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A CHARACTERISATION OF 3-JORDAN HOMOMORPHISMS ON BANACH ALGEBRAS

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Abstract

We show that, under special hypotheses, each 3-Jordan homomorphism φ between Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is a 3-homomorphism.

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1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be complex Banach algebras and $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a linear map. Then φ is called an n-homomorphism if, for all $a_1, a_2, \ldots, a_n \in \mathcal{A}$,

$$\varphi(a_1 a_2 \cdots a_n) = \varphi(a_1)\varphi(a_2)\cdots\varphi(a_n).$$

The concept of an n-homomorphism was studied for complex algebras by Hejazian *et al.* in [6]. A 2-homomorphism is just a homomorphism in the usual sense. One may refer to [2] for certain properties of 3-homomorphisms.

Eshaghi Gordji [4] introduced the concept of an n-Jordan homomorphism. A linear map φ between Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is called an n-Jordan homomorphism if

$$\varphi(a^n) = \varphi(a)^n, \quad a \in \mathcal{A}.$$

A 2-Jordan homomorphism is called simply a Jordan homomorphism.

It is obvious that each n-homomorphism is an n-Jordan homomorphism, but in general the converse is false. The converse statement may be true under certain conditions. For example, it is shown in [4] that every n-Jordan homomorphism between two commutative Banach algebras is an n-homomorphism for $n \in \{2, 3, 4\}$ and this result is extended to the case $n = 5$ in [5].

The following theorem is due to Zelazko [8]. See also [10] for another approach to the same result.

THEOREM 1.1. *Suppose that $\mathcal{A}$ is a Banach algebra, which need not be commutative, and suppose that $\mathcal{B}$ is a semisimple commutative Banach algebra. Then each Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a homomorphism.*

In [4], Eshaghi Gordji claimed a proof of the following assertion.

ASSERTION 1.2. *Suppose that $\mathcal{A}$ is a Banach algebra, which need not be commutative, and suppose that $\mathcal{B}$ is a semisimple commutative Banach algebra. Then each 3-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a 3-homomorphism.*

Assertion 1.2 is [4, Theorem 2.5] and the proof given in [4] proceeds in two steps. In the first step, it is claimed that if we replace y by $y - z$ in [4, (2.9)], we obtain [4, (2.10)]. This is true if the Banach algebra $\mathcal{A}$ is commutative, but it does not seem to follow in the general case when $\mathcal{A}$ need not be commutative. Also, it is claimed that if we replace x by $x + z$ in [4, (2.14)], then

$$h(yx^2 + yz^2 + 2yxz - x^2y - z^2y - 2xzy) = 0,$$

but this too does not seem to follow without the commutativity of $\mathcal{A}$. Since (2.10) and this last equation may not be valid, it seems that the conditions which are assumed in Assertion 1.2 do not imply that φ is a 3-homomorphism.

A linear map φ between Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is called a co-homomorphism if

$$\varphi(ab) = -\varphi(a)\varphi(b), \quad a, b \in \mathcal{A}$$

and it is called a co-Jordan homomorphism if $\varphi(a^2) = -\varphi(a)^2$ for all $a \in \mathcal{A}$.

In this paper, we prove Assertion 1.2 with the additional hypothesis that the Banach algebra $\mathcal{A}$ is unital. By [7, Lemma 6.3.2], each Jordan homomorphism is 3-Jordan, but the converse is not true. We first prove that if $\mathcal{A}$ is unital, then each 3-Jordan homomorphism from $\mathcal{A}$ into $\mathbb{C}$ is either a Jordan homomorphism or a co-Jordan homomorphism. Then we use this fact to prove our main result (Theorem 2.4 below).

2. Main results

We commence with a characterisation of a co-Jordan homomorphism.

THEOREM 2.1. *Suppose that $\mathcal{A}$ is a Banach algebra, which need not be commutative. Then each co-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathbb{C}$ is a co-homomorphism.*

PROOF. Suppose that φ is a co-Jordan homomorphism, so that $\varphi(a^2) = -\varphi(a)^2$ for all $a \in \mathcal{A}$. Replacing a by $a + b$ gives

$$\varphi(ab + ba) = -2\varphi(a)\varphi(b), \quad a, b \in \mathcal{A}. \tag{2.1}$$

Then, by (2.1),

$$\begin{aligned}
2\varphi(aba) &= \varphi[(ab + ba)a + a(ab + ba)] - \varphi[a^2b + ba^2] \\
&= -2[\varphi(ab + ba)\varphi(a) - \varphi(a^2)\varphi(b)] \\
&= -2[-2\varphi(a)^2\varphi(b) + \varphi(a)^2\varphi(b)] \\
&= 2\varphi(a)^2\varphi(b).
\end{aligned}$$

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Therefore,

$$\varphi(aba) = \varphi(a)^2\varphi(b), \quad a, b \in \mathcal{A}. \tag{2.2}$$

Let a and b be arbitrary elements of $\mathcal{A}$ and put

$$2t = \varphi(ab - ba). \tag{2.3}$$

It follows from (2.1) and (2.3) that

$$\varphi(ab) - t = -\varphi(a)\varphi(b), \quad \varphi(ba) + t = -\varphi(a)\varphi(b). \tag{2.4}$$

By (2.2)–(2.4),

$$\begin{aligned} 4t^2 &= \varphi(ab - ba)^2 = -\varphi[(ab - ba)^2] \\ &= -\varphi[(ab)^2 + (ba)^2 - ab^2a - ba^2b] \\ &= [\varphi(ab)^2 + \varphi(ba)^2 + \varphi(a)^2\varphi(b^2) + \varphi(b)^2\varphi(a^2)] \\ &= [t - \varphi(a)\varphi(b)]^2 + [-t - \varphi(a)\varphi(b)]^2 - [2\varphi(a)^2\varphi(b)^2] \\ &= 2t^2. \end{aligned}$$

Hence, $t = 0$, which proves that $\varphi(ab) = \varphi(ba)$. Therefore, by (2.1), $\varphi(ab) = -\varphi(a)\varphi(b)$ and the proof is complete. □

Lemma 2.2. *Let $\mathcal{A}$ be a unital Banach algebra with unit e and $\varphi : \mathcal{A} \longrightarrow \mathbb{C}$ be a nonzero 3-Jordan homomorphism. Then $\varphi(e) \neq 0$.*

Proof. Let φ be a nonzero 3-Jordan homomorphism, so that $\varphi(a^3) = \varphi(a)^3$ for all $a \in \mathcal{A}$. Replacing a by $a + b$ gives

$$\varphi(ab^2 + b^2a + a^2b + ba^2 + aba + bab) = 3\varphi(a)^2\varphi(b) + 3\varphi(a)\varphi(b)^2 \tag{2.5}$$

and replacing b by $-b$ in (2.5) gives

$$\varphi(ab^2 + b^2a - a^2b - ba^2 - aba + bab) = -3\varphi(a)^2\varphi(b) + 3\varphi(a)\varphi(b)^2. \tag{2.6}$$

By (2.5) and (2.6),

$$\varphi(ab^2 + b^2a + bab) = 3\varphi(a)\varphi(b)^2, \quad a, b \in \mathcal{A}. \tag{2.7}$$

Now assume that $\varphi(e) = 0$ and take $b = e$ in (2.7). It follows that $\varphi(a) = 0$ for all $a \in \mathcal{A}$, which is a contradiction. □

Lemma 2.3. *Let $\mathcal{A}$ be a unital Banach algebra with unit e and $\varphi : \mathcal{A} \longrightarrow \mathbb{C}$ be a nonzero 3-Jordan homomorphism. Then φ is either a Jordan homomorphism or a co-Jordan homomorphism.*

Proof. Let φ be a nonzero 3-Jordan homomorphism. Then, for all $a \in \mathcal{A}$,

$$\varphi(a^3) = \varphi(a)^3. \tag{2.8}$$

Replace a by $a + e$ in (2.8) to obtain

$$\varphi(a + a^2) = \varphi(a)^2\varphi(e) + \varphi(a)\varphi(e)^2. \tag{2.9}$$

Replacing a by e in (2.8) gives $\varphi(e) = \varphi(e)^3$. By Lemma 2.2, $\varphi(e) \neq 0$ and so $\varphi(e) = 1$ or $\varphi(e) = -1$. If $\varphi(e) = 1$, (2.9) gives

$$\varphi(a^2) = \varphi(a)^2$$

for all $a \in \mathcal{A}$; hence, φ is Jordan. If $\varphi(e) = -1$, (2.9) gives

$$\varphi(a^2) = -\varphi(a)^2$$

and so φ is co-Jordan. □

Now we state and prove the main theorem.

THEOREM 2.4. *Suppose that $\mathcal{A}$ is a unital Banach algebra, which need not be commutative, and suppose that $\mathcal{B}$ is a semisimple commutative Banach algebra. Then each 3-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a 3-homomorphism.*

PROOF. We first assume that $\mathcal{B} = \mathbb{C}$ and let $\varphi : \mathcal{A} \longrightarrow \mathbb{C}$ be a 3-Jordan homomorphism. By Lemma 2.3, φ is either a Jordan homomorphism or a co-Jordan homomorphism. If φ is Jordan, then by Zelazko's theorem it is a homomorphism and so it is a 3-homomorphism. If φ is co-Jordan, then by Theorem 2.1 it is a co-homomorphism, that is, for all $a, b \in \mathcal{A}$,

$$\varphi(ab) = -\varphi(a)\varphi(b).$$

Therefore,

$$\varphi(abc) = -\varphi(a)\varphi(bc) = -\varphi(a)[-\varphi(b)\varphi(c)] = \varphi(a)\varphi(b)\varphi(c)$$

for all $a, b, c \in \mathcal{A}$, and φ is 3-homomorphism.

Now suppose that $\mathcal{B}$ is semisimple and commutative. Let $\mathfrak{M}(\mathcal{B})$ be the maximal ideal space of $\mathcal{B}$ and associate with each $f \in \mathfrak{M}(\mathcal{B})$ a function $\varphi_f : \mathcal{A} \longrightarrow \mathbb{C}$ defined by

$$\varphi_f(a) := f(\varphi(a)), \quad a \in \mathcal{A}.$$

Pick $f \in \mathfrak{M}(\mathcal{B})$. It is easy to see that φ_f is a 3-Jordan homomorphism, so by the above argument it is a 3-homomorphism. Thus, by the definition of φ_f,

$$f(\varphi(abc)) = f(\varphi(a))f(\varphi(b))f(\varphi(c)) = f(\varphi(a)\varphi(b)\varphi(c)).$$

Since $f \in \mathfrak{M}(\mathcal{B})$ was arbitrary and $\mathcal{B}$ is assumed to be semisimple,

$$\varphi(abc) = \varphi(a)\varphi(b)\varphi(c)$$

for all $a, b, c \in \mathcal{A}$. This complete the proof. □

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It is well known that, on the second dual space $\mathcal{A}''$ of a Banach algebra $\mathcal{A}$, there are two multiplications, called the first and second Arens products, which make $\mathcal{A}''$ into a Banach algebra [1]. If these products coincide on $\mathcal{A}''$, then $\mathcal{A}$ is said to be Arens regular. For more information on the Arens products, one may refer to [3].

It is shown in [3] that every C^*-algebra $\mathcal{A}$ is Arens regular and semisimple. Also, the second dual of a C^*-algebra is also a C^*-algebra.

Corollary 2.5. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, where $\mathcal{A}$ need not be commutative, and suppose that $\mathcal{B}$ is commutative. Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a 3-Jordan homomorphism. Then $\varphi'' : \mathcal{A}'' \longrightarrow \mathcal{B}''$ is a 3-homomorphism.*

Proof. Suppose that $\mathcal{B}$ is a commutative C^*-algebra. Then, by [9, Lemma 1.2], $\mathcal{B}''$ is commutative and it is semisimple, because every C^*-algebra is semisimple. On the other hand, the second dual of a C^*-algebra is unital [3], so $\mathcal{A}''$ is unital. Therefore, the result follows from [10, Theorem 8] and Theorem 2.4. □

The next result follows from the preceding corollary and [2, Theorem 2.1].

Corollary 2.6. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, where $\mathcal{A}$ need not be commutative, and suppose that $\mathcal{B}$ is commutative. Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be an involution-preserving 3-Jordan homomorphism. Then $\|\varphi''\| \leq 1$.*

For a nonsemisimple Banach algebra $\mathcal{B}$, the next result characterises the 3-Jordan homomorphisms.

Theorem 2.7. *Suppose that φ is a 3-Jordan homomorphism from a unital Banach algebra $\mathcal{A}$ into a commutative Banach algebra $\mathcal{B}$ such that, for all $a, b, c \in \mathcal{A}$,*

$$\varphi(abc - acb) = 0. \tag{2.10}$$

Then φ is a 3-homomorphism.

Proof. Let e be the unit element of $\mathcal{A}$. Taking $a = e$ in (2.10) gives $\varphi(bc - cb) = 0$ for all $b, c \in \mathcal{A}$. Therefore,

$$\varphi((ab)c) = \varphi(c(ab)) = \varphi(c(ba))$$

and

$$\varphi(a(bc)) = \varphi((bc)a) = \varphi(b(ca)) = \varphi(b(ac)).$$

That is,

$$\varphi(abc) = \varphi(xyz), \tag{2.11}$$

whenever (x, y, z) is a permutation of (a, b, c). By the assumption, φ is a 3-Jordan homomorphism, that is, $\varphi(a^3) = \varphi(a)^3$ for all $a \in \mathcal{A}$. Replacing a by $a + b$ gives

$$\varphi[ab^2 + b^2a + a^2b + ba^2 + aba + bab] = 3\varphi(a)\varphi(b)^2 + 3\varphi(a)^2\varphi(b) \tag{2.12}$$

and replacing b by $-b$ in (2.12) gives

$$\varphi[ab^2 + b^2a - a^2b - ba^2 - aba + bab] = 3\varphi(a)\varphi(b)^2 - 3\varphi(a)^2\varphi(b). \tag{2.13}$$

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By (2.12) and (2.13),

$$\varphi[ab^2 + b^2a + bab] = 3\varphi(a)\varphi(b)^2. \tag{2.14}$$

Replacing b by $b - c$ in (2.14),

$$\varphi[abc + acb + bac + bca + cab + cba] = 6\varphi(a)\varphi(b)\varphi(c). \tag{2.15}$$

By (2.11) and (2.15),

$$\varphi(abc) = \varphi(a)\varphi(b)\varphi(c),$$

as required. □

In view of Assertion 1.2 and Theorem 2.4, it is natural to ask the next question.

Question 2.8. Does Assertion 1.2 hold without any additional hypothesis?

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