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The second fundamental form of the real Kaehler submanifolds

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Abstract. Let $f: M^{2n} \to \mathbb{R}^{2n+p}$, $2 \le p \le n-1$, be an isometric immersion of a Kaehler manifold into Euclidean space. Yan and Zheng (2013, *Michigan Mathematical Journal* 62, 421–441) conjectured that if the codimension is $p \le 11$, then, along any connected component of an open dense subset of M^{2n}, the submanifold is as follows: it is either foliated by holomorphic submanifolds of dimension at least $2n - 2p$ with tangent spaces in the kernel of the second fundamental form whose images are open subsets of affine vector subspaces, or it is embedded holomorphically in a Kaehler submanifold of $\mathbb{R}^{2n+p}$ of larger dimension than $2n$. This bold conjecture was proved by Dajczer and Gromoll just for codimension 3 and then by Yan and Zheng for codimension 4. In this paper, we prove that the second fundamental form of the submanifold behaves pointwise as expected in case that the conjecture is true. This result is a first fundamental step for a possible classification of the nonholomorphic Kaehler submanifolds lying with low codimension in Euclidean space. A counterexample shows that our proof does not work for higher codimension, indicating that proposing $p = 11$ in the conjecture as the largest codimension is appropriate.

1 Introduction

An isometric immersion $f: M^{2n} \to \mathbb{R}^{2n+p}$ is called a *real Kaehler submanifold* if (M^{2n}, J) is a connected Kaehler manifold of complex dimension $n \ge 2$ isometrically immersed into Euclidean space with local substantial codimension p. The latter means that the image of f restricted to any open subset of M^{2n} does not lie in a proper affine subspace of $\mathbb{R}^{2n+p}$. Moreover, when p is even, we focus in the case in which f restricted to any open subset of M^{2n} is *not* holomorphic with respect to any complex structure of the ambient space $\mathbb{R}^{2n+p}$.

Since the pioneering work by Dajczer and Gromoll [8], there has been an increasing interest in the study of the real Kaehler submanifolds. The reason, in good part, it is due because when these submanifolds are minimal then they enjoy several of the feature properties of minimal surfaces. For instance, they admit an associated one-parameter family of noncongruent isometric minimal submanifolds all with the same Gauss map.

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Another one is being the real part of its holomorphic representative. Moreover, the immersions are pluriharmonic maps and, in some cases, they admit a Weierstrass-type representation. For a partial account of results on this subject of research, as well as many references, we refer to [12].

There is plenty of knowledge on real Kaehler submanifolds $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ when the codimension is as low as $p = 1, 2$. For instance, in the hypersurface case, there is the local parametric classification obtained in [8] that can be seen in [12] as Theorem 15.14. The classification of the metrically complete submanifolds with codimension $p = 2$ follows from [9, 15]. Moreover, for both codimensions, the submanifolds carry a foliation by complex relative nullity leaves of dimension $2n - 2p$ as described next.

Let $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ be a real Kaehler submanifold, and let $L \subset N_fM(x)$ be a normal vector subspace at $x \in M^{2n}$. We denote the $\alpha_L\colon TM \times TM \to L$ the L-component of its normal vector-valued second fundamental form $\alpha\colon TM \times TM \to N_fM$ and by $\mathcal{N}(\alpha_L) \subset T_xM$ the tangent vector subspace

$$\mathcal{N}(\alpha_L) = \{Y \in T_xM\colon \alpha_L(X, Y) = 0 \text{ for any } X \in T_xM\}.$$

Then $\Delta(x) = \mathcal{N}(\alpha_{N_fM(x)})$ is called the relative nullity subspace of f at $x \in M^{2n}$. Its complex part $\Delta_c(x) = \Delta(x) \cap J\Delta(x)$ is named the *complex relative nullity* subspace whose dimension $\nu_f^c(x)$ is the *index of complex relative nullity*. It is well known that the vector subspaces $\Delta_c(x)$ form a smooth integrable distribution on any open subset of M^{2n} where $\nu_f^c(x)$ is constant. Moreover, the totally geodesic leaves are holomorphic submanifolds of M^{2n} as well as open subsets of even-dimensional affine vector subspaces of $\mathbb{R}^{2n+p}$.

Real Kaehler submanifolds in codimension at least 3 can be obtained just by considering holomorphic submanifolds of a given real Kaehler submanifold. More precisely, let $F\colon N^{2n+2m} \to \mathbb{R}^{2n+p}$, $m \geq 1$, be a real Kaehler submanifold, and then let $j\colon M^{2n} \to N^{2n+2m}$ be any holomorphic isometric immersion. Then the *composition* of isometric immersions $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ given by

$$f = F \circ j\colon M^{2n} \to \mathbb{R}^{2n+p} \tag{1.1}$$

is a real Kaehler submanifold.

It is clearly relevant to establish conditions asserting that a real Kaehler submanifold is locally a composition as in (1.1). This was achieved for $p = 3$ by Dajczer and Gromoll [10] and for $p = 4$ by Yan and Zheng [16] under the assumption that the index of complex relative nullity of $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ satisfies $\nu_f^c(x) < 2n - 2p$ at any $x \in M^{2n}$. The result in the latter paper was complemented by us in [5].

A bold conjecture by Yan and Zheng in [16] states, under the same assumption as above on the index of complex relative nullity, that any real Kaehler submanifold in codimension $p \leq 11$ is a composition as in (1.1) along connected components of an open dense subset of M^{2n}. The purpose of this paper is to walk a fundamental step in order to treat that rather challenging conjecture. We prove that the second fundamental form of the submanifold behaves pointwise as expected if the conjecture were true. Moreover, we have that our proof fails for $p = 12$, indicating that proposing $p = 11$ in the conjecture as the largest codimension seems appropriate. For codimension $p \leq 6$, our result was obtained in [2] up to some inconsistencies in the argument (see Remark 3.10). As for higher codimension, it is shown by this paper that the proof is much more difficult.

Before stating our main theorem, we roughly explain why this result turns out to be the one we expected. For this purpose, let $f = F \circ j\colon M^{2n} \to \mathbb{R}^{2n+p}$ be a composition, as in (1.1) but where F itself is not such a composition. Then the second fundamental form $\alpha^f\colon TM \times TM \to N_fM$ of f splits as the sum of the second fundamental forms α^j of j and α^F of F that one restricted to TM, and where both components need to satisfy certain conditions now discussed. On one hand, there is a vector bundle isometry $\mathcal{J} \in \Gamma(Aut(\Omega))$ such that

$$\mathcal{J}\alpha^f_\Omega(X,Y) = \alpha^f_\Omega(X,JY) \text{ for any } X,Y \in \mathfrak{X}(M), \tag{1.2}$$

where $\Omega = F_*N_jM$. In fact, being j holomorphic, we have that $j_*JX = J^N j_*X$ for any $X \in \mathfrak{X}(M)$. Differentiating once and then taking normal component yields $J^N\alpha^j(X,Y) = \alpha^j(X,JY)$. Then $\mathcal{J}F_*|_{N_jM} = F_*J^N|_{F_*N_jM}$ satisfies the requirement. On the other hand, since F is not a composition, then α^F should have a large index of complex relative nullity, and hence the same remains to be the case when it is restricted to TM.

Let $N_1(x) \subset N_fM(x)$ denote the vector subspace spanned at $x \in M^{2n}$ by the second fundamental form of f, namely, $N_1(x) = \operatorname{span}\{\alpha(X,Y)\colon X,Y \in T_xM\}$. It is usually called the first normal space of f at $x \in M^{2n}$. Then let $Q(x) \subset N_1(x)$ be the complex vector subspace defined as

$$Q(x) = \{\eta \in N_1(x)\colon \langle \eta, \alpha(Z,T)\rangle = \langle \bar\eta, \alpha(Z,JT)\rangle \text{ for any } Z,T \in T_xM\},$$

where if $\eta = \sum_{i=1}^k \alpha(X_i,Y_i)$, then $\bar\eta = \sum_{i=1}^k \alpha(X_i,JY_i)$.

Theorem 1.1 *Let $f\colon M^{2n} \to \mathbb{R}^{2n+p}$, $2 \le p \le n-1$, be a real Kaehler submanifold whose index of complex relative nullity satisfies $\nu^c_f(x_0) < 2n-2p$ at a point $x_0 \in M^{2n}$. If $p \le 11$, then the following facts hold:*

(i) *If $Q = Q(x_0)$, then* $\dim Q = \ell > 0$, *and there is an isometry $\mathcal{J} \in Aut(Q)$ such that*

$$\mathcal{J}\alpha_Q(X,Y) = \alpha_Q(X,JY) \text{ for any } X,Y \in T_{x_0}M.$$

(ii) *If $N_1(x_0) = Q \oplus P$ is an orthogonal decomposition, then $\nu^c(\alpha_P) \ge 2(n-p+\ell)$.*

If the submanifold satisfies $\dim N_1(x_0) = q < p$, then the proof of Theorem 1.1 gives a stronger result. Indeed, one can replace the assumption $p \le 11$ by $q \le 11$ and assume in part (ii) that $\nu^c_f(x_0) < 2n - 2q$.

The extrinsic assumption on index of complex relative nullity in Theorem 1.1 can be replaced by an intrinsic hypothesis, namely, there is no complex vector subspace $L^{2n-2p} \subset T_{x_0}M$ such that the sectional curvature satisfies $K_M(P) = 0$ for any plane $P^2 \subset L^{2n-2p}$. Notice that part (i) gives that $\mathcal{J}$ is a complex structure, that is, that we have $\mathcal{J}^2 = -I$. It also yields that $\alpha_Q(JX,Y) = \alpha_Q(X,JY)$ holds for any $X,Y \in T_{x_0}M$. Finally, we observe that the inequality $\nu^c_f(x) < 2n-2p$ holds in a neighborhood of x_0 in M^{2n}.

Although the above result can be seen as a validation of the Yan and Zheng conjecture at the level of the structure of the second fundamental form of the submanifold, it is a distance apart from proving that the conjecture is true. In fact, we believe that

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for codimensions $p \geq 7$, there is just one other possibility, namely, that we may have complex ruled submanifolds that are not compositions. By being complex ruled, we mean that there is a holomorphic foliation of M^{2n} such that the image by f of each leaf is part of an affine vector subspace of $\mathbb{R}^{2n+p}$, but it does not have to be part of the complex relative nullity.

An immediate application of Theorem 1.1 is the following result under a pinching curvature condition.

Theorem 1.2 *Let $f: M^{2n} \to \mathbb{R}^{2n+p}$, $2 \leq p \leq n-1$, be a real Kaehler submanifold whose index of complex relative nullity satisfies $\nu_f^c(x_0) < 2n - 2p$ at $x_0 \in M^{2n}$. If $p \leq 11$, there is a neighborhood U of x_0 such that at any point $x \in U$, there is a complex vector subspace $L^{2m} \subset T_x M$ with $m \geq n - p + \ell$ where $\dim Q(x) = \ell > 0$ such that for any complex plane $P^2 \subset L^{2m}$, the sectional curvature satisfies $K_M(P) \leq 0$.*

For $p \leq n$ and without the assumption on the index of complex relative nullity, the weaker estimate $m \geq n - p$ was given as Corollary 15.6 in [12].

Finally, we observe that if p is even and $f: M^{2n} \to \mathbb{R}^{2n+p}$ is holomorphic with respect to some complex structure in the ambient space, then $Q(x) = N_1(x)$ holds everywhere and hence both results given above are trivial.

2 Preliminaries

This section provides several basic facts used throughout the paper.

Let $\varphi: V_1 \times V_2 \to W$ denote a bilinear form between real vector spaces of finite dimension. The image of φ is the vector subspace of W defined by

$$\mathcal{S}(\varphi) = \text{span}\,\{\varphi(X, Y) \text{ for all } X \in V_1 \text{ and } Y \in V_2\},$$

whereas the (right) *nullity* of φ is the vector subspace of V_2 given by

$$\mathcal{N}(\varphi) = \{Y \in V_2 : \varphi(X, Y) = 0 \text{ for all } X \in V_1\}$$

whose dimension $\nu(\varphi)$ is the index of nullity of φ.

A vector $X \in V_1$ is called a (left) regular element of φ if $\dim \varphi_X(V_2) = \kappa(\varphi)$ where

$$\kappa(\varphi) = \max_{X \in V_1}\{\dim \varphi_X(V_2)\}$$

and $\varphi_X: V_2 \to W$ is the linear map defined by

$$\varphi_X Y = \varphi(X, Y).$$

Then $RE(\varphi) \subset V_1$ denotes the subset of regular elements of φ. Given $X \in RE(\varphi)$, then the vector subspace $N(X) = \ker \varphi_X$ satisfies

$$\dim N(X) = \dim V_2 - \kappa(\varphi). \tag{2.1}$$

Let W be endowed with an inner product of any signature. Then we denote

$$\mathcal{U}(X) = \varphi_X(V_2) \cap \varphi_X(V_2)^{\perp},$$

$\tau_\varphi(X) = \dim \mathcal{U}(X)$, and $\tau(\varphi) = \min_{X \in RE(\varphi)}\{\tau_\varphi(X)\}$.

The following result will be used throughout the paper without further reference.

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Proposition 2.1 *The following facts hold:*

(i) *The subset $RE(\varphi) \subset V_1$ is open and dense.*

(ii) *If $V_1 = V_2 = V$ and φ is symmetric, then*

$$RE^*(\varphi) = \{X \in RE(\varphi) : \varphi(X,X) \neq 0\}$$

is an open dense subset of V.

(iii) *If W is endowed with an inner product, then*

$$RE^\#(\varphi) = \{X \in RE(\varphi) : \tau_\varphi(X) = \tau(\varphi)\}$$

is an open dense subset of V_1.

Proof Part (i) is Proposition 4.4 in [12], whereas the proof of Lemma 2.1 in [11] gives part (iii). An easy argument gives part (ii), for instance, see the proof of Lemma 4.5 in [12]. ■

Let W be endowed with the inner product $\langle\langle\ ,\ \rangle\rangle$. Then the bilinear form φ is said to be *flat* if

$$\langle\langle \varphi(X,Y), \varphi(Z,T)\rangle\rangle - \langle\langle \varphi(X,T), \varphi(Z,Y)\rangle\rangle = 0$$

for any $X, Z \in V_1$ and $Y, T \in V_2$. It is said that φ is *null* if

$$\langle\langle \varphi(X,Y), \varphi(Z,T)\rangle\rangle = 0$$

for any $X, Z \in V_1$ and $Y, T \in V_2$.

Given $X \in RE(\varphi)$, we denote

$$\mathcal{L}(X) = \mathcal{S}(\varphi|_{V_1 \times N(X)}).$$

Then let $\sigma_\varphi(X) = \dim \mathcal{L}(X)$ and $\sigma(\varphi) = \min_{X \in RE(\varphi)}\{\dim \sigma_\varphi(X)\}$.

Proposition 2.2 *If $X \in RE(\varphi)$, then $\mathcal{L}(X) \subset \varphi_X(V_2)$. Moreover, if φ is flat, then*

$$\mathcal{L}(X) \subset \mathcal{U}(X) \tag{2.2}$$

and thus $\sigma(\varphi) \leq \sigma_\varphi(X) \leq \tau_\varphi(X)$.

Proof See Proposition 4.6 in [12]. ■

Let U^p be a p-dimensional vector space induced with a positive definite inner product $\langle\ ,\ \rangle$. Set $W^{p,p} = U^p \oplus U^p$, and let $\pi_1\colon W^{p,p} \to U^p$ (resp. π_2) denote taking the first (resp. second) component of $W^{p,p}$. Then let $W^{p,p}$ be endowed with the inner product $\langle\langle\ ,\ \rangle\rangle$ of signature (p,p) given by

$$\langle\langle (\xi_1,\xi_2),(\eta_1,\eta_2)\rangle\rangle = \langle \xi_1,\eta_1\rangle - \langle \xi_2,\eta_2\rangle.$$

Then $\mathcal{T} \in \operatorname{Aut}(W)$ defined by

$$\mathcal{T}(\xi,\eta) = (\eta,-\xi) \tag{2.3}$$

is a complex structure, which means that $\mathcal{T}^2 = -I$. Moreover, it holds that

$$\langle\langle \mathcal{T}\delta, \nu\rangle\rangle = \langle\langle \delta, \mathcal{T}\nu\rangle\rangle.$$

A vector subspace $L \subset W^{p,p}$ is called *degenerate* if $L \cap L^\perp \neq 0$ and *nondegenerate* if otherwise. A degenerate vector subspace $L \subset W^{p,p}$ is called *isotropic* if $L = L \cap L^\perp$.

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Proposition 2.3 *Given a vector subspace $L \subset W^{p,p}$, there is a direct sum decomposition*

$$W^{p,p} = \mathcal{U}^r \oplus \hat{\mathcal{U}}^r \oplus \mathcal{V}^{p-r,p-r}, \tag{2.4}$$

where $\mathcal{U}^r = L \cap L^\perp$, the vector subspace $\hat{\mathcal{U}}^r$ is isotropic, the vector subspace $\mathcal{U}^r \oplus \hat{\mathcal{U}}^r$ is nondegenerate, and $L \subset \mathcal{U}^r \oplus \mathcal{V}^{p-r,p-r}$, where $\mathcal{V}^{p-r,p-r} = (\mathcal{U}^r \oplus \hat{\mathcal{U}}^r)^\perp$.

Proof See Sublemma 2.3 in [3] or Corollary 4.3 in [12]. ∎

Remark 2.4 In the decomposition (2.4), only $\mathcal{U}^r$ is completely determined by L. In fact, if $\hat{\mathcal{U}}^r = \text{span}\{\xi_1, \ldots, \xi_r\}$, then any alternative description is as span $\{\xi_1 + \delta_1, \ldots, \xi_r + \delta_r\}$ where $\{\delta_1, \ldots, \delta_r\}$ is any set of vectors belonging to $\mathcal{V}^{p-r,p-r}$ that span an isotropic subspace.

Let the vector space V_2 carry a complex structure $J \in \text{Aut}(V_2)$. It is a standard fact that V_2 is even-dimensional and admits a basis of the form $\{X_j, JX_j\}_{1\le j\le n}$. Assume that the bilinear form $\varphi: V_1 \times V_2 \to W^{p,p}$ satisfies that

$$\mathcal{T}\varphi(X,Y) = \varphi(X,JY) \ \text{ for any } \ X \in V_1 \text{ and } Y \in V_2 \tag{2.5}$$

and let $W^{p,p} = \mathcal{U} \oplus \hat{\mathcal{U}} \oplus \mathcal{V}$ be the decomposition given by (2.4) for $L = \mathcal{S}(\varphi)$. Then we have $\mathcal{T}\mathcal{U} = \mathcal{U}$. In effect, if $\langle\langle \varphi(X,Y), (\xi,\bar{\xi})\rangle\rangle = 0$ for any $X \in V_1$ and $Y \in V_2$, then

$$\langle\langle \varphi(X,Y), \mathcal{T}(\xi,\bar{\xi})\rangle\rangle = \langle\langle \mathcal{T}\varphi(X,Y), (\xi,\bar{\xi})\rangle\rangle = \langle\langle \varphi(X,JY), (\xi,\bar{\xi})\rangle\rangle = 0.$$

Proposition 2.5 *The following facts hold:*

(i) *$\mathcal{T}|_{\mathcal{S}(\varphi)} \in Aut\,(\mathcal{S}(\varphi))$ and $\mathcal{T}|_{\mathcal{U}} \in Aut(\mathcal{U})$ are complex structures.*
(ii) *The vector subspaces $\mathcal{S}(\varphi)$ and $\mathcal{U}$ of $W^{p,p}$ have even dimension.*
(iii) *The vector subspace $\mathcal{N}(\varphi) \subset V_2$ is J-invariant and thus of even dimension.*
(iv) *If $\Omega = \pi_1(\mathcal{U})$, then $\dim \mathcal{U} = \dim \Omega$ and if $\varphi_\Omega = \pi_{\Omega\times\Omega} \circ \varphi$ then $\mathcal{S}(\varphi_\Omega) = \mathcal{U}$.*

Proof The considerations given above yield parts (i)–(iii). Being the subspace $\mathcal{U}$ isotropic, then $\pi_1|_{\mathcal{U}}: \mathcal{U} \to \Omega$ is an isomorphism. Since $\mathcal{T}\mathcal{U} = \mathcal{U}$ gives that $\pi_2(\mathcal{U}) = \Omega$, then part (iv) follows. ∎

Proposition 2.6 *Let the bilinear form $\varphi: V^{2n} \times V^{2n} \to W^{p,p}$ be symmetric and satisfy the condition (2.5). Then*

$$4 \dim \mathcal{S}(\varphi) \le \kappa(\varphi)(\kappa(\varphi) + 2). \tag{2.6}$$

Proof Since $\mathcal{T}|_{\varphi_X(V)}$ is a complex structure, then $\kappa(\varphi) = 2m$. Fix $X \in RE^*(\varphi)$, and let $\{X_j, JX_j\}_{1\le j\le n}$ be a basis of V^{2n} with $X_1 = X$ such that

$$\varphi_X(V) = \text{span}\{\varphi_X X_j, \varphi_X JX_j, 1 \le j \le m\}$$

and $X_r, JX_r \in \ker \varphi_X$ for $r \ge m+1$. Since Proposition 2.2 yields $\mathcal{S}(\varphi|_{V\times\ker\varphi_X}) \subset \varphi_X(V)$, then given $Z \in V^{2n}$ and $q \ge m+1$, there is $Y \in \text{span}\{X_j, JX_j, 1 \le j \le m\}$ such that

$$\varphi(Z, X_q) = \varphi(X_1, Y) \ \text{ and } \ \varphi(Z, JX_q) = \varphi(X_1, JY).$$

Being φ symmetric, we have

$$\varphi(X, JY) = \mathcal{T}\varphi(X,Y) = \mathcal{T}\varphi(Y,X) = \varphi(Y,JX) = \varphi(JX,Y)$$

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for any $X, Y \in V^{2n}$. Hence,

$$\mathcal{S}(\varphi) = \text{span}\,\{\varphi(X_i, X_j), \varphi(X_i, JX_j), 1 \le i \le j \le m\},$$

and (2.6) follows. ∎

3 The proofs

In this section, we first give a general result in the theory of flat bilinear forms tailored for our purposes in this paper. After that, we prove both results that have been stated in the Introduction.

Let $\alpha\colon V^{2n} \times V^{2n} \to U^p$ be a symmetric bilinear form, and let $J \in \text{Aut}(V)$ be a complex structure. Then let $\gamma\colon V^{2n} \times V^{2n} \to W^{p,p}$ be the associated bilinear form defined by

$$\gamma(X, Y) = (\alpha(X, Y), \alpha(X, JY)). \tag{3.1}$$

Then γ is symmetric if and only if α is *pluriharmonic* with the latter meaning that

$$\alpha(JX, Y) = \alpha(X, JY) \ \text{ for any } \ X, Y \in V^{2n}.$$

If $\mathcal{T} \in \text{Aut}(W)$ is the complex structure given by (2.3), then

$$\mathcal{T}\gamma(X, Y) = \gamma(X, JY) \ \text{ for any } \ X, Y \in V^{2n} \tag{3.2}$$

and thus Proposition 2.5 applies to γ.

Let $\beta\colon V^{2n} \times V^{2n} \to W^{p,p}$ be the bilinear form defined by

$$\begin{aligned} \beta(X, Y) &= \gamma(X, Y) + \gamma(JX, JY) \\ &= (\alpha(X, Y) + \alpha(JX, JY), \alpha(X, JY) - \alpha(JX, Y)). \end{aligned} \tag{3.3}$$

By (3.2), we have that

$$\mathcal{T}\beta(X, Y) = \beta(X, JY) \ \text{ for any } \ X, Y \in V^{2n}, \tag{3.4}$$

and hence Proposition 2.5 applies to β. Then part (iii) gives that $\nu(\beta)$ is even. We observe that $\nu(\beta)$ was called in [14] the index of pluriharmonic nullity since it satisfies

$$\mathcal{N}(\beta) = \{Y \in V^{2n} : \alpha(X, JY) = \alpha(JX, Y) \text{ for all } X \in V^{2n}\}.$$

Theorem 3.1 *Let the bilinear forms $\gamma, \beta\colon V^{2n} \times V^{2n} \to W^{p,p}$, $p \le n$, be flat and satisfy*

$$\langle\!\langle \beta(X, Y), \gamma(Z, T) \rangle\!\rangle = \langle\!\langle \beta(X, T), \gamma(Z, Y) \rangle\!\rangle \ \textit{ for any } \ X, Y, Z, T \in V^{2n}. \tag{3.5}$$

If $p \le 11$ and $\nu(\gamma) < 2n - \dim \mathcal{S}(\gamma)$, then the vector subspace $\mathcal{S}(\gamma)$ is degenerate. Moreover, if $\Omega = \pi_1(\mathcal{S}(\gamma) \cap \mathcal{S}(\gamma)^\perp)$ and $U^p = \Omega \oplus P$ is an orthogonal decomposition, then the following holds:

(i) *There is an isometric complex structure $\mathcal{J} \in End(\Omega)$ so that $\alpha_\Omega = \pi_\Omega \circ \alpha$ satisfies*

$$\mathcal{J}\alpha_\Omega(X, Y) = \alpha_\Omega(X, JY) \ \textit{ for any } \ X, Y \in V^{2n}.$$

(ii) *The bilinear form $\gamma_P = \pi_{P \times P} \circ \gamma$ is flat, the vector subspace $\mathcal{S}(\gamma_P)$ is nondegenerate, and $\nu(\gamma_P) \ge 2n - \dim \mathcal{S}(\gamma_P)$.*

The proof of Theorem 3.1 will require several lemmas.

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Lemma 3.2 *Let the bilinear form* $\gamma: V^{2n} \times V^{2n} \to W^{p,p}$ *be symmetric and flat. If* $p \leq 11$ *and* $\mathcal{S}(\gamma) = W^{p,p}$*, then*

$$\nu(\gamma) \geq 2n - \kappa(\gamma) - \sigma(\gamma) \geq 2n - 2p. \tag{3.6}$$

Proof We argue for the most difficult case $p = 11$ being the other cases similar but easier as p decreases. The first inequality in (3.6) just means that

$$\nu(\gamma) \geq 2n - \kappa(\gamma) - \sigma_\gamma(X) \ \text{ for any } \ X \in RE(\gamma).$$

Thus, for what follows, we fix $X \in RE(\gamma)$ and prove the latter. Proposition 2.3 yields

$$W^{p,p} = \mathcal{U}^\tau(X) \oplus \hat{\mathcal{U}}^\tau(X) \oplus \mathcal{V}^{p-\tau,p-\tau}(X), \tag{3.7}$$

where $\mathcal{U}^\tau(X) = \gamma_X(V) \cap \gamma_X(V)^\perp$, $\gamma_X(V) \subset \mathcal{U}^\tau(X) \oplus \mathcal{V}^{p-\tau,p-\tau}(X)$, and $\tau = \tau_\gamma(X)$ for simplicity. Thus, $\kappa(\gamma) \leq 2p - \tau$. Then (2.2) yields $\kappa(\gamma) + \sigma_\gamma(X) \leq \kappa(\gamma) + \tau \leq 2p$, which gives the second inequality in (3.6).

The vector subspace $\mathcal{U}^\tau(X)$ is zero or is by Proposition 2.5 isotropic of even dimension. It follows from (2.2) that $0 \leq \sigma \leq \tau \leq 10$ where $\sigma = \sigma_\gamma(X)$ for simplicity of notation.

If $\sigma = 0$, that is, we have $N(X) = \mathcal{N}(\gamma)$ and then (3.6) follows from (2.1). Hence, we assume $\sigma > 0$. Moreover, using first part (iii) and then part (ii) of Proposition 2.5, we obtain that σ is even. Thus, henceforth, we assume $\sigma \geq 2$.

In view of (2.2), there is a decomposition

$$\mathcal{U}^\tau(X) \oplus \hat{\mathcal{U}}^\tau(X) = \mathcal{L}^\sigma(X) \oplus \hat{\mathcal{L}}^\sigma(X) \oplus \mathcal{V}_0^{\tau-\sigma,\tau-\sigma}, \tag{3.8}$$

where $\hat{\mathcal{L}}^\sigma(X) \subset \hat{\mathcal{U}}^\tau(X)$ is such that the vector subspace $\mathcal{V}_0^{\tau-\sigma,\tau-\sigma} = (\mathcal{L}^\sigma(X) \oplus \hat{\mathcal{L}}^\sigma(X))^\perp$ is nondegenerate. We denote $\hat{\gamma} = \pi_{\hat{\mathcal{L}}^\sigma(X)} \circ \gamma$ and show that $\mathcal{T}\hat{\gamma}(Y,Z) = \hat{\gamma}(Y,JZ)$, that is, that

$$\mathcal{T}|_{\mathcal{S}(\hat{\gamma})} \hat{\gamma}_Y Z = \hat{\gamma}_Y JZ \ \text{ for any } \ Y, Z \in V^{2n}. \tag{3.9}$$

Hence, $\mathcal{T}|_{\mathcal{S}(\hat{\gamma})}$ is a complex structure and $\kappa_0 = \kappa(\hat{\gamma})$ is even. Part (iii) of Proposition 2.5 gives that $N(X)$ is J-invariant. If $(\xi, \bar{\xi}) \in \mathcal{L}^\sigma(X)$, then part (i) of Proposition 2.5 applied to $\varphi = \gamma|_{V \times N(X)}$ yields that $\mathcal{T}(\xi, \bar{\xi}) \in \mathcal{L}^\sigma(X)$. Using (3.7) and (3.8), we have

$$\begin{aligned}\langle\langle \mathcal{T}\hat{\gamma}(Y,Z), (\xi, \bar{\xi}) \rangle\rangle &= \langle\langle \hat{\gamma}(Y,Z), \mathcal{T}(\xi, \bar{\xi}) \rangle\rangle = \langle\langle \gamma(Y,Z), \mathcal{T}(\xi, \bar{\xi}) \rangle\rangle = \langle\langle \gamma(Y,JZ), (\xi, \bar{\xi}) \rangle\rangle \\ &= \langle\langle \hat{\gamma}(Y,JZ), (\xi, \bar{\xi}) \rangle\rangle\end{aligned}$$

for any $(\xi, \bar{\xi}) \in \mathcal{L}^\sigma(X)$ and this gives (3.9).

We have that

$$\sigma \leq \tau(\gamma). \tag{3.10}$$

In effect, it follows from (2.1) that the dimension of $N(Y)$ on $RE(\gamma)$ is constant. Then, by continuity, $\sigma \leq \sigma_\varphi(Y)$ in a neighborhood of X in $RE(\gamma)$. On the other hand, we obtain from (2.2) that $\sigma_\varphi(Y) \leq \tau(\gamma)$ for any $Y \in RE^\#(\gamma)$ which is open and dense in V^{2n}. Then (3.10) follows.

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Claim *Given* $Z \in V^{2n}$, *then* $\dim \gamma_Z(N(X))$ *is even and*

$$\dim \gamma_Z(N(X)) \le p - \kappa_0 - \tau(\gamma) + \sigma \le p - \kappa_0. \tag{3.11}$$

■

That $\dim \gamma_Z(N(X))$ is even follows from parts (ii) and (iii) of Proposition 2.5, whereas (3.10) yields the second inequality in (3.11).

To prove the first inequality in (3.11), it suffices to argue for $Z \in RE^{\#}(\gamma) \cap RE(\hat{\gamma})$ since this subset of V^{2n} is open and dense. Let $V_0 \subset V^{2n}$ be the vector subspace $V_0 = \gamma_Z^{-1}(\mathcal{L}^\sigma(X))$ and $s_0 = \dim \gamma_Z(V_0)$. Since $N(X) \subset V_0$ by (2.2), then $r \le s_0$, where $r = \dim \gamma_Z(N(X))$. Because $Z \in RE(\hat{\gamma})$, there is a vector subspace $V_1^{\kappa_0} \subset V^{2n}$ satisfying $\hat{\gamma}_Z(V_1) = \hat{\gamma}_Z(V)$. Since any vector in $\gamma_Z(V_1)$ has a nonzero $\hat{\mathcal{L}}^\sigma(X)$-component, then

$$\gamma_Z(V_0) \cap \gamma_Z(V_1) = 0. \tag{3.12}$$

Let $Y_0 \in V_0$ satisfy $\gamma_Z Y_0 \in \mathcal{U}^{\bar{\tau}}(Z)$, where $\bar{\tau} = \tau(\gamma)$ for simplicity of notation. Since $\gamma_Z(V_0) \subset \mathcal{L}^\sigma(X)$ and $\hat{\gamma}_Z(V_1) \subset \hat{\mathcal{L}}^\sigma(X)$, then using (3.8), we have

$$\langle\langle \gamma_Z Y_0, \hat{\gamma}_Z(V_1)\rangle\rangle = \langle\langle \gamma_Z Y_0, \gamma_Z(V_1)\rangle\rangle = 0.$$

Hence,

$$\dim \gamma_Z(V_0) \cap \mathcal{U}^{\bar{\tau}}(Z) \le \sigma - \kappa_0. \tag{3.13}$$

Let $Y_1 \in V_1^{\kappa_0}$ satisfy $\gamma_Z Y_1 \in \mathcal{U}^{\bar{\tau}}(Z)$. Since $\gamma_Z(V_0) \subset \mathcal{L}^\sigma(X)$ and $\hat{\gamma}_Z(V_1) \subset \hat{\mathcal{L}}^\sigma(X)$, then (3.8) gives

$$\langle\langle \hat{\gamma}_Z Y_1, \gamma_Z(V_0)\rangle\rangle = \langle\langle \gamma_Z Y_1, \gamma_Z(V_0)\rangle\rangle = 0.$$

Hence, $\dim \pi_{\hat{\mathcal{L}}(X)}(\gamma_Z(V_1) \cap \mathcal{U}^{\bar{\tau}}(Z)) \le \sigma - s_0$. But since $V_1^{\kappa_0}$ has been chosen to satisfy that $\pi_{\hat{\mathcal{L}}(X)}|_{\gamma_Z(V_1)}$ is injective, then

$$\dim \gamma_Z(V_1) \cap \mathcal{U}^{\bar{\tau}}(Z) \le \sigma - s_0. \tag{3.14}$$

The decomposition (3.7) for Z yields $\gamma_Z(V) \subset \mathcal{U}^{\bar{\tau}}(Z) \oplus \mathcal{V}^{p-\bar{\tau},p-\bar{\tau}}(Z)$. Let the vector subspace $\mathcal{R} \subset \gamma_Z(V)$ be such that $\gamma_Z(V) = (\gamma_Z(V) \cap \mathcal{U}^{\bar{\tau}}(Z)) \oplus \mathcal{R}$. Since any vector in $\mathcal{R}$ has a nonzero $\mathcal{V}^{p-\bar{\tau},p-\bar{\tau}}(Z)$-component, then $\pi_{\mathcal{V}(Z)}|_{\mathcal{R}}$ is injective.

Set $\mathcal{S} = \pi_{\mathcal{V}(Z)}(\gamma_Z(V_0) \cap \mathcal{R})$ and $\hat{\mathcal{S}} = \pi_{\mathcal{V}(Z)}(\gamma_Z(V_1) \cap \mathcal{R})$. Since $\dim \gamma_Z(V_0) = s_0$ and $\dim \gamma_Z(V_1) = \kappa_0$, it follows from (3.13) and (3.14) that $\dim \mathcal{S}, \dim \hat{\mathcal{S}} \ge \kappa_0 - \sigma + s_0$. Let $\delta \in \mathcal{S} \cap \hat{\mathcal{S}}$. Then $\delta = \pi_{\mathcal{V}(Z)}(\gamma_Z Y_i)$, where $Y_i \in V_i$ and $\gamma_Z Y_i \in \mathcal{R}$, $i = 0, 1$. By (3.12) and the injectivity of $\pi_{\mathcal{V}(Z)}|_{\mathcal{R}}$, we have that $\gamma_Z Y_1 = \gamma_Z Y_0 = 0$. Thus, $\delta = 0$, and hence

$$\dim \mathcal{S} \oplus \hat{\mathcal{S}} \ge 2(\kappa_0 - \sigma + s_0).$$

Since $r \le s_0$, then that

$$2(\kappa_0 - \sigma + r) \le 2(\kappa_0 - \sigma + s_0) \le \dim \mathcal{V}^{p-\bar{\tau},p-\bar{\tau}}(Z)$$

concludes the proof of the claim.

Since $\mathcal{S}(\gamma) = W^{p,p}$, it holds that

$$\mathcal{S}(\hat{\gamma}) = \hat{\mathcal{L}}^\sigma(X). \tag{3.15}$$

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From (3.15) and $\sigma \geq 2$, we have $\hat{\gamma} \neq 0$. Since α is pluriharmonic, then γ symmetric and hence also is $\hat{\gamma}$. Thus, (2.6), (3.9), and (3.15) yield that

$$4\sigma \leq \kappa_0(\kappa_0 + 2). \tag{3.16}$$

Case $\kappa_0 = \sigma$. This says that $\hat{\gamma}_Z(V) = \hat{\mathcal{L}}^\sigma(X)$ for any $Z \in RE(\hat{\gamma})$. Given $Z \in RE(\hat{\gamma})$, set $\gamma_1 = \gamma_Z|_{N(X)}: N(X) \to \mathcal{L}^\sigma(X)$ and $N_1 = \ker \gamma_1$. Then $\dim N_1 \geq \dim N(X) - \sigma$. On one hand, if $\eta \in N(X)$ and $Y \in V^{2n}$, it follows from (3.8) that $\gamma_Y \eta = 0$ if and only if $\langle\langle \gamma_Y \eta, \hat{\gamma}_Z(V)\rangle\rangle = 0$. On the other hand, from (3.7), (3.8), and the flatness of γ, we obtain

$$\langle\langle \gamma_Y \eta, \hat{\gamma}_Z(V)\rangle\rangle = \langle\langle \gamma_Y \eta, \gamma_Z(V)\rangle\rangle = \langle\langle \gamma_Y(V), \gamma_Z \eta\rangle\rangle = 0$$

for any $\eta \in N_1$ and $Y \in V^{2n}$. Thus, $N_1 = \mathcal{N}(\gamma)$. Now, (2.1) yields

$$\nu(\gamma) = \dim N_1 \geq \dim N(X) - \sigma = 2n - \kappa(\gamma) - \sigma \tag{3.17}$$

and gives (3.6).

We have seen that κ_0 and $2 \leq \sigma \leq 10$ are both even. Since $\kappa_0 \leq \sigma$ and the case of equality has already been considered, then we assume that $\kappa_0 < \sigma$. Hence, in view of (3.16), it remains to consider the cases $(\kappa_0, \sigma) = (4,6), (6,8), (6,10)$, and $(8,10)$.

Cases $(6,8)$ *and* $(8,10)$. By (3.9), the vector subspace $\hat{\gamma}_R(V) \cap \hat{\gamma}_S(V)$ is $\mathcal{T}|_{\mathcal{S}(\hat{\gamma})}$-invariant for any $R, S \in V^{2n}$ and thus of even dimension. Then, by (3.15), there are $Z_1, Z_2 \in RE(\hat{\gamma})$ such that

$$\hat{\mathcal{L}}^\sigma(X) = \hat{\gamma}_{Z_1}(V) + \hat{\gamma}_{Z_2}(V). \tag{3.18}$$

If $\eta \in N(X)$ and $Y \in V^{2n}$, it follows from (3.8) and (3.18) that $\gamma_Y \eta = 0$ if and only if $\langle\langle \gamma_Y \eta, \hat{\gamma}_{Z_j}(V)\rangle\rangle = 0$ for $j = 1,2$. Set $\gamma_1 = \gamma_{Z_1}|_{N(X)}: N(X) \to \mathcal{L}^\sigma(X)$, $N_1 = \ker \gamma_1$, $\gamma_2 = \gamma_{Z_2}|_{N_1}: N_1 \to \mathcal{L}^\sigma(X)$, and $N_2 = \ker \gamma_2$. Then $N_2 = \mathcal{N}(\gamma)$ since from (3.7), (3.8), and the flatness of γ, we have

$$\begin{aligned}\langle\langle \gamma_Y \eta, \hat{\gamma}_{Z_j}(V)\rangle\rangle &= \langle\langle \gamma_Y \eta, \gamma_{Z_j}(V)\rangle\rangle \\ &= \langle\langle \gamma_Y(V), \gamma_{Z_j}\eta\rangle\rangle = 0, \ \ j = 1,2,\end{aligned}$$

for any $\eta \in N_2$ and $Y \in V^{2n}$. From the claim above, $\dim \gamma_{Z_j}(N(X)) \leq 4$, $j = 1,2$, and

$$\nu(\gamma) = \dim N_2 \geq \dim N_1 - 4 \geq \dim N(X) - 8 \geq 2n - \kappa(\gamma) - \sigma$$

as wished.

Case $(6,10)$. If we have $Z_1, Z_2 \in RE(\hat{\gamma})$ such that (3.18) holds, then a similar argument as in the previous case gives (3.6). Otherwise, by (3.15), there are $Z_1, Z_2, Z_3 \in RE(\hat{\gamma})$ such that

$$\hat{\mathcal{L}}^{10}(X) = \hat{\gamma}_{Z_1}(V) + \hat{\gamma}_{Z_2}(V) + \hat{\gamma}_{Z_3}(V)$$

and $\dim(\hat{\gamma}_{Z_1}(V) + \hat{\gamma}_{Z_2}(V)) = 8$. Set $\gamma_1 = \gamma_{Z_1}|_{N(X)}: N(X) \to \mathcal{L}^{10}(X)$, $N_1 = \ker \gamma_1$, $\gamma_2 = \gamma_{Z_2}|_{N_1}: N_1 \to \mathcal{L}^{10}(X)$, $N_2 = \ker \gamma_2$, $\gamma_3 = \gamma_{Z_3}|_{N_2}: N_2 \to \mathcal{L}^{10}(X)$, and $N_3 = \ker \gamma_3$. From (3.7), (3.8), and the flatness of γ, we have

$$\begin{aligned}\langle\langle \gamma_{Z_3} \eta_2, \hat{\gamma}_{Z_j}(V)\rangle\rangle &= \langle\langle \gamma_{Z_3} \eta_2, \gamma_{Z_j}(V)\rangle\rangle \\ &= \langle\langle \gamma_{Z_3}(V), \gamma_{Z_j}\eta_2)\rangle\rangle = 0, \ \ j = 1,2,\end{aligned}$$

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for $\eta_2 \in N_2$ and $Y \in V^{2n}$. Hence, $\dim \gamma_{Z_3}(N_2) \leq 2$. Moreover, as in the previous case, we obtain $N_3 = \mathcal{N}(\gamma)$. From the claim above, we have $\dim \gamma_{Z_j}(N(X)) \leq 4$, $j = 1, 2$, and

$$\nu(\gamma) = \dim N_3 \geq \dim N_2 - 2 \geq \dim N_1 - 6 \geq \dim N(X) - 10 = 2n - \kappa(\gamma) - \sigma$$

as wished.

Case $(4, 6)$. Given $Z_1 \in RE(\hat{\gamma})$, by (3.15), there is $Z_2 \in RE(\hat{\gamma})$ such that (3.18) holds. Suppose that there is $Z_1 \in RE(\hat{\gamma})$ such that $\dim \gamma_{Z_1}(N(X)) \leq 4$. Since $\tau(\gamma)$ is even, by (3.11), this always holds if $\tau(\gamma) > 6$. Set $\gamma_1 = \gamma_{Z_1}|_{N(X)}: N(X) \to \mathcal{L}^6(X)$ and $N_1 = \ker \gamma_1$. From (3.7), (3.8), and the flatness of γ, we obtain

$$\begin{aligned}\langle\langle \gamma_{Z_2}\eta_1, \hat{\gamma}_{Z_1}(V)\rangle\rangle &= \langle\langle \gamma_{Z_2}\eta_1, \gamma_{Z_1}(V)\rangle\rangle \\ &= \langle\langle \gamma_{Z_2}(V), \gamma_{Z_1}\eta_1)\rangle\rangle = 0\end{aligned}$$

for any $\eta_1 \in N_1$. Since $\kappa_0 = 4$, then $\dim \gamma_{Z_2}(N_1) \leq 2$.

If $\eta \in N(X)$ and $Y \in V^{2n}$, it follows from (3.8) and (3.18) that $\gamma_Y \eta = 0$ if and only if $\langle\langle \gamma_Y \eta, \hat{\gamma}_{Z_j}(V)\rangle\rangle = 0$ for $j = 1, 2$. Set $\gamma_2 = \gamma_{Z_2}|_{N_1}: N_1 \to \mathcal{L}^6(X)$ and $N_2 = \ker \gamma_2$. As above, we obtain that $N_2 = \mathcal{N}(\gamma)$. Now, (2.1) yields

$$\nu(\gamma) = \dim N_2 \geq \dim N_1 - 2 \geq \dim N(X) - 6 = 2n - \kappa(\gamma) - \sigma$$

as wished.

By the above, it remains to consider the case when $\tau(\gamma) = 6$ and

$$\gamma_Z(N(X)) = \mathcal{L}^6(X) \text{ for any } Z \in RE(\hat{\gamma}). \tag{3.19}$$

If $Y \in RE(\gamma)$, then $\sigma_\gamma(Y) \leq \tau(\gamma) = 6$ by (3.10). Suppose that there is $Y \in RE(\gamma)$ such that $\sigma_\gamma(Y) \leq 4$. From (3.16), we are in case $\kappa_0 = \sigma$ for Y and thus $\nu(\gamma) \geq 2n - k(\gamma) - \sigma_\gamma(Y)$. Since $\sigma_\gamma(Y) < 6 = \sigma$, then (3.6) also holds for X.

In view of the above, we assume further that $\sigma_\gamma(Y) = 6$ for any $Y \in RE(\gamma)$. Now, let $Z_1 \in RE(\gamma) \cap RE(\hat{\gamma})$ and then let $\tilde{\gamma}: V^{2n} \times V^{2n} \to \hat{\mathcal{L}}^6(Z_1)$ stand for taking the $\hat{\mathcal{L}}^6(Z_1)$-component of γ. Suppose that there is $Z_2 \in RE(\tilde{\gamma})$ such that $\tilde{\gamma}_{Z_2}(V) = \hat{\mathcal{L}}^6(Z_1)$. Under this assumption for Z_1, we are in the situation analyzed in Case $\kappa_0 = \sigma$ and thus (3.6) holds for Z_1. Since $\sigma_\gamma(Z_1) = \sigma$, it also holds for X.

In view of (3.16), we now also assume that $\dim \tilde{\gamma}_{Z_2}(V) = 4$ for any $Z_2 \in RE(\tilde{\gamma})$. If $\dim \gamma_{Z_2}(N(Z_1)) \leq 4$ for some $Z_2 \in RE(\tilde{\gamma})$, then the initial part of the proof of this case gives that (3.6) holds for Z_1 and then also for X since $\sigma_\gamma(Z_1) = \sigma$. Hence, we assume that $\gamma_{Z_2}(N(Z_1)) = \mathcal{L}(Z_1)$ for any $Z_2 \in RE(\tilde{\gamma})$.

The remaining case to consider is when there are $Z_1, Z_2 \in RE(\gamma) \cap RE(\hat{\gamma})$ and $Z_2 \in RE(\tilde{\gamma})$ for which (3.18) holds, $\sigma_\gamma(Z_j) = 6$, $j = 1, 2$, $\gamma_{Z_2}(N(Z_1)) = \mathcal{L}(Z_1)$, and $\dim \tilde{\gamma}_{Z_2}(V) = 4$. To conclude the proof, we show that this situation is not possible. Hence, suppose otherwise. In particular, we have $\mathcal{L}(Z_1) \subset \gamma_{Z_2}(V)$. From (3.19), we obtain that $\mathcal{L}(X) \subset \gamma_{Z_j}(V)$, $j = 1, 2$. Thus, given $\eta_0 \in \mathcal{L}(X)$, there are $Y_1, Y_2 \in V^{2n}$ such that $\eta_0 = \gamma_{Z_1} Y_1 = \gamma_{Z_2} Y_2$. Let $\xi_0 \in \mathcal{L}(X)$ and $\xi_j \in \mathcal{L}(Z_j)$, $j = 1, 2$. Then

$$\langle\langle \xi_0 + \xi_1 + \xi_2, \eta_0\rangle\rangle = \langle\langle \xi_1, \gamma_{Z_1} Y_1\rangle\rangle + \langle\langle \xi_2, \gamma_{Z_2} Y_2\rangle\rangle = 0.$$

If $\eta_1 \in \mathcal{L}(Z_1)$, then $\langle\langle \xi_0, \eta_1\rangle\rangle = 0$ since $\mathcal{L}(X) \subset \gamma_{Z_1}(V)$ and $\mathcal{L}(Z_1) \subset \mathcal{U}(Z_1)$. Let $Y_3 \in V^{2n}$ be such that $\eta_1 = \gamma_{Z_2} Y_3$. Since $\mathcal{L}(Z_2) \subset \mathcal{U}(Z_2)$, then

$$\langle\langle \xi_0 + \xi_1 + \xi_2, \eta_1\rangle\rangle = \langle\langle \xi_2, \gamma_{Z_2} Y_3\rangle\rangle = 0.$$

If $\eta_2 \in \mathcal{L}(Z_2)$, then $\langle\langle \xi_j, \eta_2 \rangle\rangle = 0$, $j = 0, 1$, since $\mathcal{L}(X) \subset \gamma_{Z_2}(V)$, $\mathcal{L}(Z_1) \subset \gamma_{Z_2}(V)$ and $\mathcal{L}(Z_2) \subset \mathcal{U}(Z_2)$. Thus, $\langle\langle \xi_0 + \xi_1 + \xi_2, \eta_2 \rangle\rangle = 0$. Hence, $\mathcal{L}(X) + \mathcal{L}(Z_1) + \mathcal{L}(Z_2)$ is an isotropic vector subspace.

We argue that $\dim \mathcal{L}(X) \cap \mathcal{L}(Z_j) = \dim \mathcal{L}(Z_1) \cap \mathcal{L}(Z_2) = 2$. On one hand, we have that $\mathcal{L}(X) \cap \mathcal{L}(Z_j) \neq 0$ since otherwise the vector subspace $\mathcal{L}(X) \oplus \mathcal{L}(Z_j)$ would be isotropic of dimension 12 which is not possible. On the other hand, we have

$$\langle\langle \xi, \hat{\gamma}_{Z_j}(V) \rangle\rangle = \langle\langle \xi, \gamma_{Z_j}(V) \rangle\rangle = 0$$

for any $\xi \in \mathcal{L}(X) \cap \mathcal{L}(Z_j)$. Since $\kappa_0 = 4$, it follows from part (ii) of Proposition 2.5 that $\dim \mathcal{L}(X) \cap \mathcal{L}(Z_j) = 2$. Having that $\mathcal{L}(Z_1) \oplus \mathcal{L}(Z_2) \subset \gamma_{Z_2}(V)$ is isotropic yields $\mathcal{L}(Z_1) \cap \mathcal{L}(Z_2) \neq 0$. If $\xi \in \mathcal{L}(Z_1) \cap \mathcal{L}(Z_2)$, then

$$\langle\langle \xi, \tilde{\gamma}_{Z_2}(V) \rangle\rangle = \langle\langle \xi, \gamma_{Z_2}(V) \rangle\rangle = 0,$$

where the second equality follows from $\mathcal{L}(Z_2) \subset \mathcal{U}(Z_2)$. Since $\dim \tilde{\gamma}_{Z_2}(V) = 4$, then $\mathcal{L}(Z_1) \cap \mathcal{L}(Z_2) = 2$. We have shown that $\mathcal{L}(X) + \mathcal{L}(Z_1) + \mathcal{L}(Z_2)$ has dimension 12, but this is a contradiction.

Remark 3.3 The estimate $\nu(\gamma) \geq 2n - 2p$ is Proposition 10 in [1]. A counterexample constructed in [1] shows that this estimate is false already for $p = 12$.

Henceforward, $U^p = U_1^s \oplus U_2^{p-s}$ is an orthogonal decomposition where

$$U_1^s = \mathcal{S}(\pi_1 \circ \beta).$$

Lemma 3.4 *If (3.5) holds, then*

$$\mathcal{S}(\beta) = U_1^s \oplus U_1^s \tag{3.20}$$

and $\mathcal{N}(\beta) = \mathcal{N}(\gamma_{U_1})$, *where* $\gamma_{U_1} = \pi_{U_1 \times U_1} \circ \gamma$.

Proof We have that

$$\beta(X, Y) = (\xi, \eta) \iff \beta(Y, X) = (\xi, -\eta) \iff \beta(X, JY) = (\eta, -\xi).$$

Thus, if $(\xi, \eta) = \sum_k \beta(X_k, Y_k)$, then

$$\sum_k \beta(Y_k, X_k) = (\xi, -\eta), \;\; \sum_k \beta(X_k, JY_k) = (\eta, -\xi), \;\; \sum_k \beta(JY_k, X_k) = (\eta, \xi).$$

Hence, if $(\xi, \eta) \in \mathcal{S}(\beta)$, then $(\xi, 0), (0, \xi), (\eta, 0) \in \mathcal{S}(\beta)$ and thus $\mathcal{S}(\beta) \subset U_1 \oplus U_1$. On the other hand, if $(\xi, \eta) \in U_1 \oplus U_1$, there are $\bar{\xi}, \bar{\eta} \in U^p$ so that $(\xi, \bar{\xi}), (\eta, \bar{\eta}) \in \mathcal{S}(\beta)$ and thus $(\xi, \eta) \in \mathcal{S}(\beta)$, which proves (3.20).

From (3.5), (3.20), and $(U_2 \oplus U_2)^\perp = U_1 \oplus U_1$, we obtain $\mathcal{S}(\gamma|_{V \times \mathcal{N}(\beta)}) \subset U_2 \oplus U_2$. Then $\langle\langle \gamma(X, Y), (\xi, \bar{\xi}) \rangle\rangle = 0$ if $X \in V^{2n}$, $Y \in \mathcal{N}(\beta)$, and $\xi, \bar{\xi} \in U_1^s$. Thus, $\mathcal{N}(\beta) \subset \mathcal{N}(\gamma_{U_1})$. On the other hand, if $S \in \mathcal{N}(\gamma_{U_1})$, then $\beta(X, S) = \gamma_{U_2}(X, S) + \gamma_{U_2}(JX, JS) = 0$ by (3.20) for any $X \in V^{2n}$. ∎

Lemma 3.5 *Let the bilinear form* $\beta\colon V^{2n} \times V^{2n} \to W^{p,p}$, $p \leq n$, *be flat. Then*

$$\nu(\beta) = 2n - \kappa(\beta). \tag{3.21}$$

Moreover, if $\kappa(\beta) = 2p$, *there is a basis* $\{X_i, JX_i\}_{1 \leq i \leq n}$ *of* V^{2n} *such that:*

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(i) $\mathcal{N}(\beta) = span\,\{X_j, JX_j,\ p+1 \le j \le n\}$.
(ii) $\beta(Y_i, Y_j) = 0$ *if* $i \neq j$ *and* $Y_k \in span\,\{X_k, JX_k\}$ *for* $k = i, j$.
(iii) $\{\beta(X_j, X_j), \beta(X_j, JX_j)\}_{1\le j\le p}$ *is an orthonormal basis of* $W^{p,p}$.

Proof Proposition 7 in [1] gives (3.21). The remaining of the statement is Proposition 2.6 in [4] as well as Lemma 7 in [13]. ■

Let $\theta\colon V^{2n} \times V^{2n} \to W^{p,p}$ be the pluriharmonic symmetric bilinear form defined as

$$\theta(X, Y) = \gamma(X, Y) - \gamma(JX, JY). \tag{3.22}$$

It follows from (3.2) that

$$\mathcal{T}\theta(X, Y) = \theta(X, JY) \ \text{ for any } \ X, Y \in V^{2n}. \tag{3.23}$$

If the condition (3.5) holds, then also

$$\langle\langle \beta(X, Y), \theta(Z, T)\rangle\rangle = \langle\langle \beta(X, T), \theta(Z, Y)\rangle\rangle \ \text{ for any } \ X, Y, Z, T \in V^{2n}. \tag{3.24}$$

In fact, we have using (3.2) and (3.4) that

$$\begin{aligned}
\langle\langle \beta(X, Y), \theta(Z, T)\rangle\rangle &= \langle\langle \beta(X, Y), \gamma(Z, T)\rangle\rangle - \langle\langle \beta(X, Y), \gamma(JZ, JT)\rangle\rangle\\
&= \langle\langle \beta(X, T), \gamma(Z, Y)\rangle\rangle - \langle\langle \beta(X, JT), \gamma(JZ, Y)\rangle\rangle\\
&= \langle\langle \beta(X, T), \gamma(Z, Y)\rangle\rangle - \langle\langle \mathcal{T}\beta(X, T), \gamma(JZ, Y)\rangle\rangle\\
&= \langle\langle \beta(X, T), \gamma(Z, Y)\rangle\rangle - \langle\langle \beta(X, T), \gamma(JZ, JY)\rangle\rangle\\
&= \langle\langle \beta(X, T), \theta(Z, Y)\rangle\rangle.
\end{aligned}$$

If γ is flat, then also is θ. In effect, using (3.2), we have

$$\begin{aligned}
\langle\langle \theta(X, Y), \theta(Z, T)\rangle\rangle &= \langle\langle \gamma(X, Y), \gamma(Z, T)\rangle\rangle - \langle\langle \gamma(X, Y), \gamma(JZ, JT)\rangle\rangle\\
&\quad - \langle\langle \gamma(JX, JY), \gamma(Z, T)\rangle\rangle + \langle\langle \gamma(JX, JY), \gamma(JZ, JT)\rangle\rangle\\
&= \langle\langle \gamma(X, T), \gamma(Z, Y)\rangle\rangle - \langle\langle \gamma(X, JT), \gamma(JZ, Y)\rangle\rangle\\
&\quad - \langle\langle \gamma(JX, T), \gamma(Z, JY)\rangle\rangle + \langle\langle \gamma(JX, JT), \gamma(JZ, JY)\rangle\rangle\\
&= \langle\langle \gamma(X, T), \gamma(Z, Y)\rangle\rangle - \langle\langle \gamma(X, T), \gamma(JZ, JY)\rangle\rangle\\
&\quad - \langle\langle \gamma(JX, JT), \gamma(Z, Y)\rangle\rangle + \langle\langle \gamma(JX, JT), \gamma(JZ, JY)\rangle\rangle\\
&= \langle\langle \theta(X, T), \theta(Z, Y)\rangle\rangle.
\end{aligned}$$

Since $2\gamma = \beta + \theta$, then $\mathcal{N}(\beta) \cap \mathcal{N}(\theta) \subset \mathcal{N}(\gamma)$, whereas the opposite inclusion follows from (3.3), (3.22), and that $\mathcal{N}(\gamma)$ is J-invariant. Therefore,

$$\mathcal{N}(\gamma) = \mathcal{N}(\beta) \cap \mathcal{N}(\theta), \tag{3.25}$$

and, in particular, we have that $\nu(\beta) \ge \nu(\gamma)$.

Lemma 3.6 *Let* $\gamma, \beta\colon V^{2n} \times V^{2n} \to W^{p,p}$, $p \le n$, *be flat and satisfy the condition* (3.5). *If* $\nu(\beta) = 2n - 2s$, *then the bilinear forms* $\theta_j = \pi_{U_j \times U_j} \circ \theta$, $j = 1, 2$, *are flat.*

Proof By (3.20) and Lemma 3.5, there is a basis $\{X_j, JX_j\}_{1\le j\le n}$ of V^{2n} satisfying that $\mathcal{N}(\beta) = \text{span}\,\{X_j, JX_j,\ s+1 \le j \le n\}$, that

$$\beta(X_i, X_j) = 0 = \beta(X_i, JX_j) \ \text{ if } \ i \neq j \tag{3.26}$$

and that $\{\beta(X_j, X_j), \beta(X_j, JX_j)\}_{1\le j\le s}$ is an orthonormal basis of $U_1^s \oplus U_1^s$.

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From (3.20), we have that (3.24) is equivalent to

$$\langle\langle\beta(X,Y),\theta_1(Z,T)\rangle\rangle = \langle\langle\beta(X,T),\theta_1(Z,Y)\rangle\rangle \text{ for any } X,Y,Z,T\in V^{2n}. \tag{3.27}$$

In particular, we obtain using (3.26) that

$$\theta_1(X_i,X_i),\theta_1(X_i,JX_i)\in\text{span}\,\{\beta(X_i,X_i),\beta(X_i,JX_i)\}.$$

Moreover, since θ_1 is symmetric, it follows from (3.26) and (3.27) for $k\neq\ell$ that

$$\langle\langle\beta(X_j,X_j),\theta_1(X_k,X_\ell)\rangle\rangle = \langle\langle\beta(X_j,X_\ell),\theta_1(X_k,X_j)\rangle\rangle = \langle\langle\beta(X_j,X_k),\theta_1(X_\ell,X_j)\rangle\rangle = 0,$$

$$\langle\langle\beta(X_j,JX_j),\theta_1(X_k,X_\ell)\rangle\rangle = \langle\langle\beta(X_j,X_\ell),\theta_1(X_k,JX_j)\rangle\rangle = \langle\langle\beta(X_j,X_k),\theta_1(X_\ell,JX_j)\rangle\rangle = 0$$

and thus

$$\theta_1(X_k,X_\ell) = 0 = \theta_1(X_k,JX_\ell) \text{ if } k\neq\ell.$$

We have that

$$\mathcal{T}\gamma_{U_1}(X,Y) = \mathcal{T}(\alpha_{U_1}(X,Y),\alpha_{U_1}(X,JY)) = (\alpha_{U_1}(X,JY),-\alpha_{U_1}(X,Y)) = \gamma_{U_1}(X,JY)$$

for any $X,Y\in V^{2n}$. Then

$$\mathcal{T}\theta_1(X,Y) = \mathcal{T}(\gamma_{U_1}(X,Y)-\gamma_{U_1}(JX,JY)) = \gamma_{U_1}(X,JY)+\gamma_{U_1}(JX,Y) = \theta_1(X,JY),$$

and hence

$$\langle\langle\theta_1(X_i,X_i),\theta_1(JX_i,JX_i)\rangle\rangle = \langle\langle\theta_1(X_i,JX_i),\theta_1(X_i,JX_i)\rangle\rangle.$$

We have shown that θ_1 and θ are flat. Since $\theta=\theta_1\oplus\theta_2$, then also θ_2 is flat. ∎

Lemma 3.7 *Let $\gamma,\beta\colon V^{2n}\times V^{2n}\to W^{p,p}$, $p\leq n$, be flat and satisfy the condition (3.5). If $\mathcal{S}(\gamma)=W^{p,p}$ and $p-s\leq 9$, the flat bilinear form $\varphi=\theta|_{V\times\mathcal{N}(\beta)}\colon V^{2n}\times\mathcal{N}(\beta)\to W^{p,p}$ satisfies*

$$0\leq\nu(\beta)-\nu(\gamma)\leq\kappa(\varphi)+\sigma(\varphi)\leq 2p-2s. \tag{3.28}$$

Proof It follows from (3.20) and (3.24) that $\mathcal{S}(\varphi)\subset U_2^{p-s}\oplus U_2^{p-s}\subset W^{p,p}$. Thus, φ is seen in the sequel as a map

$$\varphi\colon V^{2n}\times\mathcal{N}(\beta)\to U_2^{p-s}\oplus U_2^{p-s}. \tag{3.29}$$

To obtain the proof, it suffices to show for any $X\in RE(\varphi)$ that we have

$$0\leq\nu(\beta)-\nu(\gamma)\leq\kappa(\varphi)+\sigma_\varphi(X)\leq 2p-2s. \tag{3.30}$$

Fix $X\in RE(\varphi)$ and set $\sigma=\sigma_\varphi(X)$ for simplicity. Proposition 2.3 gives a decomposition

$$U_2^{p-s}\oplus U_2^{p-s} = \mathcal{U}^\tau(X)\oplus\hat{\mathcal{U}}^\tau(X)\oplus\mathcal{V}^{p-s-\tau,p-s-\tau}, \tag{3.31}$$

where $\mathcal{U}^\tau(X)=\varphi_X(\mathcal{N}(\beta))\cap\varphi_X(\mathcal{N}(\beta))^\perp$ with $\tau=\tau_\varphi(X)$ for simplicity of notation and $\varphi_X(\mathcal{N}(\beta))\subset\mathcal{U}^\tau(X)\oplus\mathcal{V}^{p-s-\tau,p-s-\tau}$. Thus, $\kappa(\varphi)\leq 2p-2s-\tau$. From (2.2), we obtain that $\sigma(\varphi)\leq\sigma\leq\tau$ and the last inequality in (3.30) follows.

We have that $0 \leq \sigma \leq \tau \leq p - s \leq 9$. From (3.23) and the J-invariance of $\mathcal{N}(\beta)$, we obtain that $\mathcal{T}\varphi(Y,Z) = \varphi(Y,JZ)$ for any $Y \in V^{2n}$ and $Z \in \mathcal{N}(\beta)$. Thus, part (iii) of Proposition 2.5 gives that $N(X) = \ker \varphi_X$ is J-invariant and hence part (ii) that σ is even. Therefore, we have that $0 \leq \sigma \leq 8$.

Case $\sigma = 0$. Since $\theta(Y, N(X)) = 0$ for any $Y \in V^{2n}$, then $N(X) \subset \mathcal{N}(\beta) \cap \mathcal{N}(\theta) = \mathcal{N}(\gamma)$ from (3.25). On the other hand, it is a general fact that

$$\dim N(X) = \nu(\beta) - \kappa(\varphi), \tag{3.32}$$

and since $\sigma = 0$, then (3.30) follows from (3.32).

Henceforward, we assume that $\sigma \geq 2$. As in (3.8), we have a decomposition

$$\mathcal{U}^\tau(X) \oplus \hat{\mathcal{U}}^\tau(X) = \mathcal{L}^\sigma(X) \oplus \hat{\mathcal{L}}^\sigma(X) \oplus \mathcal{V}_0^{\tau-\sigma,\tau-\sigma}. \tag{3.33}$$

We claim that the symmetric bilinear form defined by $\hat{\theta} = \pi_{\hat{\mathcal{L}}^\sigma(X)} \circ \theta$ satisfies

$$\mathcal{S}(\hat{\theta}) = \hat{\mathcal{L}}^\sigma(X). \tag{3.34}$$

In fact, if otherwise, there is $0 \neq \eta = \sum_{j=1}^r \varphi(Y_j, S_j) \in \mathcal{L}^\sigma(X)$ with $Y_1, \dots, Y_r \in V^{2n}$ and $S_1, \dots, S_r \in N(X)$ such that

$$0 = \langle\langle \eta, \hat{\theta}(Z,T) \rangle\rangle = \sum_{j=1}^r \langle\langle \theta(Y_j, S_j), \theta(Z,T) \rangle\rangle \tag{3.35}$$

for any $Z, T \in V^{2n}$. Since $2\gamma = \beta + \theta$, we obtain from (3.24) and (3.35) that

$$2\langle\langle \eta, \gamma(Z,T) \rangle\rangle = \sum_{j=1}^r \langle\langle \theta(Y_j,S_j), \beta(Z,T) + \theta(Z,T) \rangle\rangle = \sum_{j=1}^r \langle\langle \theta(Y_j,S_j), \theta(Z,T) \rangle\rangle = 0$$

for any $Z, T \in V^{2n}$. Since $0 \neq \eta = 2\sum_{j=1}^r \gamma(Y_j, S_j)$, this is a contradiction by the assumption for $\mathcal{S}(\gamma)$ that proves the claim.

Part (i) of Proposition 2.5 gives that $\mathcal{T}|_{\mathcal{L}^\sigma(X)} \in \operatorname{Aut}(\mathcal{L}^\sigma(X))$ is a complex structure. Then, from (3.31) and (3.33), we obtain for any $(\xi, \bar{\xi}) \in \mathcal{L}^\sigma(X)$ that

$$\langle\langle \mathcal{T}\hat{\theta}(Z,Y), (\xi,\bar{\xi}) \rangle\rangle = \langle\langle \theta(Z,Y), \mathcal{T}(\xi,\bar{\xi}) \rangle\rangle = \langle\langle \theta(Z,JY), (\xi,\bar{\xi}) \rangle\rangle = \langle\langle \hat{\theta}(Z,JY), (\xi,\bar{\xi}) \rangle\rangle,$$

which says that

$$\mathcal{T}\hat{\theta}_Z Y = \hat{\theta}_Z JY \tag{3.36}$$

for any $Z, Y \in V^{2n}$. Hence, if $Z \in RE(\hat{\theta})$, then $\kappa_0 = \kappa(\hat{\theta})$ is even. Being θ symmetric then also is $\hat{\theta}$, and it follows from (2.6) and (3.34) that

$$4\sigma \leq \kappa_0(\kappa_0 + 2). \tag{3.37}$$

Therefore, if $\sigma = 2, 4$, then $\sigma = \kappa_0$, and if $\sigma = 6, 8$, then either $\sigma = \kappa_0$ or $\sigma = \kappa_0 + 2$.

Case $\sigma = \kappa_0$. Given $Z \in RE(\hat{\theta})$, we have $\hat{\theta}_Z(V) = \hat{\mathcal{L}}^\sigma(X)$. Since $\varphi(Y,\eta) \in \mathcal{L}^\sigma(X)$, if $Y \in V^{2n}$ and $\eta \in N(X)$, then

$$\varphi(Y,\eta) = 0 \text{ if and only if } \langle\langle \varphi(Y,\eta), \hat{\theta}_Z T \rangle\rangle = 0 \tag{3.38}$$

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for any $T \in V^{2n}$. Set $\theta_1 = \theta_Z|_{N(X)}: N(X) \to \mathcal{L}^\sigma(X)$ and $N_1 = \ker \theta_1$. From (3.31), (3.33), and the flatness of θ, we obtain

$$\langle\langle \varphi(Y,\delta), \hat{\theta}_Z T\rangle\rangle = \langle\langle \theta(Y,\delta), \theta(Z,T)\rangle\rangle = \langle\langle \theta(Y,T), \theta(Z,\delta)\rangle\rangle = 0$$

for any $\delta \in N_1$ and $Y, T \in V^{2n}$. Hence, from (3.38), we have $N_1 \subset \mathcal{N}(\beta) \cap \mathcal{N}(\theta) = \mathcal{N}(\gamma)$. Then (3.25) and (3.32) give

$$\nu(\gamma) \geq \dim N_1 \geq \dim N(X) - \sigma = \nu(\beta) - \kappa(\varphi) - \sigma,$$

and (3.30) follows.

Case $\sigma = \kappa_0 + 2$. Suppose that there is $Z \in RE(\varphi)$ such that $\mathcal{L}(Z) = \mathcal{S}(\varphi|_{V\times N(Z)})$ satisfies $\sigma_\varphi(Z) \leq 4$. Then, by (3.37), for such $Z \in RE(\varphi)$, we are in Case $\sigma = \kappa_0$. Hence,

$$\nu(\beta) - \nu(\gamma) \leq \kappa(\varphi) + \sigma_\varphi(Z) \leq \kappa(\varphi) + 4 < \kappa(\varphi) + \sigma,$$

and since $\sigma \geq 6$, then (3.30) holds. Thus, henceforward, we assume that $\sigma_\varphi(Z) \geq 6$ for any $Z \in RE(\varphi)$.

If $Z_1, Z_2 \in RE(\hat{\theta})$, then (3.36) gives that $\hat{\theta}_{Z_1}(V) \cap \hat{\theta}_{Z_2}(V)$ is $\mathcal{T}$-invariant and therefore of even dimension. Given $Z_1 \in RE(\hat{\theta})$, then by (3.34) there is $Z_2 \in RE(\hat{\theta})$ such that we have $\hat{\mathcal{L}}^\sigma(X) = \hat{\theta}_{Z_1}(V) + \hat{\theta}_{Z_2}(V)$.

Since $\varphi(Y,\eta) \in \mathcal{L}^\sigma(X)$, if $\eta \in N(X)$ and $Y \in V^{2n}$, then

$$\varphi(Y,\eta) = 0 \text{ if and only if } \langle\langle \varphi(Y,\eta), \hat{\theta}_{Z_j} T\rangle\rangle = 0,\ j = 1,2, \tag{3.39}$$

for any $T \in V^{2n}$. If $\theta_1 = \theta_{Z_1}|_{N(X)}: N(X) \to \mathcal{L}^\sigma(X)$ and $N_1 = \ker \theta_1$, then

$$\dim N(X) = \dim N_1 + \dim \theta_1(N(X)). \tag{3.40}$$

If $\theta_2 = \theta_{Z_2}|_{N_1}: N_1 \to \mathcal{L}^\sigma(X)$ and $N_2 = \ker \theta_2$, we have from (3.31) and (3.33) that

$$\langle\langle \theta_2 \delta_1, \hat{\theta}_{Z_1} Y\rangle\rangle = \langle\langle \theta(Z_2,\delta_1), \theta_{Z_1} Y\rangle\rangle = \langle\langle \theta(Z_2,Y), \theta(Z_1,\delta_1)\rangle\rangle = 0$$

for any $\delta_1 \in N_1$ and $Y \in V^{2n}$. Thus, $\dim \theta_2(N_1) \leq \sigma - \kappa_0 = 2$ and hence

$$\dim N_1 \leq \dim N_2 + 2. \tag{3.41}$$

It follows from (3.31) and (3.33) that

$$\langle\langle \varphi(Y,\delta_2), \hat{\theta}_{Z_j} T\rangle\rangle = \langle\langle \theta(Y,\delta_2), \theta(Z_j,T)\rangle\rangle = \langle\langle \theta(Y,T), \theta(Z_j,\delta_2)\rangle\rangle = 0$$

for any $\delta_2 \in N_2$, $Y, T \in V^{2n}$ and $j = 1,2$. Then, from (3.39), we have $N_2 \subset \mathcal{N}(\beta) \cap \mathcal{N}(\theta)$. Hence, using (3.25), (3.32), (3.40), and (3.41), we obtain

$$\begin{aligned}\nu(\gamma) &\geq \dim N_2 \geq \dim N_1 - 2 = \dim N(X) - \dim \theta_1(N(X)) - 2\\ &= \nu(\beta) - \kappa(\varphi) - \dim \theta_1(N(X)) - 2.\end{aligned}$$

Since $\sigma = \kappa_0 + 2$, then in order from the above to have (3.30), it is necessary to show that there is $Z \in RE(\hat{\theta})$ such that $\dim \theta_Z(N(X)) \leq \kappa_0$. Arguing by contradiction, assume that $\dim \theta_Z(N(X)) > \kappa_0$ for any $Z \in RE(\hat{\theta})$. Since $N(X)$ is J-invariant, then $\theta_Z(N(X))$ has even dimension and thus the assumption means that $\theta_Z(N(X)) = \mathcal{L}^\sigma(X)$ for any $Z \in RE(\hat{\theta})$. Let us take $Z \in RE(\varphi) \cap RE(\theta) \cap RE(\hat{\theta})$. Then $N(Z) =$

$\ker \varphi_Z$ satisfies $N(Z) \subset \ker \theta_Z$. From (2.2) and (3.29), we have

(3.42) $$\mathcal{L}^{\sigma_\varphi(Z)}(Z) \subset \mathcal{R} = \theta_Z(V) \cap \theta_Z(V)^\perp \cap (U_2^{p-s} \oplus U_2^{p-s}),$$

and from (3.31) and (3.33) that there is a decomposition

$$U_2^{p-s} \oplus U_2^{p-s} = \mathcal{L}^\sigma(X) \oplus \hat{\mathcal{L}}^\sigma(X) \oplus \mathcal{V}_0^{\tau-\sigma,\tau-\sigma} \oplus \mathcal{V}^{p-s-\tau,p-s-\tau}.$$

Thus, if $\delta \in \mathcal{R}$, then $\delta = \delta_1 + \delta_2 + \delta_3$, where $\delta_1 \in \mathcal{L}^\sigma(X)$, $\delta_2 \in \hat{\mathcal{L}}^\sigma(X)$, and $\delta_3 \in \mathcal{V}_0 \oplus \mathcal{V}$. Since $\delta \in \theta_Z(V)^\perp$ and $\theta_Z(N(X)) = \mathcal{L}^\sigma(X)$, then

$$0 = \langle\langle \delta, \theta_Z(N(X))\rangle\rangle = \langle\langle \delta_2, \mathcal{L}^\sigma(X)\rangle\rangle,$$

and hence $\delta_2 = 0$. Thus, if $\delta^1, \delta^2 \in \mathcal{R}$, we have

$$0 = \langle\langle \delta^1, \delta^2\rangle\rangle = \langle\langle \delta_1^1 + \delta_3^1, \delta_1^2 + \delta_3^2\rangle\rangle = \langle\langle \delta_3^1, \delta_3^2\rangle\rangle.$$

Hence, if $\pi: \mathcal{R} \to \mathcal{V}_0 \oplus \mathcal{V}$ is defined by $\pi(\delta) = \delta_3$, then the vector subspace $\pi(\mathcal{R}) \subset \mathcal{V}_0 \oplus \mathcal{V}$ is isotropic and thus $\dim \pi(\mathcal{R}) \leq p - s - \sigma$. Since $p - s \leq 9$ by assumption and $\sigma \geq 6$, then $\dim \pi(\mathcal{R}) \leq 3$. From (3.42), we have $\dim \mathcal{R} \geq \sigma_\varphi(Z)$, and hence $\dim \ker \pi \geq \sigma_\varphi(Z) - 3 \geq 3$. On the other hand, we have that $\ker \pi \subset \mathcal{L}^\sigma(X) \cap \mathcal{R}$. Since $\mathcal{R} \subset \theta_Z(V)^\perp$, we obtain from (3.31) and (3.33) that

$$\langle\langle \zeta, \hat{\theta}_Z Y\rangle\rangle = \langle\langle \zeta, \theta_Z Y\rangle\rangle = 0$$

for any $\zeta \in \ker \pi$ and $Y \in V^{2n}$. Therefore, that $\ker \pi \subset \mathcal{L}^\sigma(X)$, $\hat{\theta}_Z(V) \subset \hat{\mathcal{L}}^\sigma(X)$ and that $\dim \hat{\theta}_Z(V) = \kappa_0 = \sigma - 2$ yield $\dim \ker \pi \leq 2$, and we reached a contradiction. ∎

Lemma 3.8 *Let $\gamma, \beta: V^{2n} \times V^{2n} \to W^{p,p}$, $p \leq n$, be flat and satisfy the condition (3.5). If $p \leq 11$ and $\mathcal{S}(\gamma) = W^{p,p}$, then $\nu(\gamma) \geq 2n - 2p$.*

Proof First, assume that $s \geq 2$ in which case (3.28) holds. Let $\varphi: V \times \mathcal{N}(\beta) \to U_2^{p-s} \oplus U_2^{p-s}$ be given by (3.29), and let $X \in RE(\varphi)$ satisfy $\sigma_\varphi(X) = \sigma(\varphi)$. By (3.31), we have that

$$U_2^{p-s} \oplus U_2^{p-s} = \mathcal{U}^\tau(X) \oplus \hat{\mathcal{U}}^\tau(X) \oplus \mathcal{V}^{p-s-\tau,p-s-\tau}$$

with $\varphi_X(\mathcal{N}(\beta)) \subset \mathcal{U}^\tau(X) \oplus \mathcal{V}^{p-s-\tau,p-s-\tau}$. Then $\kappa(\varphi) \leq 2p - 2s - \tau$. On the other hand, it follows from (3.20) and (3.21) that $\nu(\beta) \geq 2n - 2s$, whereas by (3.28) we have that $\nu(\beta) - \nu(\gamma) \leq \kappa(\varphi) + \sigma(\varphi)$. Hence,

$$2n - 2s - \nu(\gamma) \leq \nu(\beta) - \nu(\gamma) \leq \kappa(\varphi) + \sigma(\varphi) \leq 2p - 2s - \tau + \sigma(\varphi).$$

Since $\tau \geq \sigma(\varphi)$, by (2.2), then $\nu(\gamma) \geq 2n - 2p$ as we wished.

If $s = 0$, then $\beta = 0$, that is, α is pluriharmonic and then Lemma 3.2 gives the result. Thus, it remains to consider the case $s = 1$ and thus $\beta \neq 0$. Part (ii) of Proposition 2.5 yields that the vector space $\beta_X(V)$ is even-dimensional. Thus, $\kappa(\beta) = 2$ and then (3.21) gives that $\nu(\beta) = 2n - 2$. Then, by Lemma 3.6, we have that $\theta = \theta_1 + \theta_2$, where the bilinear form $\theta_2: V^{2n} \times V^{2n} \to U_2^{p-1} \oplus U_2^{p-1}$ is flat.

We claim that θ_2 satisfies the assumptions of Lemma 3.2. From (3.20) and $2\gamma = \beta + \theta$, we have

$$\theta_2(X,Y) = 2\gamma_2(X,Y) = 2(\alpha_{U_2}(X,Y), \alpha_{U_2}(X,JY)).$$

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Since $\mathcal{S}(\gamma) = W^{p,p}$, by assumption, then $\mathcal{S}(\theta_2) = U_2^{p-1} \oplus U_2^{p-1}$. If follows from (3.20) that the symmetric bilinear form α_{U_2} is pluriharmonic and hence also is θ_2.

By Lemma 3.2, we have $\nu(\theta_2) \geq 2n - 2p + 2$. Then (3.22) yields $\mathcal{N}(\gamma_{U_1}) \subset \mathcal{N}(\theta_1)$, whereas Lemma 3.4 yields that $\mathcal{N}(\beta) \subset \mathcal{N}(\theta_1)$. Then (3.25) gives

$$\mathcal{N}(\gamma) = \mathcal{N}(\beta) \cap \mathcal{N}(\theta_1) \cap \mathcal{N}(\theta_2) = \mathcal{N}(\beta) \cap \mathcal{N}(\theta_2).$$

Hence, we have

$$2n \geq \dim(\mathcal{N}(\beta) + \mathcal{N}(\theta_2)) = \nu(\beta) + \nu(\theta_2) - \dim \mathcal{N}(\beta) \cap \mathcal{N}(\theta_2) = \nu(\beta) + \nu(\theta_2) - \nu(\gamma),$$

and since $\nu(\beta) = 2n - 2$, we conclude that $\nu(\gamma) \geq 2n - 2p$. ∎

Remark 3.9 The estimate given by Lemma 3.8 is sharp. For instance, if we take as α in (3.1) and (3.3) the second fundamental form of a product of real Kaehler hypersurfaces, then the hypotheses of the lemma are satisfied and we have equality in the estimate.

Proof of Theorem 3.1 By Lemma 3.8, the vector subspace $\mathcal{S}(\gamma)$ is degenerate since, if otherwise, then by (3.2) it is of the form $\mathcal{S}(\gamma) = W_1^{q,q} \subset W^{p,p}$ and then Lemma 3.8 yields a contradiction. Parts (ii) and (iv) of Proposition 2.5 give, respectively, that $\dim \Omega \geq 2$ and that $\gamma_\Omega = \pi_{\Omega \oplus \Omega} \circ \gamma$ satisfies $\mathcal{S}(\gamma_\Omega) = \mathcal{S}(\gamma) \cap \mathcal{S}(\gamma)^\perp$. Hence,

$$0 = \langle\langle \gamma_\Omega(X,Y), \gamma_\Omega(Z,T) \rangle\rangle = \langle \alpha_\Omega(X,Y), \alpha_\Omega(Z,T) \rangle - \langle \alpha_\Omega(X,JY), \alpha_\Omega(Z,JT) \rangle$$

for any $X, Y, Z, T \in V^{2n}$. Thus, the complex structure $\mathcal{J} \in \mathrm{End}(\mathcal{S}(\alpha_\Omega))$ defined by $\mathcal{J}\alpha_\Omega(X,Y) = \alpha_\Omega(X,JY)$ is an isometry. Part (iv) of Proposition 2.5 gives that $\mathcal{S}(\alpha_\Omega) = \Omega$ and hence $\mathcal{J} \in \mathrm{Aut}(\Omega)$ is a complex structure. In particular, we have that α_Ω is pluriharmonic. Then $\mathcal{S}(\beta) \subset P \oplus P$ and hence $\beta = \beta_P = \pi_{P\times P} \circ \beta$. Thus, if $\gamma_P = \pi_{P \times P} \circ \gamma$ and $\gamma = \gamma_\Omega \oplus \gamma_P$, then

$$\langle\langle \gamma_P(X,Y), \beta_P(Z,T) \rangle\rangle = \langle\langle \gamma(X,Y), \beta(Z,T) \rangle\rangle \text{ for any } X, Y, Z, T \in V^{2n}.$$

Hence, γ_P and β_P satisfy the condition (3.5). Since γ is flat and the bilinear form γ_Ω is null, then also γ_P is flat. Then, by (3.2), we have that $\mathcal{S}(\gamma_P) = W^{q_1,q_1}$ and the remaining of the proof follows from Lemma 3.8. ∎

We now prove the results stated in the Introduction.

Proof of Theorem 1.1 Let the bilinear forms $\gamma, \beta\colon T_{x_0}M \times T_{x_0}M \to N_1^f(x_0) \oplus N_1^f(x_0)$ be defined by (3.1) and (3.3) in terms of the second fundamental form α of f at $x_0 \in M^{2n}$. We endow $N_1^f(x_0) \oplus N_1^f(x_0)$ with the inner product defined by

$$\langle\langle (\xi, \bar{\xi}), (\eta, \bar{\eta}) \rangle\rangle = \langle \xi, \eta \rangle_{N_1^f(x_0)} - \langle \bar{\xi}, \bar{\eta} \rangle_{N_1^f(x_0)}.$$

We claim that γ and β are flat and that (3.5) holds. For a Kaehler manifold, it is a standard fact that the curvature tensor $x \in M^{2n}$ satisfies $R(X,Y)JZ = JR(X,Y)Z$ for any $X, Y, Z \in T_xM$. From this and the Gauss equation for f, we obtain

$$\begin{aligned}\langle\langle \gamma(X,T), \gamma(Z,Y) \rangle\rangle &= \langle \alpha(X,T), \alpha(Z,Y) \rangle - \langle \alpha(X,JT), \alpha(Z,JY) \rangle \\ &= \langle R(X,Z)Y, T \rangle + \langle \alpha(X,Y), \alpha(Z,T) \rangle - \langle R(X,Z)JY, JT \rangle - \langle \alpha(X,JY), \alpha(Z,JT) \rangle \\ &= \langle\langle \gamma(X,Y), \gamma(Z,T) \rangle\rangle.\end{aligned}$$

Since γ satisfies (3.2), then using (2.3), we have

$$\begin{aligned}\langle\!\langle\gamma(X,T),\beta(Z,Y)\rangle\!\rangle &= \langle\!\langle\gamma(X,T),\gamma(Z,Y)\rangle\!\rangle + \langle\!\langle\gamma(X,T),\gamma(JZ,JY)\rangle\!\rangle\\ &= \langle\!\langle\gamma(X,Y),\gamma(Z,T)\rangle\!\rangle + \langle\!\langle\gamma(X,T),\mathcal{T}\gamma(JZ,Y)\rangle\!\rangle\\ &= \langle\!\langle\gamma(X,Y),\gamma(Z,T)\rangle\!\rangle + \langle\!\langle\gamma(X,JT),\gamma(JZ,Y)\rangle\!\rangle\\ &= \langle\!\langle\gamma(X,Y),\gamma(Z,T)\rangle\!\rangle + \langle\!\langle\gamma(X,Y),\gamma(JZ,JT)\rangle\!\rangle\\ &= \langle\!\langle\gamma(X,Y),\beta(Z,T)\rangle\!\rangle.\end{aligned}$$

Using (3.5) and since $\beta(JX,Y) = -\beta(X,JY)$ from (3.3), then

$$\begin{aligned}&\langle\!\langle\gamma(JX,JY),\beta(Z,T)\rangle\!\rangle = \langle\!\langle\gamma(JX,T),\beta(Z,JY)\rangle\!\rangle = -\langle\!\langle\gamma(JX,T),\beta(JZ,Y)\rangle\!\rangle\\ &= -\langle\!\langle\gamma(JX,Y),\beta(JZ,T)\rangle\!\rangle = \langle\!\langle\gamma(JX,Y),\beta(Z,JT)\rangle\!\rangle = \langle\!\langle\gamma(JX,JT),\beta(Z,Y)\rangle\!\rangle.\end{aligned}$$

Then, by (3.3), we have

$$\begin{aligned}&\langle\!\langle\beta(X,Y),\beta(Z,T)\rangle\!\rangle = \langle\!\langle\gamma(X,Y),\beta(Z,T)\rangle\!\rangle + \langle\!\langle\gamma(JX,JY),\beta(Z,T)\rangle\!\rangle\\ &= \langle\!\langle\gamma(X,T),\beta(Z,Y)\rangle\!\rangle + \langle\!\langle\gamma(JX,JT),\beta(Z,Y)\rangle\!\rangle = \langle\!\langle\beta(X,T),\beta(Z,Y)\rangle\!\rangle,\end{aligned}$$

and the claim has been proved. The proof now follows from Theorem 3.1 since we have that $\Delta_c(x_0) = \mathcal{N}(\gamma)$ and $Q(x_0) = \Omega$. ∎

Remark 3.10 The proof of Theorem 1.1 in [2] makes use of Theorem 1 in [6], but that result does not hold for $p = 6$ as was clarified in [7]. Nevertheless, it was also established in [7] that Theorem 1 in [6] still holds for $p = 6$ under a slightly stronger assumption which happens to be satisfied in our case of real Kaehler submanifolds.

Proof of Theorem 1.2 Theorem 1.1 and the Gauss equation for f give

$$\begin{aligned}K(X,JX) &= \langle\alpha(X,X),\alpha(JX,JX)\rangle - \|\alpha(X,JX)\|^2\\ &= -\|\alpha_Q(X,X)\|^2 - \|\alpha_Q(X,JX)\|^2 \le 0\end{aligned}$$

for any $X \in \mathcal{N}(\alpha_P)$. ∎

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