $n = B$. Let us first reduce the problem to the case of pairwise relatively prime $a$'s.

Let $d_{ij} = (a_i, a_j)$, $a_i = b_i d_{ij} d_{ik}$, so that $(b_1, b_2) = (b_2, b_3) = (b_3, b_1) = 1$. Then we have

THEOREM 2.

$$B(a_1, a_2, a_3) = d_{12}d_{23}d_{31}B(b_1, b_2, b_3). \tag{6}$$

*Proof.* First we show that if we write $d = d_{12}$, $\bar{b}_1 = d_{13}b_1$, $\bar{b}_2 = d_{23}b_2$ so that $(d, a_3) = (\bar{b}_1, \bar{b}_2) = 1$, then

$$B(d\bar{b}_1, d\bar{b}_2, a_3) = dB(\bar{b}_1, \bar{b}_2, a_3). \tag{7}$$

Suppose that $dB(\bar{b}_1, \bar{b}_2, a_3) = d\bar{b}_1x + d\bar{b}_2y + a_3z$, $x, y, z > 0$. Then since $(d, a_3) = 1$, we must have $z = wd$, $w > 0$, so that $B(\bar{b}_1, \bar{b}_2, a_3) = \bar{b}_1x + \bar{b}_2y$ $a_3w$, $x, y, w > 0$, a contradiction to the definition of $B(\bar{b}_1, \bar{b}_2, a_3)$. In addition, for any positive integer $m > 0$, we show that

$$dB(\bar{b}_1, \bar{b}_2, a_3) + m = d\bar{b}_1x + d\bar{b}_2y + a_3z, \qquad x, y, z > 0. \tag{8}$$

We apply a result from **(2)**.

LEMMA 1 (Brauer). *Let $a$ and $b$ be relatively prime positive integers. Then every positive integer $m$ divisible neither by $a$ nor by $b$ is representable either in the form*

$$m = au + bv, \qquad u > 0, v > 0, \tag{9}$$

*or*

$$m = ab - au - bv, \qquad b > u > 0, a > v > 0. \tag{10}$$

Letting $d = a$ and $a_3 = b$ in Lemma 1, if (9) holds, we have

$$\begin{aligned} dB(\bar{b}_1, \bar{b}_2, a_3) + m &= d(B(\bar{b}_1, \bar{b}_2, a_3) + u) + va_3 \\ &= d\bar{b}_1x + d\bar{b}_2y + a_3(dz + v) \end{aligned} \tag{11}$$

by the definition of $B(\bar{b}_1, \bar{b}_2, a_3)$, giving (8).

If (10) holds, we have $0 < u < a_3$, and $0 < v < d$, so that

$$d(B(\bar{b}_1, \bar{b}_2, a_3) + a_3 - u) - va_3 = d\bar{b}_1x + d\bar{b}_2y + (dz - v)a_3, \tag{12}$$

for $x$, $y$, and $(dz - v) > 0$, giving (8).

Finally, if $m = ud$, then (8) follows directly. If $m = va_3$, write $m = da_3$ $+ (v - d)a_3$ giving (8). Thus (7) holds. Applying the method of obtaining (7) twice more gives (6) and Theorem 2.

We have thus reduced the problem to where the $a$'s are pairwise relatively prime. For the moment let $a_1 > a_2 > a_3$. If

$$a_1 = ua_2 + va_3, \qquad u, v > 0, \tag{13}$$

then $B(a_1, a_2, a_3) = a_2a_3 + a_1$ as Brauer showed in **(2)**. Otherwise

$$B(a_1, a_2, a_3) < a_ia_j + a_k. \tag{14}$$

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**4. An expression for $B(a_1, a_2, a_3)$.** We develop a symmetric expression for $B(a_1, a_2, a_3)$ for the case of pairwise relatively prime $a$'s where each $a_i \neq xa_j + ya_k$, $x > 0$, $y > 0$. Later we show that this same form of expression gives the general solution for $t = 3$.

DEFINITION. Let $L_i$ = the minimum positive $K_i$ satisfying

$$K_i a_i = v_{ij} a_j + v_{ik} a_k, \qquad v_{ij} \geqslant 0, v_{ik} \geqslant 0, \qquad i = 1, 2, 3. \tag{15}$$

Such a number exists since $B(a_j, a_k) = a_j a_k < K a_i$ for large $K$.

THEOREM 3. *Given*

$$(a_1, a_2) = (a_2, a_3) = (a_3, a_1) = 1 \tag{16}$$

*and*

$$L_i > 1, \qquad i = 1, 2, 3 \tag{17}$$

*and*

$$L_i a_i = x_{ij} a_j + x_{ik} a_k, \tag{15'}$$

*then the $x_{ij}$ are uniquely defined and*

$$x_{ij} > 0. \tag{18}$$

Since $L_i > 1$, it follows from (10) and (16) that

$$a_i = a_j a_k - v_{ij} a_j - v_{ik} a_k \tag{19}$$

where $0 < v_{ij} < a_k$, $0 < v_{ik} < a_j$. Thus $v_{ik} a_k + a_i = (a_k - v_{ij}) a_j \geqslant L_j a_j$ and so by symmetry

$$L_j < a_k, \qquad \text{for each } j \neq k. \tag{20}$$

If $x_{ji} = 0$, then $L_i a_i = x_{jk} a_k$ and by (16) $L_j = m a_k$, a contradiction to (20). This gives (18). Also the $x_{ij}$ are uniquely determined since if $L_i a_i = x_{ij} a_j + x_{ik} a_k = z_{ij} a_j + z_{ik} a_k$, then by (16) we have $x_{ij} = z_{ij} + m a_k$ and $x_{ik} = z_{ik} - m a_j$. If $m > 0$, $x_{ij} > a_k$. But then for some $d > 0$, $L_i a_i = (a_k + d) a_j + x_{ik} a_k$ and by (19) we get $(L_i - 1) a_i = (d + v_{ij}) a_j + (x_{ik} + v_{ik}) a_k$, contradicting the definition of $L_i$. Similarly, for $m < 0$.

For $t = 3$ and (16) and (17) we show that there are just two numbers $N$ with properties (3) and (4) so that $B$ is the larger of these numbers. From (5) such a number $N$ has representations of the form

$$N = y_{ij} a_j + y_{ik} a_k \qquad i = 1, 2, 3. \tag{5'}$$

Next observe that from (18) we have

$$y_{ij} \leqslant L_j \tag{21}$$

since otherwise for some $d_j > 0$ we would have $N = (L_j + d_j) a_j + y_{ik} a_k = x_{ji} a_i + d_j a_j + (x_{jk} + y_{ik}) a_k$, contradicting (3). From (20) and (21) we have

$$y_{kj} < a_k, \qquad y_{kj} < a_i. \tag{22}$$

Next we show that the representations (5′) for $N$ are unique for each $i$. For otherwise $y_{ki}a_i + y_{kj}a_j = z_{ki}a_i + z_{kj}a_j$ and from (16) and (22), $y_{kj} - z_{kj} = ma_i$, $m \leqslant 0$, and $y_{ki} - z_{ki} = ma$, $m \geqslant 0$, so that $m = 0$ and $y_{kj} = z_{kj}$, etc.

From (5′) and Theorem 1 we now have unique representations of $N$ of the form

$$N = y_{ki}a_i + y_{kj}a_j = y_{ij}a_j + y_{ik}a_k = y_{jk}a_k + y_{ji}a_i.$$

If $y_{kj} = y_{ij}$, then $y_{ki} = ma_k$, contradicting (22). Thus either $y_{kj} < y_{ij}$ or $y_{kj} > y_{ij}$.

*Case* 1. If

$$y_{kj} < y_{ij} \tag{23}$$

then $y_{ki}a_i = (y_{ij} - y_{kj})a_j + y_{ik}a_k$ so that $y_{ki} \geqslant L_i$. Thus by (21) we have

$$y_{ki} = L_i. \tag{24}$$

Then by (24) and (5′)

$$N = L_ia_i + y_{kj}a_j = y_{ji}a_i + y_{jk}a_k$$

or $(L_i - y_{ji})a_i + y_{kj}a_j = y_{jk}a_k$, where $L_i \geqslant y_{ji}$ by (21). If $L_i = y_{ji}$ then $y_{kj} = ma_k$, contradicting (22), so that $L_i - y_{ji} > 0$ and $y_{jk} \geqslant L_k$ by the definition of $L_k$. But then $y_{jk} = L_k$ by (21). Thus (23) implies that $y_{ki} = L_i$, $y_{jk} = L_k$, and cyclically, $y_{ij} = L_j$. But then by (15′)

$$N = (x_{ij} + y_{kj})a_j + x_{ik}a_k = L_ja_j + y_{ik}a_k$$

and by the uniqueness of these representations and by cyclic permutation of subscripts, we have

$$y_{ik} = x_{ik} \tag{25}$$

and

$$L_j = x_{ij} + x_{kj}. \tag{26}$$

Thus if $y_{kj} < y_{ij}$, we get a unique number $N$ where

$$N = L_ia_i + x_{kj}a_j \tag{27}$$

with cyclic permutations of subscripts.

*Case* 2. If

$$y_{kj} > y_{ij}, \tag{28}$$

we get another number where by symmetry

$$N' = L_ia_i + x_{jk}a_k \tag{29}$$

with cyclic permutations of subscripts. $N \neq N'$ since otherwise $x_{jk}a_k = x_{kj}a_j$ which implies $x_{jk} \geqslant a_j$, which by (25) contradicts (22). Note that these two

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numbers are the only numbers with properties (3) and (4) for (16), (17), and $t = 3$. Since $B$ is the largest number with property (3), it satisfies (4) so that $B$ is the maximum of $N$ and $N'$ and we have

THEOREM 4. *Given* (16) *and* (17), *then for cyclic permutations of subscripts*

$$B(a_1, a_2, a_3) = L_i a_i + \max (x_{kj}a_j, x_{jk}a_k) \tag{30}$$

*and* (26) *holds.*

Also it is easy to verify that $C$, the corresponding best bound for $G(a_1, a_2, a_3)$, satisfies

$$C(a_1, a_2, a_3) + a_1 + a_2 + a_3 = B(a_1, a_2, a_3). \tag{31}$$

**5. A computing algorithm for $L_i$ and $x_{ij}$.** Thus we have shown that finding $B$ is equivalent to finding the set of positive integers $L_i$ and $x_{ij}$ exhibited in the form of a matrix of detached coefficients of the three equations (15′) as follows:

| $a_1$ | $a_2$ | $a_3$ |
|---|---|---|
| $-L_1$ | $x_{12}$ | $x_{13}$ |
| $x_{21}$ | $-L_2$ | $x_{23}$ |
| $x_{31}$ | $x_{32}$ | $-L_3$ |

In order to develop a simple computing algorithm for these numbers, we need the following result.

LEMMA 2. *Given* $(a_1, a_2) = (a_2, a_3) = (a_3, a_1) = 1$, *then any system of integers* $K_i > 1$ *and* $v_{ij} > 0$ (*not necessarily* $L_i$ *and* $x_{ij}$) *satisfying* (15) *and* (26) $K_i = v_{ji} + v_{ki}$, *implies that*

$$K_iK_j - v_{ij}v_{ji} = v_{ji}v_{kj} + v_{ki}K_j = \lambda a_k \geqslant a_k \tag{32}$$

*for some positive integer* $\lambda$.

If we write

$$v_{jk}(K_i a_i - v_{ij}a_j) = v_{ik}v_{jk}a_k = v_{ik}(K_j a_j - v_{ji}a_i),$$

then

$$(v_{jk}K_i + v_{ik}v_{ji})a_i = (v_{ik}K_j + v_{jk}v_{ij})a_j$$

and (32) follows by (16) and (26).

Furthermore, we have

THEOREM 5. *If* (16) *and* (17) *hold, then the* $L_i$ *and* $x_{ij}$ *in Theorem* 4 *are characterized by the equations* (15′) *and* (26), *and*

$$L_iL_j + x_{ij}x_{ji} = a_k, \tag{33}$$

*for cyclic permutations of subscripts. That is,* $\lambda = 1$ *in* (32).

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*Proof.* Suppose a system of $K_i$ and $v_{ij}$ satisfy (15), (26), and (33) where at least one $K_i > L_i$, the minimum positive integer satisfying (15).

*Case* 1. If $K_1 = L_1$, $K_2 = L_2$, then $K_3 = L_3$ by (26) and Theorem 3.

*Case* 2. Suppose $K_1 = L_1$, but $K_2 > L_2$, $K_3 > L_3$.

Then $x_{12} = v_{12}$ and $x_{13} = v_{13}$ by Theorem 3 and by (15), (26), and (33) $a_1 = K_2K_3 - v_{32}v_{23} = K_2K_3 - (K_2 - x_{12})(K_3 - x_{13}) = x_{12}K_3 + x_{13}K_2 - x_{12}x_{13} > x_{12}L_3 + x_{13}L_2 - x_{12}x_{13} = L_2L_3 - x_{32}x_{23} \geqslant a_1$ by (32), a contradiction to the assumption that $K_2 > L_2$, $K_3 > L_3$.

*Case* 3. If $K_1 > L_1, L_2 > K_2, K_3 > L_3$, then first observe that either $v_{ij} > x_{ij}$ or $v_{ik} > x_{ik}$, but not both. For suppose $v_{ij} > x_{ij}$ and $v_{ik} > x_{ik}$. By (33) $v_{ij}v_{jk} + K_jv_{ik} = a_i \leqslant x_{ij}x_{jk} + L_jx_{ik}$ by (32). Thus $v_{jk} < x_{jk}$. Similarly $v_{ij}K_k + v_{ik}v_{kj} = a_i \leqslant x_{ij}L_k + x_{ik}x_{kj}$ so that $v_{kj} < x_{kj}$. But then $a_i \leqslant L_jL_k - x_{jk}x_{kj} < K_jK_k - v_{jk}v_{kj} = a_i$, a contradiction.

In addition either $v_{ji} > x_{ji}$ or $v_{ki} > x_{ki}$ but not both. For suppose $v_{ji} > x_{ji}$ and $v_{ki} > x_{ji}$. By the previous remark $v_{jk} < x_{jk}$, $v_{kj} < x_{kj}$, leading to the same contradiction obtained above. Thus either $v_{12}$, $v_{23}$, $v_{31}$, or $v_{21}$, $v_{32}$, $v_{13}$ are larger than the corresponding $x$'s. That is $v_{ij} > x_{ij}$ for cyclic permutations of subscripts.

Suppose $v_{21}$, $v_{32}$, $v_{13}$ are larger than $x_{21}$, $x_{32}$, $x_{13}$ respectively. Then by (26)

$$(K_2 - L_2)a_2 + (x_{23} - v_{23})a_3 = (v_{21} - x_{21})a_1 \geqslant L_1a_1$$

by the definition of $L_1$. Thus $v_{21} > L_1$ and by cyclic permutation of subscripts $v_{32} > L_2$, $v_{13} > L_3$.

Finally $a_3 \leqslant L_1L_2 - x_{12}x_{21} < L_1L_2 < v_{21}v_{32} < v_{21}v_{32} + K_2v_{31} = a_3$, a contradiction.

Thus $\lambda = 1$ in (32) implies that $K_i = L_i$, $v_{ij} = x_{ij}$.

Conversely, $\lambda = 1$ in (32), for $K_i = L_i$, $v_{ij} = x_{ij}$ etc. By the following computing algorithm we can always find sets of $K_i$ and $v_{ij}$ with $\lambda = 1$ in (32). Thus they are the desired $L_i$ and $x_{ij}$. Moreover since the $x_{ij}$ are unique by Theorem 3, $\lambda$ is unique and must equal 1.

The usefulness of Theorem 5 is apparent since it will be easier to find $K$'s and $v$'s satisfying (15), (26), and (33) rather than find minimal solutions to (15).

The algorithm follows. First we solve for any $a_k$ in terms of $a_i$ and $a_j$; for instance, for $k = 3$, giving

$$v_{21}a_1 - K_2a_2 + a_3 = 0 \tag{34}$$

with $0 < v_{21} < a_2$, $0 < K_2 < a_1$ by (10), easily done for example as in **(4)**.

Next construct

$$-K_1a_1 + v_{12}a_2 + v_{13}a_3 = 0 \tag{35}$$

where

$$v_{13} = \left[\frac{a_1}{K_2}\right], \qquad K_1 = a_2 - v_{21}v_{13}, a_1 = K_2v_{13} + v_{12}$$

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so that $\lambda = 1$ in (32). If $K_1 > v_{21}$, then $K_1 = L_1$, $K_2 = L_2$, and $L_3$ can be found by (26). Then apply Theorem 4 for $B(a_1, a_2, a_3)$. If $K_1 \leqslant v_{21}$, note that $K_1 \nmid v_{21}$. For if $K_1 | v_{21}$, then since $K_1 = a_2 - v_{21}v_{13}$, $K_1 | a_2$. But then in (34) $K_1 | a_3$. Thus $(a_2, a_3) \geqslant K_1 > 1$, by (17) a contradiction.

Therefore if $K_1 < v_{21}$ we can construct another equation

$$(v_{21} - pK_1)a_1 - (K_2 - pv_{12})a_2 + (1 + pv_{13})a_3 = 0 \tag{36}$$

with

$$p = \left[\frac{v_{21}}{K_1}\right].$$

Since $v_{21} - pK_1 > 0$, $K_2' = K_2 - pv_{12}$ forms a smaller value of $K_2$ in (34).

Note that the pair of equations giving the smallest values of $K_1$ and $K_2$ will still give $\lambda = 1$ in (32). At each stage we repeat the above generating of a smaller $K_1$ or $K_2$ until eventually $K_1 = L_1$, $K_2 = L_2$. By Theorem 5 this will come about when we obtain equations of the type (34) and (35) with $K_1 > v_{21}$ and $K_2 > v_{12}$.

To illustrate we find $B(137, 251, 256)$. First calculate that

$$a_1 - 75a_2 + 73a_3 = 0.$$

Then by the algorithm we obtain

$$\begin{aligned} 3a_1 + 31a_2 - 32a_3 &= 0, \\ 7a_1 - 13a_2 + 9a_3 &= 0, \\ 17a_1 + 5a_2 - 14a_3 &= 0. \end{aligned}$$

Thus the matrix of detached coefficients is

| $a_1$ | $a_2$ | $a_3$ |
|---|---|---|
| −24 | 8 | 5 |
| 7 | −13 | 9 |
| 17 | 5 | −14 |

and $B = 24a_1 + 9a_3 = 5{,}592$.

It should be pointed out that solving for (34) is not always necessary. Many computational short cuts become apparent after some practice. Note that the suggested algorithm is not merely numerical but gives algebraic relations as well, enabling one to solve all previously solved special cases for $t = 3$ by a unified approach. For example, see the end of the next section.

**6. Extensions and restatement of basic theorem.** Even if $L_3 = 1$, the statement of Theorems 4 and 5 still holds, dropping the minimality condition on the $L_i$. In this case, $B = a_1a_2 + a_3$, see **(2)**. But the matrix of coefficients is

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| $a_1$ | $a_2$ | $a_3$ |
|---|---|---|
| $-a_2$ | $a_1$ | 0 |
| $a_2 - x_{31}$ | $-a_1 - x_{32}$ | 1 |
| $x_{31}$ | $x_{32}$ | $-1$ |

with $x_{31} < a_2$. Then $\lambda = 1$, so that Theorem 5 gives the same result $a_1a_2 + a_3$.

Next we show that Theorems 4 and 5 hold even though the $a_i$'s are not reduced to a pairwise relatively prime set $b_1, b_2, b_3$.

We compare the $L$'s and $x_{ij}$'s associated with $a_1, a_2, a_3$ with those $L'$'s and $x_{ij}'$'s associated with $b_1, b_2, b_3$. From (15′), $L_ia_i = x_{ij}a_j + x_{ik}a_k$, we see that $d_{jk}|L_i$, $d_{ij}|x_{ik}$, $d_{ik}|x_{ij}$. Thus, setting $L_i = d_{jk}L_i'$, $x_{ik} = d_{ij}x_{ik}'$, we have

$$L_jL_k - x_{jk}x_{kj} = a_i \quad \text{if and only if} \quad L_j'L_k' - x_{jk}'x_{kj}' = b_i, \tag{35}$$

since $d_{ij}d_{ik}(L_j'L_k' - x_{jk}'x_{kj}') = d_{ij}d_{ik}b_i = a_i$.

Finally, all these results can be collected in the following form:

THEOREM 6. *For $(a_1, a_2, a_3) = 1$, define $B$ to be the largest number not of the form $xa_1 + ya_2 + za_3$, $x, y, z > 0$. Then for cyclic permutation of subscripts*

$$B = L_ia_i + \max\,(x_{jk}a_k, x_{kj}a_j),$$

*where*

$$L_ia_i = x_{ij}a_j + x_{ik}a_k, \qquad L_i > 0,\ x_{ij} \geqslant 0,\ x_{ik} \geqslant 0, \quad L_i = x_{ji} + x_{ki}$$

*and*

$$L_iL_j - x_{ij}x_{ji} = a_k.$$

The $L$'s and $x$'s can be found either by the computing algorithm discussed in §5, modified to solve first for $d_{ij}a_k$ in terms of $a_i$ and $a_j$, or by first applying Theorem 2.

In conclusion, observe that the special cases previously obtained for $t = 3$ can be derived directly from the results of this paper.

*Example.* We can extend the results stated in **(5)** for $B(a, a + 1, a + z)$. Write $a = kz - u$, $0 \leqslant u < z$, $k \geqslant 1$, $z \geqslant 2$. Then for $u \leqslant k + 1$ the coefficient matrix is

| $a = a_1 = kz - u$ | $a_2 = kz - u + 1$ | $a_3 = kz - u + z$ |
|---|---|---|
| $-(z + k - u)$ | $z - u$ | $k - 1$ |
| $z - 1$ | $-z$ | 1 |
| $k + 1 - u$ | $u$ | $-k$ |

If $u \leqslant 1$, then

$$B = L_3a_3 + x_{12}a_2 = \left(\frac{a + u}{z}\right)(a + z) + (z - u)(a + 1).$$

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To correspond to the notation of **(5)**, we solve for $C + 1 = B + 1 - \sum a_i$. Then

$$C + 1 = \left(\frac{a + u}{z}\right) a + (z - 2 - u)a.$$

If $u > 1$, then $B = L_3 a_3 + x_{21} a_1 = k(a + z) + (z - 1)a$, and

$$C + 1 = \left[\frac{a + 1}{z}\right] (a + z) + (z - 3)a$$

since

$$\left(\frac{a + u}{z}\right) = \left[\frac{a + 1}{z}\right] + 1.$$

For $t > 3$, Theorem 1 holds and the author has verified that relations analogous to Theorem 4 hold in many cases. However, this will be the subject of a later paper.

## References


1. P. T. Bateman, *Remark on a recent note on linear forms*, Amer. Math. Monthly, *65* (1958), 517–518.
2. A. T. Brauer, *On a problem of partitions—I*, Amer. J. Math., *64* (1942), 299–312.
3. A. T. Brauer and B. M. Seelbinder, *On a problem of partitions—II*, Amer. J. Math., *76* (1954), 343–346.
4. R. J. Levit, *A minimum solution for a diophantine equation*, Amer. Math. Monthly, *63* (1956), 646–651.
5. J. B. Roberts, *Note on linear forms*, Proc. Amer. Math. Soc. (1956), 465–469.
6. ——— *On a diophantine problem*, Can. J. Math., *9* (1957), 219–223.


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