BULL. AUSTRAL. MATH. SOC. 03E25, 03E35, 04A25
VOL. 52 (1995) [373–376]

ELEMENTARY ABELIAN P-GROUPS REVISITED

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For each prime p, a Fraenkel-Mostowski model is constructed in which there are two elementary Abelian p-groups with the same cardinality that are not isomorphic.

INTRODUCTION

In a paper published in 1977 Hickman [1] proves that for each prime $p \geqslant 5$ there is a Fraenkel-Mostowski (FM) model in which the statement,
S(p): There exist two elementary Abelian p-groups with the same cardinality that are not isomorphic.
is true. Hickman says it is not known whether there exists such a model if $p = 2$ or 3. (It is an easy exercise in ZFC to prove the negation of S(p) for every prime p.) In this note, for each prime p, we show that there is an FM model in which S(p) is true. Our result extends Hickman's to the primes 2 and 3, but since the construction is quite a bit different than Hickman's, we present the result here. It follows from the Jech-Sochor transfer results that the result is transferable to ZF, that is, for each prime p, Con(ZF + S(p)).

An elementary Abelian p-group is an Abelian group in which all non-identity elements have order p. First note that an elementary Abelian p-group is just a vector space over the p-element field $\{0, 1, \dots, p-1\}$. Further, every vector space isomorphism is a group isomorphism and every group isomorphism is a vector space isomorphism.

THE MODEL

Let $\mathcal{N}$ be a model of ZFU + AC whose set of atoms A is written as a disjoint union $A = B \cup C \cup R \cup S \cup T$ and the sets B, C, R, S, and T are indexed as follows:

$$B = \{ b_{n,\lambda} : n \in \omega \wedge \lambda < \aleph_1 \},$$
$$C = \{ c_{n,\lambda} : n \in \omega \wedge \lambda < \aleph_1 \},$$
$$R = \{ r_{n,\lambda} : n \in \omega \wedge \lambda < \aleph_1 \},$$
$$S = \{ s_{n,\lambda} : n \in \omega \wedge \lambda < \aleph_1 \},$$
$$T = \{ t_{n,\lambda} : n \in \omega \wedge \lambda < \aleph_1 \}.$$

Received 10th January, 1995

For each $\lambda < \aleph_1$, we shall let $A_\lambda = B_\lambda \cup C_\lambda \cup R_\lambda \cup S_\lambda \cup T_\lambda$ where $B_\lambda = \{b_{n,\lambda} : n \in \omega\}$, $C_\lambda = \{c_{n,\lambda} : n \in \omega\}$ and so on. And let ϕ_λ be the permutation of A which is the product of the cycles $\phi_\lambda = \prod_{n\in\omega} (b_{n,\lambda}, c_{n,\lambda})(r_{n,\lambda}, s_{n,\lambda})$. (So that $\phi_\lambda(t_{n,\lambda}) = t_{n,\lambda}$ and ϕ_λ is the identity on $A - A_\lambda$.) We let G be the group generated by the permutations ϕ_λ for $\lambda < \aleph_1$, Γ the filter of subgroups of G determined by finite supports and $\mathcal{M}$ the model determined by $\mathcal{N}$, G and Γ.

Note that for each $\lambda < \aleph_1$, if $\psi \in G$ fixes any element of $B_\lambda \cup C_\lambda \cup R_\lambda \cup S_\lambda$ then ψ fixes A_λ pointwise. Hence for each $x \in \mathcal{M}$, there is some finite set $\{\lambda_1, \dots, \lambda_n\} \subset \aleph_1$ such that $\mathrm{fix}_G\left(\bigcup_{i=1}^n A_{\lambda_i}\right) =_{\mathrm{def}} \{\phi \in G : \phi \text{ fixes } \bigcup_{i=1}^n A_{\lambda_i} \text{ pointwise}\}$ is in Γ and $\forall \phi \in \mathrm{fix}_G\left(\bigcup_{i=1}^n A_{\lambda_i}\right)$, $\phi(x) = x$. For the rest of the argument we shall refer to $\{\lambda_1, \dots, \lambda_n\}$ as a support of x.

We now define elementary Abelian p-groups G_1 and G_2 (with $p = 2$) both of which are in $\mathcal{M}$ with empty support: G_1 is the vector space over the two element field $\{0,1\}$ with basis $B \cup C$ and G_2 is the vector space over the two element field with basis $R \cup S \cup T$. (More formally, we could define G_1 to be the set of all functions $\mathbf{v} : B \cup C \to \{0,1\}$ such that $\mathbf{v}^{-1}(1)$ is finite, together with the operation $+$ of coordinatewise addition mod 2, similarly for G_2.) If $\mathbf{v} \in G_2$ and $\mathbf{v} = \mathbf{v}_1 + \cdots + \mathbf{v}_n$ where $\mathbf{v}_1, \dots, \mathbf{v}_n \in R \cup S \cup T$, we shall say that $\mathbf{v}_1, \dots, \mathbf{v}_n$ *occur* in $\mathbf{v}$.

Since $\forall \gamma, \lambda < \aleph_1$ and $\forall n \in \omega$, $\phi_\gamma(t_{n,\lambda}) = t_{n,\lambda}$ (and since $t_{n,\lambda} + t_{n,\lambda} = 0$) we have:

Lemma 1. *For all $\mathbf{v} \in G_2$, for all $\lambda, \gamma < \aleph_1$ and for all $n \in \omega$, $t_{n,\lambda}$ does not occur in $\mathbf{v} + \phi_\gamma(\mathbf{v})$.*

Theorem 2. *In $\mathcal{M}$, $|G_1| = |G_2|$.*

Proof: It is clear that the function $H_1^* : B \cup C \to R \cup S$ defined by $H_1^*(b_{n,\lambda}) = r_{n,\lambda}$ and $H_1^*(c_{n,\lambda}) = s_{n,\lambda}$ is in $\mathcal{M}$ and has empty support. We note that H_1^* is a one to one function from a basis for G_1 onto an independent subset of G_2 and therefore can be extended uniquely to a one to one homomorphism H_1 from G_1 into G_2. Similarly, if we define $H_2^* : R \cup S \cup T \to G_1$ by $H_2^*(r_{n,\lambda}) = b_{2n,\lambda}$, $H_2^*(s_{n,\lambda}) = c_{2n,\lambda}$ and $H_2^*(t_{n,\lambda}) = b_{2n+1,\lambda} + c_{2n+1,\lambda}$ then

(1) H_2^* has empty support. (Note that $\phi_\lambda(t_{n,\lambda}) = t_{n,\lambda}$ and
$\phi_\lambda(b_{2n+1,\lambda} + c_{2n+1,\lambda}) = c_{2n+1,\lambda} + b_{2n+1,\lambda} = b_{2n+1,\lambda} + c_{2n+1,\lambda}$.)

(2) H_2^* maps a basis for G_2 one to one onto an independent subset of G_1.

Therefore, H_2^* has empty support and can be extended uniquely to a one to one homomorphism H_2 from G_2 into G_1. Since $|G_1| \leqslant |G_2|$ and $|G_2| \leqslant |G_1|$ in $\mathcal{M}$, we conclude that $|G_1| = |G_2|$ in $\mathcal{M}$. ∎

THEOREM 3. *In $\mathcal{M}$, G_1 and G_2 are not isomorphic.*

PROOF: We argue by contradiction that there is no isomorphism in $\mathcal{M}$ from G_1 onto G_2. Suppose that H is such an isomorphism with support $E = \{\lambda_1, \dots, \lambda_n\}$. The set $X = \bigcup_{j=1}^{n} \left(B_{\lambda_j} \cup C_{\lambda_j}\right)$ is countable and hence there exists $n_0 \in \omega$ and $\lambda_0 < \aleph_1$ such that t_{n_0,λ_0} occurs in no element of $H''X (= \{H(x) : x \in X\})$.

Since H is an isomorphism, $H''(B \cup C)$ is a basis for G_2 and therefore,

$$t_{n_0,\lambda_0} = \mathbf{v}_1 + \cdots + \mathbf{v}_k + \mathbf{u}_1 + \cdots + \mathbf{u}_r \tag{3}$$

where $\mathbf{v}_1, \dots, \mathbf{v}_k \in H''((B \cup C) - X)$, $\mathbf{u}_1, \dots, \mathbf{u}_r \in H''X$ and the elements $\mathbf{v}_1, \dots, \mathbf{v}_k$, $\mathbf{u}_1, \dots, \mathbf{u}_r$ are pairwise distinct. Furthermore, except for the order in which the elements are written, this is the only way of writing t_{n_0,λ_0} as a sum of pairwise distinct elements from $H''(B \cup C)$. By our choice of t_{n_0,λ_0}, t_{n_0,λ_0} does not occur in $\mathbf{u}_1 + \cdots + \mathbf{u}_r$ (and hence $k > 0$). We shall arrive at a contradiction (and hence complete the proof) by showing that t_{n_0,λ_0} does not occur in $\mathbf{v}_1 + \cdots + \mathbf{v}_k$. ∎

LEMMA 4. *Assume $1 \leqslant j \leqslant k$, then there is a λ not in E and an i with $1 \leqslant i \leqslant k$ and $i \neq j$ such that $\phi_\lambda(\mathbf{v}_j) = \mathbf{v}_i$.*

PROOF: Assume that $\mathbf{v}_j = H(b_{n,\lambda})$ where $\lambda \notin E$. (The proof is similar if $\mathbf{v}_j = H(c_{n,\lambda})$ where $\lambda \notin E$.) Since ϕ_λ fixes X pointwise, $\phi_\lambda(H) = H$. Further $\phi_\lambda(b_{n,\lambda}) = c_{n,\lambda}$ so

$$\phi_\lambda(\mathbf{v}_j) = \phi_\lambda(H(b_{n,\lambda})) = H(\phi_\lambda(b_{n,\lambda})) = H(c_{n,\lambda}) \neq H(b_{n,\lambda}) = \mathbf{v}_j \tag{4}$$

since H is one to one. Applying ϕ_λ to both sides of the equality (3) gives

$$t_{n_0,\lambda_0} = \phi_\lambda(\mathbf{v}_1) + \cdots + \phi_\lambda(\mathbf{v}_k) + \mathbf{u}_1 + \cdots + \mathbf{u}_r. \tag{5}$$

(ϕ_λ fixes t_{n_0,λ_0} and fixes X pointwise. Therefore, ϕ_λ fixes $H''X$ pointwise.)

Since $\phi_\lambda(B \cup C) = B \cup C$, we have $\phi_\lambda(H''(B \cup C)) = H''(B \cup C)$. Therefore (5) expresses t_{n_0,λ_0} as a sum of vectors from the basis $H''(B \cup C)$. Hence the right hand sides of (3) and (5) contain the same terms and so $\phi_\lambda(\mathbf{v}_j) = \mathbf{v}_i$ for some i such that $1 \leqslant i \leqslant k$. By (4), $\phi_\lambda(\mathbf{v}_j) \neq \mathbf{v}_j$, hence $i \neq j$. This completes the proof of Lemma 4.

It follows that $\mathbf{v}_1 + \cdots + \mathbf{v}_k$ can be written as a sum of vectors of the form $\mathbf{v}_j + \phi_\lambda(\mathbf{v}_j)$. Since (by Lemma 1) t_{n_0,λ_0} does not occur in $\mathbf{v}_j + \phi_\lambda(\mathbf{v}_j)$ it follows that t_{n_0,λ_0} does not occur in $\mathbf{v}_1 + \cdots + \mathbf{v}_k$, which completes the proof of Theorem 3. ∎

It is easy to see how the result can be generalised to any prime p. We shall indicate the construction for $p = 3$. Let $A = B \cup C \cup D \cup R \cup S \cup T \cup W$. The group G is generated, by $\phi_\lambda = \prod_{n \in \omega} (a_{n,\lambda}, b_{n,\lambda}, c_{n,\lambda})(r_{n,\lambda}, s_{n,\lambda}, t_{n,\lambda})$, $\lambda < \aleph_1$, where $\phi_\lambda(w_{n,\lambda}) = w_{n,\lambda}$ for all

$n \in \omega$. Supports are again finite. The elementary Abelian p-groups, are defined so that G_1 is the vector space over the three element field $\{0,1,2\}$ with basis $B \cup C \cup D$ and G_2 is the vector space over the same three element field with basis $R \cup S \cup T \cup W$. The proofs of the lemmas are similar to that given above using the fact that for all $\mathbf{v} \in G_2$, $w_{n,\lambda}$ does not occur in $\mathbf{v} + \phi_\lambda(\mathbf{v}) + \phi_\lambda^2(\mathbf{v})$. The proofs of Theorems 2 and 3 are also similar to the proofs given above.

References

[1] J.S. Hickman, 'A remark on elementary Abelian groups', *Bull. Austral. Math. Soc.* **16** (1977), 213–217.

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https://doi.org/10.1017/S0004972700014866 Published online by Cambridge University Press