Ergod. Th. & Dynam. Sys., page 1 of 16 © The Author(s), 2024. Published by Cambridge University Press.
doi:10.1017/etds.2024.82

Global rigidity of smooth $\mathbb{Z} \ltimes_\lambda \mathbb{R}$-actions on $\mathbb{T}^2$

CHANGGUANG DONG† and YI SHI‡

† *Chern Institute of Mathematics and LPMC, Nankai University, Tianjin 300071, Tianjin, China*
(e-mail: dongchg@nankai.edu.cn)
‡ *School of Mathematics, Sichuan University, Chengdu 610065, Sichuan, China*
(e-mail: shiyi@scu.edu.cn)

(*Received* 21 *March* 2024 *and accepted in revised form* 13 *September* 2024)

Abstract. For $\lambda > 1$, we consider the locally free $\mathbb{Z} \ltimes_\lambda \mathbb{R}$ actions on $\mathbb{T}^2$. We show that if the action is C^r with $r \geq 2$, then it is $C^{r-\epsilon}$-conjugate to an affine action generated by a hyperbolic automorphism and a linear translation flow along the expanding eigen-direction of the automorphism. In contrast, there exists a $C^{1+\alpha}$-action which is semi-conjugate, but not topologically conjugate to an affine action.

Key words: global rigidity, solvable group action, Anosov diffeomorphism, smooth conjugacy
2020 Mathematics Subject Classification: 37C85 (Primary); 37C15 (Secondary)

1. *Introduction*

In dynamical systems, rigidity phenomena have been extensively studied over the past decades. In particular, there have been a tremendous number of advancements, with fascinating applications not only in dynamical systems, but also in other fields such as geometry, number theory, etc. These include, but are not limited to, orbit closure rigidity (e.g. [**6**, **21**, **38**]), measure rigidity (e.g. [**16**, **17**, **32**, **38**]), local rigidity (e.g. [**10**, **11**, **18**]), global rigidity (e.g. [**12**, **42**]) etc. This article is a contribution to the global rigidity program. Namely, we would like to understand, describe, and classify all the actions of a specific group on a specific manifold.

The group we are considering is a special type of so-called abelian-by-cyclic group. It is given as follows. Let $\lambda > 1$ and $G_\lambda := \mathbb{Z} \ltimes_\lambda \mathbb{R}$ be defined by the following group relation:

$$(m, t) \circ (n, s) = (m + n, \lambda^m s + t) \quad \text{for any } (m, t), (n, s) \in \mathbb{Z} \times \mathbb{R}.$$

We are interested in the action of G_λ on the 2-torus $\mathbb{T}^2$ by smooth diffeomorphisms. Typical examples of such actions are affine actions. Let $A \in \mathrm{GL}(2, \mathbb{Z})$ with an eigenvalue $\lambda > 1$ and corresponding unit eigenvector v with $Av = \lambda v$. (To clarify, $\mathrm{GL}(2, \mathbb{Z})$ refers to the group of 2×2 matrices over $\mathbb{Z}$ with determinant ± 1.) Then, for every constant $a > 0$, it is easy to see that the automorphism A (induced by the matrix A on $\mathbb{T}^2$, which we denote by A for simplicity) together with the flow generated by av (the flow direction is v with constant velocity a) on $\mathbb{T}^2$ generate the group G_λ, and hence this gives an affine solvable action of G_λ.

One may wonder whether there exist other smooth G_λ actions on $\mathbb{T}^2$ up to smooth conjugacy. To answer this question, we show that any C^{2+} locally free action of G_λ on $\mathbb{T}^2$ is smoothly conjugate to an affine action.

THEOREM 1.1. *Let $\lambda > 1$ and $r \geq 2$. Suppose that $\rho : \mathbb{Z} \ltimes_\lambda \mathbb{R} \to \mathrm{Diff}^r(\mathbb{T}^2)$ is a locally free action; then it is $C^{r-\epsilon}$ conjugate to an affine action for any $\epsilon > 0$. More precisely, there exist:*

- *a hyperbolic automorphism $A \in \mathrm{GL}(2, \mathbb{Z})$ where λ is the unstable eigenvalue of A;*
- *a flow v_t generated by the unit unstable vector field of A,*

such that ρ is $C^{r-\epsilon}$ conjugate to the group action generated by $\{A, v_{at}\}$ for some $a \in \mathbb{R} \setminus \{0\}$. In particular, if the action ρ is C^∞-smooth, then it is C^∞-smoothly conjugate to an affine action.

Remark 1.2. If the action ρ is orientation preserving, then the hyperbolic automorphism obtained in Theorem 1.1 is induced by a hyperbolic element in $\mathrm{SL}(2, \mathbb{Z})$ instead of $\mathrm{GL}(2, \mathbb{Z})$.

Recently, there has been an increasing interest in the study of rigidity properties for actions of abelian-by-cyclic groups, see [**1**, **7**, **8**, **29**, **35**, **44**]. In the literature, there has been much more attention to higher rank abelian group actions and the well-known Zimmer program (classifying actions of higher rank Lie groups/lattices). This may be the case because of the following. The abelian actions have lots of symmetry either along Lyapunov foliations in the ambient space or from the structure of an acting group and, more importantly, there are many deep applications in the Diophantine approximation etc. As for the Zimmer program, it aims to classify higher rank Lie groups/lattices acting on low-dimensional manifolds, which brings together many fields such as group theory, dynamics, and rigidity. In contrast, the group we consider here does not seem to have certain properties like symmetry or rigidity (or super rigidity), so it is commonly known that, in general, one should not expect any rigidity phenomenon for such group actions. Nevertheless, it is quite surprising, as we state in Theorem 1.1, that when restricted to some special manifolds (say $\mathbb{T}^2$), it is still possible to obtain the rigidity result.

There are a few interesting works that are related to ours. We list some of them here. By considering the same acting group, in [**35**], a local rigidity result on $\mathbb{T}^d$ is proven under some additional conditions (Diophantine+Anosov). Additionally, in the Lie group setting [**1**, **43**], the authors obtained a few local rigidity results for certain special solvable group actions, under different conditions. We also note that in [**7**, **8**, **29**, **44**], the authors studied various discrete abelian-by-cyclic group actions, and showed certain local/global rigidity. Additionally, we refer to [**2**, **18**, **29**] and the references therein for more details about these works.

https://doi.org/10.1017/etds.2024.82 Published online by Cambridge University Press

From another point of view, our work fits in the smooth linearization program of Anosov diffeomorphisms on a torus, which aims to obtain global rigidity under certain conditions. In [13, 14], de la Llave obtained smooth conjugacy on $\mathbb{T}^2$ under the assumptions that the Anosov diffeomorphisms are topological conjugate and the Lyapunov exponents of the corresponding periodic orbits are the same. Since then, this has been generalized to Anosov diffeomorphisms on higher dimensional tori under similar conditions, see for example [15, 24] and the references therein. In [19], it has been shown that an Anosov diffeomorphism on a four-dimensional torus that preserves a symplectic form is C^∞-conjugate to a linear automorphism, provided the stable and unstable foliations are C^∞. Around the same time, in a series of works [3–5], the authors considered higher dimensional Anosov diffeomorphisms/flows, and obtained C^∞ conjugacy to algebraic models under the smoothness conditions of both stable and unstable distributions, together with conditions of some contact/symplectic structure or preserving a C^∞ connection. Compared with these results, Theorem 1.1 is new in the sense that we only assume the smoothness of one of the stable and unstable distributions (as well as the group relation).

Now let us explain briefly our argument. We obtain certain hyperbolicity by combining Denjoy's theory for circle maps and the geometry of the invariant foliations and then, via Franks [20], we get topological conjugacy. After that, we obtain the rigidity of Lyapunov exponents, which can be approximated by those on periodic orbits. From here, we can complete the proof by using [13, 14, 30]. Our technique shares some similarity to [44]; however, neither do we use a Kolmogorov–Arnold–Moser iterative scheme nor assume *a priori* any hyperbolicity of the action or Diophantine condition on the rotation number (vector), all of which are heavily relied on in [35, 44]. Let us remark that, to extend our argument in the higher dimensional manifold, a result analogous to Herman's result for pseudo rotations seems to be necessary.

We would like to emphasize that the regularity assumption, that is, $r \geq 2$, is crucial in the proof. In particular, the assertion in Theorem 1.1 cannot be obtained if the action ρ is only $C^{1+\alpha}$-smooth for some $\alpha \in [0, 1)$. We have the following example of a $C^{1+\alpha} G_\lambda$-action which is not topologically conjugate to a linear model.

Example 1.3. Let $A \in \mathrm{GL}(2, \mathbb{Z})$ be a hyperbolic automorphism on $\mathbb{T}^2$. One can carry out the DA (derived from Anosov) construction in a small neighborhood of a fixed point of A (for details see [41, (9.4d)] and [39, Ch. 8.8]) to obtain a diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ satisfying the conditions:

- f is partially hyperbolic with the splitting $T\mathbb{T}^2 = E^{cs} \oplus E^u$, which admits a $C^{1+\alpha}$-smooth unstable foliation $\mathcal{F}^u$ tangent to E^u and a linear foliation tangent to E^{cs}, which is the stable foliation of A;
- the Ω-set of f consists of a source and a hyperbolic expanding attractor.

The fact that f preserves the linear stable foliation of A implies that $\|Df|_{E^u(x)}\| = \lambda$ for every $x \in \mathbb{T}^2$ by taking an adapted metric. Let ϕ_t be a flow going through the unstable foliation $\mathcal{F}^u$ of f with constant flow speed preserving the linear stable foliation of A. Then the action $\rho : \mathbb{Z} \ltimes_\lambda \mathbb{R} \to \mathrm{Diff}^{1+\alpha}(\mathbb{T}^2)$ is defined by the fact that

$$f = \rho(1, 0) \quad \text{and} \quad \phi_t = \rho(0, t) \quad \text{satisfies } f \circ \phi_t = \phi_{\lambda t} \circ f.$$

https://doi.org/10.1017/etds.2024.82 Published online by Cambridge University Press

Since f is not topologically conjugate to A, this action is not topologically conjugate to any affine actions.

As pointed out to us by the anonymous referee, one may construct a similar example from [23, §5]. We also want to remark that, under the existence of Anosov diffeomorphisms, it is possible to obtain a smooth conjugacy result under a weaker regularity condition.

2. *Invariant foliation and linear action*

Let $f = \rho(1, 0)$ and $\phi_t = \rho(0, t)$. By the group relation,

$$f \circ \phi_t = \phi_{\lambda t} \circ f. \tag{2.1}$$

Let $\mathcal{X}$ be the vector field generating ϕ_t, namely

$$\mathcal{X}(x) = \frac{d}{dt}|_{t=0}\phi_t(x).$$

Notice that by our assumption, $\mathcal{X}$ is a smooth vector field and $\mathcal{X}(x) \neq 0$ for every $x \in \mathbb{T}^2$.

We have the following important observation.

LEMMA 2.1. *There exists a constant $C \geq 1$ such that, for any $n \in \mathbb{N}$ and $x \in \mathbb{T}^2$,*

$$C^{-1}\lambda^n \leq \|Df^n|_{\mathcal{X}(x)}\| \leq C\lambda^n.$$

In particular, if we denote by $\mathcal{F}^u$ the foliation generated by ϕ_t, then for any ergodic measure μ of f, the Lyapunov exponent of f on $\mathcal{F}^u$ is $\log \lambda$.

Proof. From (2.1), we have $f \circ \phi_t \circ f^{-1}(x) = \phi_{\lambda t}(x)$. By taking derivatives on both sides with respect to t, we have

$$Df|_{\mathcal{X}(f^{-1}x)} \cdot \mathcal{X}(f^{-1}x) = \lambda\mathcal{X}(x).$$

Hence,

$$Df^n|_{\mathcal{X}(x)} \cdot \mathcal{X}(x) = \lambda^n\mathcal{X}(f^n(x));$$

therefore,

$$\|Df^n|_{\mathcal{X}(x)}\| = \lambda^n \frac{\|\mathcal{X}(f^n(x))\|}{\|\mathcal{X}(x)\|}.$$

The proof is complete by setting $C = \max_x\{\|\mathcal{X}(x)\|\}/\min_x\{\|\mathcal{X}(x)\|\}$. □

LEMMA 2.2. *Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a diffeomorphism which preserves a one-dimensional expanding foliation $\mathcal{F}^u$: there exists $C_0 > 0$, $\lambda_0 > 1$ such that*

$$f(\mathcal{F}^u(x)) = \mathcal{F}^u(f(x)) \quad \text{and} \quad \|Df^n|_{T\mathcal{F}^u(x)}\| \geq C_0 \cdot \lambda_0^n \quad \text{for all } x \in \mathbb{T}^2, \text{ for all } n \geq 0.$$

Then, the expanding foliation $\mathcal{F}^u$ satisfies the conditions that:

(1) *$\mathcal{F}^u$ is a suspension foliation with irrational rotation numbers, that is, there exists a simple closed circle $\gamma : \mathbb{S}^1 \to \mathbb{T}^2$ which transversally intersects every leaf of $\mathcal{F}^u$, and the holonomy map induced by $\mathcal{F}^u$ on γ has an irrational rotation number;*

(2) *if $\mathcal{F}^u$ is C^2, then it is minimal on $\mathbb{T}^2$;*

https://doi.org/10.1017/etds.2024.82 Published online by Cambridge University Press

(3) *the lifting foliation $\tilde{\mathcal{F}}^u$ of $\mathcal{F}^u$ is quasi-isometric on $\mathbb{R}^2$, that is, there exist constants $a, b > 0$ such that, for all $x \in \mathbb{R}^2$ and $y \in \tilde{\mathcal{F}}^u(x)$,*

$$d_{\tilde{\mathcal{F}}^u}(x, y) \leq a \cdot d_{\mathbb{R}^2}(x, y) + b. \tag{2.2}$$

Here, $d_{\mathbb{R}^2}$, $d_{\tilde{\mathcal{F}}^u}$ are distance functions on $\mathbb{R}^2$ and leaves of $\tilde{\mathcal{F}}^u$, respectively.

Proof. Since f is uniformly expanding along $\mathcal{F}^u$, f^{-1} uniformly contracts leaves of $\mathcal{F}^u$. This implies that $\mathcal{F}^u$ has no closed leaves. Otherwise, assume $\gamma^u \in \mathcal{F}^u$ is a closed leaf. Then, the length of $f^{-n}(\gamma^u)$ tends to zero as $n \to +\infty$. By taking a subsequence if necessary, $f^{-k_n}(\gamma^u) \to z \in \mathbb{T}^2$, which implies that the leaf $\mathcal{F}^u(z) = \{z\}$ and contradicts with the condition on the foliation $\mathcal{F}^u$.

Since $\mathcal{F}^u$ has no closed leaves, [27, Theorem 4.3.3] shows that $\mathcal{F}^u$ is a suspension foliation with an irrational rotation number, that is, there exists a simple closed circle $\gamma : \mathbb{S}^1 \to \mathbb{T}^2$ which transversally intersects every leaf of $\mathcal{F}^u$, and the holonomy map induced by $\mathcal{F}^u$ on γ has an irrational rotation number. In particular, if $\mathcal{F}^u$ is C^2-smooth, then Denjoy's theorem shows that $\mathcal{F}^u$ is minimal. This proves items (1) and (2).

Finally, we show that the lifting foliation $\tilde{\mathcal{F}}^u$ is quasi-isometric. Since γ is a simple closed curve that intersects every leaf of $\mathcal{F}^u$, there exists $a_1 > 0$ such that, for every $x \in \gamma$ and $y \in \mathcal{F}^u(x) \cap \gamma$ which contains no point in γ between x and y in $\mathcal{F}^u(x)$, we have

$$d_{\mathcal{F}^u}(x, y) < a_1.$$

Since γ is a simple closed curve in $\mathbb{T}^2$ and transverse to the one-dimensional foliation $\mathcal{F}^u$, we have that:

- γ is homotopically non-trivial (otherwise it bounds a disk and Poincare–Hopf theorem implies $\mathcal{F}^u$ has singularities);
- there exists $(k, l) \in \mathbb{Z}^2$ with k, l coprime, such that for every lift $\tilde{\gamma} \subset \mathbb{R}^2$ of γ,
$$\tilde{\gamma} = \tilde{\gamma} + (k, l);$$
- there exists $b_1 > 0$ such that, for every $x \in \tilde{\gamma}$, the line $L(x) = \{x + t \cdot (k, l) : t \in \mathbb{R}\}$, satisfies that the Hausdorff distance
$$d_H(\tilde{\gamma}, L(x)) = \max\left\{\max_{z\in\tilde{\gamma}}\{d_{\mathbb{R}^2}(z, L(x))\}, \max_{z\in L(x)}\{d_{\mathbb{R}^2}(z, \tilde{\gamma})\}\right\} < b_1.$$

Taking $N_1 \in \mathbb{N}$ large enough such that $3b_1 \ll N_1 \cdot \|(l, -k)\|$, then

$$\begin{aligned}
&\inf\{d_{\mathbb{R}^2}(z, y) : z \in \tilde{\gamma}, y \in (\tilde{\gamma} + N_1(l, -k))\} \\
&\quad \geq \inf\{d_{\mathbb{R}^2}(z, y) : z \in L(x), y \in L(x + N_1(l, -k))\} \\
&\qquad - d_H(\tilde{\gamma}, L(x)) - d_H(\tilde{\gamma} + N_1(l, -k), L(x + N_1(l, -k))) \\
&\quad \geq N_1 \cdot \|(l, -k)\| - b_1 - b_1 \\
&\quad \geq b_1.
\end{aligned}$$

There exists $N_2 > 0$ (which depends on γ only), such that, for every $x \in \tilde{\gamma}$ and $y \in \tilde{\mathcal{F}}^u(x) \cap (\tilde{\gamma} + N_1(l, -k))$, the segment $[x, y]^u \subset \mathcal{F}^u(x)$ with endpoints x, y intersects $\tilde{\gamma} + \mathbb{Z}^2$ (all lifts of γ) with $(N_2 + 1)$-points. Thus, we have

$$d_{\mathbb{R}^2}(x, y) \geq b_1 \quad \text{and} \quad d_{\tilde{\mathcal{F}}^u}(x, y) \leq N_2 \cdot a_1.$$

https://doi.org/10.1017/etds.2024.82 Published online by Cambridge University Press

For every $x \in \mathbb{R}^2$ and $y \in \tilde{\mathcal{F}}^u(x)$ with $d_{\tilde{\mathcal{F}}^u}(x, y)$ large enough, there exist:

- $x_1 \in \tilde{\mathcal{F}}^u(x) \cap \tilde{\gamma}_1$ with $d_{\tilde{\mathcal{F}}^u}(x, x_1) < a_1$, where $\tilde{\gamma}_1$ is a lift of γ;
- $m \in \mathbb{Z}$ with $|m|$ large, and $y_1 \in \tilde{\mathcal{F}}^u(x) \cap (\tilde{\gamma}_1 + mN_1(l, -k))$ such that $d_{\tilde{\mathcal{F}}^u}(y, y_1) < N_2 \cdot a_1$.

Then, we have

$$d_{\mathbb{R}^2}(x, y) \geq |m| \cdot b_1 - (N_2 + 1)a_1 \quad \text{and} \quad d_{\tilde{\mathcal{F}}^u}(x, y) \leq |m| \cdot N_2 \cdot a_1 + (N_2 + 1)a_1,$$

which implies

$$d_{\tilde{\mathcal{F}}^u}(x, y) \leq \left[\frac{N_2 \cdot a_1}{b_1}\right] \cdot d_{\mathbb{R}^2}(x, y) + \left[\frac{N_2(N_2 + 1)a_1^2}{b_1} + (N_2 + 1)a_1\right].$$

This proves that $\tilde{\mathcal{F}}^u$ is quasi-isometric. □

LEMMA 2.3. *Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a diffeomorphism which preserves a one-dimensional expanding foliation $\mathcal{F}^u$, and let $f_* : H_1(\mathbb{T}^2) \to H_1(\mathbb{T}^2)$ be the induced map of f on the first homology group of $\mathbb{T}^2$. Then $f_* := A \in \mathrm{GL}(2, \mathbb{Z})$ is hyperbolic.*

Proof. Assume A is not hyperbolic. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of f. Then $F(x) = Ax + G(x)$ for every $x \in \mathbb{R}^2$, where $G : \mathbb{R}^2 \to \mathbb{R}^2$ is a $\mathbb{Z}^2$-periodic continuous function:

$$G(x + m) = G(x) \quad \text{for all } x \in \mathbb{R}^2, \text{ for all } m \in \mathbb{Z}^2.$$

In particular, there exists $C_0 > 0$ such that $\|G(x)\| \leq C_0$ for every $x \in \mathbb{R}^2$.

For any bounded set $\gamma \subset \mathbb{R}^2$, denote $\|\gamma\| = \sup_{x \in \gamma}\{\|x\|\}$. Then we inductively have

$$\begin{aligned}
\|F(\gamma)\| &= \|A(\gamma) + G(\gamma)\| \leq \|A\| \cdot \|\gamma\| + C_0 \\
\|F^2(\gamma)\| &= \|F(A(\gamma) + G(\gamma))\| \leq \|A^2(\gamma) + A \circ G(\gamma)\| + C_0 \\
&\leq \|A^2\| \cdot \|\gamma\| + C_0\|A\| + C_0 \\
&\cdots\cdots \\
\|F^k(\gamma)\| &\leq \|A^k\| \cdot \|\gamma\| + C_0 \cdot \left(\sum_{i=0}^{k-1} \|A^i\|\right).
\end{aligned}$$

Since A is not hyperbolic, $\|F^k(\gamma)\|$ has at most polynomial growth rate in $\mathbb{R}^2$ with respect to k.

However, if we take a segment $\gamma^u \subset \tilde{\mathcal{F}}^u(x)$ for any $x \in \mathbb{R}^2$ with two endpoints $x, y \in \gamma^u$, then

$$d_{\tilde{\mathcal{F}}^u}(F^n(x), F^n(y)) \geq C^{-1}\lambda^n \cdot d_{\tilde{\mathcal{F}}^u}(x, y).$$

The third item of Lemma 2.2 shows that $\tilde{\mathcal{F}}^u$ is quasi-isometric, and thus we have

$$d_{\mathbb{R}^2}(F^n(x), F^n(y)) \geq \frac{1}{a} \cdot (d_{\tilde{\mathcal{F}}^u}(F^n(x), F^n(y)) - b) \geq \frac{1}{a} \cdot (C^{-1}\lambda^n \cdot d_{\tilde{\mathcal{F}}^u}(x, y) - b).$$

https://doi.org/10.1017/etds.2024.82 Published online by Cambridge University Press

This implies

$$\|F^n(\gamma^u)\| \geq \frac{1}{2} \cdot d_{\mathbb{R}^2}(F^n(x), F^n(y)) \geq \frac{1}{2a} \cdot (C^{-1}\lambda^n \cdot d_{\tilde{\mathcal{F}}^u}(x, y) - b),$$

which has exponential growth rate. This is a contradiction, so $f_* = A \in \mathrm{GL}(2, \mathbb{Z})$ is hyperbolic. $\square$

Remark 2.4. We can apply Lemmas 2.2 and 2.3 to $f = \rho(1, 0)$ which shows that $f_* \in \mathrm{GL}(2, \mathbb{Z})$ is hyperbolic. Notice here that we only use the fact that f preserves an expanding foliation. In particular, neither the fact that f's expanding rate is constant along $\mathcal{F}^u$, as in Lemma 2.1, nor the fact that the action ρ is C^2-smooth (hence, $\mathcal{F}^u$ is C^2-smooth) are used.

3. *Topological conjugacy on $\mathbb{T}^2$*

In this section, we prove the following proposition, which implies $f = \rho(1, 0)$ is topologically conjugate to its linearization $f_* \in \mathrm{GL}(2, \mathbb{Z})$. We want to mention that here we only need f to be expanding along $\mathcal{F}^u$ (we do not assume f has constant expanding rate as in Lemma 2.1) and $\mathcal{F}^u$ is minimal which can be deduced from C^2-smoothness of $\mathcal{F}^u$.

PROPOSITION 3.1. *Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a diffeomorphism which preserves a one-dimensional expanding foliation $\mathcal{F}^u$ where $\mathcal{F}^u$ is minimal. Let $A = f_* : H_1(\mathbb{T}^2) \to H_1(\mathbb{T}^2)$ be the induced map of f which is hyperbolic by Lemma 2.3. Then, f is topologically conjugate to A by a homeomorphism $h : \mathbb{T}^2 \to \mathbb{T}^2$, where h is homotopic to the identity.*

Moreover, the conjugacy h maps the foliation $\mathcal{F}^u$ generated by ϕ_t to the linear expanding foliation L^u of A on $\mathbb{T}^2$, namely $h(\mathcal{F}^u) = L^u$.

First, we state the following well-known semi-conjugacy theorem for toral diffeomorphisms proven by Franks [20].

THEOREM 3.2. [20] *Suppose f is a diffeomorphism on $\mathbb{T}^2$, and F is a lift of f on $\mathbb{R}^2$. Assume that $f_* = A$ is hyperbolic. Then there exists a continuous surjective map $H : \mathbb{R}^2 \to \mathbb{R}^2$ such that:*

- *$H(x + m) = H(x) + m$ for any $x \in \mathbb{R}^2$ and $m \in \mathbb{Z}^2$;*
- *there exists a constant $K > 0$ such that $\|H - \mathrm{Id}\|_{C^0} < K$;*
- *$H \circ F(x) = A \circ H(x)$ for any $x \in \mathbb{R}^2$.*

Moreover, let $h : \mathbb{T}^2 \to \mathbb{T}^2$ be the projection of H on $\mathbb{T}^2$. Then h is continuous, surjective, and satisfies $h \circ f = A \circ h$ on $\mathbb{T}^2$.

By Lemma 2.3, we can apply Theorem 3.2 to $f = \rho(1, 0)$ in our context, and thus obtain that H, h satisfies the properties in Theorem 3.2.

Proof of Proposition 3.1. Since $H : \mathbb{R}^2 \to \mathbb{R}^2$ is $\mathbb{Z}^2$-periodic which induces a continuous surjective map $h : \mathbb{T}^2 \to \mathbb{T}^2$, we only need to show that H is injective, which will guarantee that both H and h are homeomorphisms.

We have the following claim which implies that $h(\mathcal{F}^u) = L^u$.

CLAIM 3.3. *Let $\tilde{L}^u$ be the expanding line foliation of A on $\mathbb{R}^2$. For any $x \in \mathbb{R}^2$, the map H satisfies*

$$H(\tilde{\mathcal{F}}^u(x)) = \tilde{L}^u(H(x))$$

and it is a homeomorphism.

Proof of the claim. First, we have $H(\tilde{\mathcal{F}}^u(x)) \subset \tilde{L}^u(H(x))$. For any $y \in \tilde{\mathcal{F}}^u(x)$, by (2.1),

$$d_{\mathbb{R}^2}(F^{-n}(x), F^{-n}(y)) \to 0 \quad \text{as } n \to +\infty.$$

Together with $\|H - \mathrm{Id}\|_{C^0} < K$, this implies that there exists $C > 0$ such that

$$d_{\mathbb{R}^2}(H \circ F^{-n}(x), H \circ F^{-n}(y)) < C \quad \text{for all } n \geq 0.$$

Then, by the semi-conjugacy $H \circ F(x) = A \circ H(x)$, we have

$$d_{\mathbb{R}^2}(A^{-n}(H(x)), A^{-n}(H(y))) < C.$$

Hence, $H(y) \in \tilde{L}^u(H(x))$.

Moreover, $H : \tilde{\mathcal{F}}^u(x) \to \tilde{L}^u(H(x))$ is injective. Actually, for any $y, z \in \tilde{\mathcal{F}}^u(x)$, since $\tilde{\mathcal{F}}^u$ is quasi-isometric,

$$d_{\mathbb{R}^2}(F^n(y), F^n(z)) \geq \frac{1}{a}(d_{\tilde{\mathcal{F}}^u}(F^n(y), F^n(z)) - b) \to \infty \quad \text{as } n \to +\infty.$$

If $H(y) = H(z)$, then $H \circ F^n(y) = A^n \circ H(y) = A^n \circ H(z) = H \circ F^n(z)$ and hence

$$d_{\mathbb{R}^2}(F^n(y), F^n(z)) \leq d_{\mathbb{R}^2}(F^n(y), H \circ F^n(y)) + d_{\mathbb{R}^2}(H \circ F^n(z), F^n(z)) \leq 2K \quad \text{for all } n \geq 0.$$

This is a contradiction, and thus $H(y) \neq H(z)$ and $H : \tilde{\mathcal{F}}^u(x) \to \tilde{L}^u(H(x))$ is injective.

Finally, since H is continuous and $\tilde{\mathcal{F}}^u(x)$ is simply connected, it follows that

$$H(\tilde{\mathcal{F}}^u(x)) \subset \tilde{L}^u(H(x))$$

is also simply connected. This together with $\|H - \mathrm{Id}\|_{C^0} < K$ implies that

$$H(\tilde{\mathcal{F}}^u(x)) = \tilde{L}^u(H(x)),$$

proving that H is surjective.

Therefore, $H : \tilde{\mathcal{F}}^u(x) \to \tilde{L}^u(H(x))$ is a homeomorphism for every $x \in \mathbb{R}^2$ as claimed. □

To complete the proof, we consider the corresponding quotient maps. Recall that $F : \mathbb{R}^2 \to \mathbb{R}^2$ preserving the foliation $\tilde{\mathcal{F}}^u$ and $A : \mathbb{R}^2 \to \mathbb{R}^2$ preserving $\tilde{L}^u$. Since H maps the foliation $\tilde{\mathcal{F}}^u(x)$ onto the foliation $\tilde{L}^u(H(x))$, namely

$$H(\tilde{\mathcal{F}}^u(x)) = \tilde{L}^u(H(x))$$

https://doi.org/10.1017/etds.2024.82 Published online by Cambridge University Press

for every $x \in \mathbb{R}^2$, the commutative diagram $H \circ F = A \circ H$ reduces to a diagram of the corresponding quotient maps on quotient spaces $\mathbb{R}^2/\tilde{\mathcal{F}}^u$ and $\mathbb{R}^2/\tilde{L}^u$. Namely, we have the following diagram:

$$\begin{tikzcd} \mathbb{R}^2/\tilde{\mathcal{F}}^u \arrow[r, "\hat{F}"] \arrow[d, "\hat{H}"'] & \mathbb{R}^2/\tilde{\mathcal{F}}^u \arrow[d, "\hat{H}"] \\ \mathbb{R}^2/\tilde{L}^u \arrow[r, "\hat{A}"] & \mathbb{R}^2/\tilde{L}^u \end{tikzcd} \tag{3.1}$$

where $\hat{F}$ (respectively $\hat{A}$, $\hat{H}$) is induced by F (respectively A, H) on the quotient space. Since by Lemma 2.2 both $\mathcal{F}^u$ and $L^u = \pi(\tilde{L}^u)$ are irrational minimal foliations on $\mathbb{T}^2$, both quotient spaces $\mathbb{R}^2/\tilde{\mathcal{F}}^u$ and $\mathbb{R}^2/\tilde{L}^u$ are necessarily isomorphic to $\mathbb{R}$. We denote $\mathbb{R}_{\tilde{\mathcal{F}}^u} = \mathbb{R}^2/\tilde{\mathcal{F}}^u$. Notice that the quotient space $\mathbb{R}^2/\tilde{L}^u$ is equal to the stable leaf $\tilde{L}^s(0)$ of A at $0 \in \mathbb{R}^2$: $\tilde{L}^s(0) = \mathbb{R}^2/\tilde{L}^u$, and the quotient map $\hat{A} = A$ on $\tilde{L}^s(0)$. Thus, the diagram in (3.1) induces a new diagram

$$\begin{tikzcd} \mathbb{R}_{\tilde{\mathcal{F}}^u} = \mathbb{R}^2/\tilde{\mathcal{F}}^u \arrow[r, "\hat{F}"] \arrow[d, "\hat{H}"'] & \mathbb{R}_{\tilde{\mathcal{F}}^u} = \mathbb{R}^2/\tilde{\mathcal{F}}^u \arrow[d, "\hat{H}"] \\ \tilde{L}^s(0) = \mathbb{R}^2/\tilde{L}^u \arrow[r, "\hat{A}=A"] & \tilde{L}^s(0) = \mathbb{R}^2/\tilde{L}^u \end{tikzcd} \tag{3.2}$$

where $\hat{A} = A : \tilde{L}^s(0) \to \tilde{L}^s(0)$ is the linear contracting map $A(w) = \lambda^{-1}w$ for every $w \in \tilde{L}^s(0) \subset \mathbb{R}^2$.

We have the following claim.

CLAIM 3.4. *We fix an orientation on $\mathbb{R}_{\tilde{\mathcal{F}}^u}$ and the induced orientation on $\tilde{L}^s(0)$ by H. Then the quotient map $\hat{H} : \mathbb{R}_{\tilde{\mathcal{F}}^u} \to \tilde{L}^s(0)$ satisfies:*

(a1) *$\hat{H}$ is orientation-preserving and increasing;*
(a2) *$\hat{H}$ is a bijection.*

Proof of the claim. Since $\|H - \mathrm{Id}\|_{C^0} < K$, the orientation of $\mathbb{R}_{\tilde{\mathcal{F}}^u}$ induces an orientation on $\tilde{L}^s(0)$ by H globally on $\mathbb{R}^2$. Since $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism, the quotient map $\hat{F} : \mathbb{R}_{\tilde{\mathcal{F}}^u} \to \mathbb{R}_{\tilde{\mathcal{F}}^u}$ is a homeomorphism. Since iterating $\hat{F}$ is necessary, we can assume that $\hat{F} : \mathbb{R}_{\tilde{\mathcal{F}}^u} \to \mathbb{R}_{\tilde{\mathcal{F}}^u}$ preserves the orientation and so does $A : \tilde{L}^s(0) \to \tilde{L}^s(0)$.

For claim (a1), assume otherwise that two points $\hat{x}, \hat{y} \in \mathbb{R}_{\tilde{\mathcal{F}}^u}$ satisfy $\hat{x} < \hat{y}$ and $\hat{H}(\hat{x}) > \hat{H}(\hat{y})$ in $\tilde{L}^s(0)$. Since $\hat{F}$ preserves the orientation, for $\tilde{\mathcal{F}}^u(x) = \hat{x}$ and $\tilde{\mathcal{F}}^u(y) = \hat{y}$, we have

$$\hat{F}^{-n}(\hat{x}) < \hat{F}^{-n}(\hat{y}) \quad \text{and} \quad F^{-n}(\tilde{\mathcal{F}}^u(x)) < F^{-n}(\tilde{\mathcal{F}}^u(y)) \quad \text{for all } n > 0.$$

For $\hat{H}(\hat{x}) > \hat{H}(\hat{y})$ and $\hat{H}(\hat{x}) = \tilde{L}^u(H(x)) > \hat{H}(\hat{y}) = \tilde{L}^u(H(y))$, since A^{-1} is uniformly expanding along $\tilde{L}^s(0)$, we have

$$\hat{A}^{-n}(\hat{H}(\hat{x})) - \hat{A}^{-n}(\hat{H}(\hat{y})) \to +\infty \quad \text{as } n \to +\infty.$$

This is equivalent to that the Hausdorff distance between $A^{-n}(\tilde{L}^u(H(x)))$ and $A^{-n}(\tilde{L}^u(H(y)))$ tends to infinity as $n \to +\infty$ and $A^{-n}(\tilde{L}^u(H(x))) > A^{-n}(\tilde{L}^u(H(y)))$.

https://doi.org/10.1017/etds.2024.82 Published online by Cambridge University Press

However, since $F^{-n}(\tilde{\mathcal{F}}^u(x)) < F^{-n}(\tilde{\mathcal{F}}^u(y))$ for every $n > 0$, the semi-conjugation

$$A^{-n}(\tilde{L}^u(H(x))) = H \circ F^{-n}(\tilde{\mathcal{F}}^u(x)), A^{-n}(\tilde{L}^u(H(y))) = H \circ F^{-n}(\tilde{\mathcal{F}}^u(y)),$$

and $\|H - \mathrm{Id}\|_{C_0} < K$ shows that $A^{-n}(\tilde{L}^u(H(x))$ has $2K$-bounded distance with the negative component of $\mathbb{R}^2 \setminus A^{-n}(\tilde{L}^u(H(y)))$ for every $n > 0$. This contradicts the fact that the Hausdorff distance between $A^{-n}(\tilde{L}^u(H(x)))$ and $A^{-n}(\tilde{L}^u(H(y)))$ tends to infinity as $n \to +\infty$ and

$$A^{-n}(\tilde{L}^u(H(x))) > A^{-n}(\tilde{L}^u(H(y))).$$

This proves claim (a1).

For claim (a2), the surjective part of $\hat{H}$ comes from the fact that $H : \mathbb{R}^2 \to \mathbb{R}^2$ is surjective. We only need to show that $\hat{H}$ is injective. Otherwise, $\hat{H}(\hat{x}) = \hat{H}(\hat{y})$, meaning that both $\tilde{\mathcal{F}}^u(x)$ and $\tilde{\mathcal{F}}^u(y)$ are mapped to a single line $\tilde{L}^u(H(x)) = \tilde{L}^u(H(y))$. Since $\hat{H}$ is orientation-preserving and increasing, it follows that H maps the region $R_{x,y} \subset \mathbb{R}^2$ bounded by $\tilde{\mathcal{F}}^u(x)$ and $\tilde{\mathcal{F}}^u(y)$ in $\mathbb{R}^2$ to $\tilde{L}^u(H(x))$. However, since $\tilde{\mathcal{F}}^u$ is an irrational minimal foliation, we have

$$\mathbb{T}^2 = \pi(R_{x,y}) \quad \text{and} \quad \mathbb{T}^2 = h(\mathbb{T}^2) = h \circ \pi(R_{x,y})) = \pi(H(R_{x,y})) = \pi(\tilde{L}^u(H(x))).$$

This is a contradiction since $\pi(\tilde{L}^u(H(x)))$ is a single one-dimensional leaf in $\mathbb{T}^2$. This proves the claim. □

Finally, the injectivity of H follows from the injectivity of both $H|_{\tilde{\mathcal{F}}^u(x)}$ for every $x \in \mathbb{R}^2$ and $\hat{H}$. Thus, $H : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism. □

Remark 3.5. We can apply this proposition directly to $f = \rho(1, 0)$ in our solvable action. Notice that we only need the fact that f is uniformly expanding along $\mathcal{F}^u$ and $\mathcal{F}^u$ is minimal to get topological conjugacy; neither the fact that f's expanding rate is constant along $\mathcal{F}^u$ as in Lemma 2.1 nor the fact that $\mathcal{F}^u$ is C^2-smooth is used. Surprisingly, these two conditions will be crucial to get smooth conjugacy in Proposition 4.4.

4. *Smooth conjugacy*

By Proposition 3.1, $f : \mathbb{T}^2 \to \mathbb{T}^2$ is topologically conjugate to the hyperbolic automorphism $A = f_* \in \mathrm{GL}(2, \mathbb{Z})$ by a homeomorphism $h : \mathbb{T}^2 \to \mathbb{T}^2$ where $h \circ f = A \circ h$.

By Lemma 2.1, f is uniformly expanding with constant Lyapunov exponent $\log \lambda$ for every ergodic measure along the C^2-foliation $\mathcal{F}^u$, which is the orbit foliation of ϕ_t. Moreover, the conjugacy $h : \mathbb{T}^2 \to \mathbb{T}^2$ satisfies $h(\mathcal{F}^u) = L^u$. So we have the following proposition.

PROPOSITION 4.1. *The diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ is partially hyperbolic $T\mathbb{T}^2 = E^{cs} \oplus E^u$ with $E^u = T\mathcal{F}^u$. Moreover, we have:*

- *for every ergodic measure μ of f, the Lyapunov exponent of μ along E^{cs} is non-positive;*
- *there exists an f-invariant foliation $\mathcal{F}^{cs}$ tangent to E^{cs}, and the conjugacy h maps $\mathcal{F}^{cs}$ to the linear stable foliation L^s of A.*

https://doi.org/10.1017/etds.2024.82 Published online by Cambridge University Press

Proof. By Lemma 2.1, for every periodic point p of f, f has one positive Lyapunov exponent along $\mathcal{F}^u$ which is $\log \lambda$. We denote it as $\lambda^u(p) = \log \lambda$. By Proposition 3.1, f is topologically conjugate to A by h and $h(\mathcal{F}^u) = L^u$. Since A is uniformly contracting along the transversal direction of L^u, f is topologically contracting in the transversal direction of $\mathcal{F}^u$. Thus, p has another Lyapunov exponent $\lambda^{cs}(p) \le 0$.

Moreover, the periodic measures of A are dense in the space of ergodic measures of A. By the topological conjugacy, the periodic measures of f are also dense in the space of ergodic measures of f. Thus, for every ergodic measure μ of f, f has two Lyapunov exponents

$$\lambda^{cs}(\mu) \le 0 < \lambda^u(\mu) = \log \lambda.$$

Now, we only need to show that f admits a dominated splitting, which implies that f is partially hyperbolic. That is the following claim.

CLAIM 4.2. *There exists a Df-dominated splitting $T\mathbb{T}^2 = E^{cs} \oplus E^u$ with $E^u = T\mathcal{F}^u$, that is, the splitting is continuous, Df-invariant, and there exist two constants $0 < \eta < 1$, $C > 1$, such that*

$$\frac{\|Df^n|_{E^{cs}(x)}\|}{\|Df^n|_{E^u(x)}\|} \le C \cdot \eta^n \quad \textit{for all } x \in \mathbb{T}^2, n \ge 0.$$

Proof of the claim. The proof follows exactly the same form as in [**25**, Proposition 5.9]. Since $\mathcal{F}^u$ is a C^2-foliation, the Df-invariant $E^u = T\mathcal{F}^u$ is a C^1-bundle. We have a C^1-smooth splitting $T\mathbb{T}^2 = E^u \oplus E^\perp$ where $E^\perp$ is perpendicular to E^u. We take continuous families of unit vectors in $\{e^u(x), e^\perp(x)\}_{x\in\mathbb{T}^2}$ in E^u, $E^\perp$, respectively, which form a C^1 base on $T\mathbb{T}^2$.

Since F is C^2-smooth, there exist three families of C^1-functions $\{A(x)\}_{x\in\mathbb{T}^2}$, $\{B(x)\}_{x\in\mathbb{T}^2}$, and $\{C(x)\}_{x\in\mathbb{T}^2}$, such that in the base $\{e^u(x), e^\perp(x)\}_{x\in\mathbb{T}^2}$,

$$Df(x) = \begin{pmatrix} A(x) & B(x) \\ 0 & C(x) \end{pmatrix} \quad \text{for all } x \in \mathbb{T}^2.$$

Then, we have

$$Df(e^u(x)) = A(x)e^u(fx), \quad \text{and} \quad proj^\perp \circ Df(e^\perp(x)) = C(x)e^\perp(fx),$$

where $proj^\perp : T\mathbb{T}^2 \to E^\perp$ is the projection through E^u.

For every $x \in \mathbb{T}^2$ and $n \ge 1$, we introduce the following notation for the cocycles:

$$A^n(x) = \prod_{i=0}^{n-1} A(f^i(x)) \quad \text{and} \quad C^n(x) = \prod_{i=0}^{n-1} C(f^i(x)).$$

Lemma 2.1 shows that there exists $C > 1$, such that $|A^n(x)| \ge C^{-1}\lambda^n$ for every $x \in \mathbb{T}^2$ and $n \in \mathbb{N}$. Moreover, for every $\epsilon > 0$ and every periodic point p of f, since the other Lyapunov exponent of p is non-positive, we have

$$\lim_{n\to+\infty} \frac{1}{n} \log |C^n(p)| \le \epsilon.$$

https://doi.org/10.1017/etds.2024.82 Published online by Cambridge University Press

Since periodic measures are dense in all invariant measures, we have

$$\lim_{n\to+\infty}\frac{1}{n}\log|C^n(x)|\le\epsilon \quad \text{for all } x\in\mathbb{T}^2.$$

We fix $0<\epsilon\ll\log\lambda$, and [31, Theorem 1.3] shows that there exists some $N=N(\epsilon)$, such that

$$|C^n(x)|\le\exp(n\epsilon) \quad \text{for all } n\ge N, \text{ for all } x\in\mathbb{T}^2.$$

Finally, since $B(x)$ varies C^1-smoothly with respect to $x\in\mathbb{T}^2$ and is uniformly bounded, there exists a continuous cone field $\{\mathcal{C}(x)\}_{x\in\mathbb{T}^2}$ containing E^u, such that

$$Df(\overline{\mathcal{C}(x)})\subset\mathcal{C}(f(x)) \quad \text{for all } x\in\mathbb{T}^2.$$

Therefore, by the cone-field criterion [9, Theorem 2.6], there exists a dominated splitting

$$T\mathbb{T}^2=E^{cs}\oplus E^u \quad \text{with } T\mathcal{F}^u=E^u.$$

This proves the claim. □

Finally, [37, Proposition 4.A.7] shows that a partially hyperbolic diffeomorphism on $\mathbb{T}^2$ is dynamically coherent, that is, there exists an f-invariant foliation $\mathcal{F}^{cs}$ tangent to E^{cs}. Moreover, by the topological conjugacy $h\circ f=A\circ h$, the foliation $h(\mathcal{F}^{cs})$ is A-invariant and transverse to $L^u=h(\mathcal{F}^u)$, which is unique and $L^s=h(\mathcal{F}^{cs})$. □

Remark 4.3. The fact $L^s=h(\mathcal{F}^{cs})$ directly implies that the foliation $\mathcal{F}^{cs}$ is topologically contracting by f, that is, for every segment $\gamma\subset\mathcal{F}^{cs}(x)$ for some $x\in\mathbb{T}^2$, the length $|f^n(\gamma)|\to 0$ as $n\to+\infty$.

The following proposition shows that if the unstable foliation of $\mathcal{F}^u$ is C^2, then f is uniformly contracting along E^{cs} with constant Lyapunov exponent $-\log\lambda$. The proof is almost the same as [26], see also [22, 36], but we include the proof for completeness.

PROPOSITION 4.4. *For every periodic point p of f, the Lyapunov exponent $\lambda^{cs}(p)$ of f along E^{cs} is equal to $-\log\lambda$. In particular, the diffeomorphism $f\in\mathrm{Diff}^r(\mathbb{T}^2)$ with $r\ge 2$ is Anosov and the conjugacy $h:\mathbb{T}^2\to\mathbb{T}^2$ with $h\circ f=A\circ h$ is $C^{r-\epsilon}$-smooth.*

Proof. First of all, let $\mu_{\max}$ be the measure with maximal entropy of f, which is also the measure with maximal entropy of f^{-1}. Since f is topologically conjugate to A, the measure entropy of $\mu_{\max}$ associated to f^{-1} is equal to the topological entropy of A which is $\log\lambda$. From Ruelle's inequality, the largest Lyapunov exponent of f^{-1} in $\mu_{\max}$ satisfies

$$\log\lambda\le\lambda^+(\mu_{\max},f^{-1})=-\lambda^{cs}(\mu_{\max},f).$$

From the density of periodic measures, there exists a sequence of periodic points p_n whose periodic measures converge to $\mu_{\max}$. Then, we have

$$\lim_{n\to\infty}\lambda^{cs}(p_n)=\lambda^{cs}(\mu_{\max},f)\le-\log\lambda. \tag{4.1}$$

In particular, $\lambda^{cs}(p_n)<0$ and p_n is hyperbolic for n large enough.

https://doi.org/10.1017/etds.2024.82 Published online by Cambridge University Press

CLAIM 4.5. *For every pair of hyperbolic periodic points $p, q \in \mathrm{Per}(f)$, we have $\lambda^{cs}(p) = \lambda^{cs}(q)$.*

Proof of the claim. Since ϕ_t is C^2, when restricted to a smooth transversal cross section which is diffeomorphic to a unit circle $\mathbb{S}^1$, the induced map $\hat{\phi}$ is a C^2 irrational rotation, whose rotation number is a degree-2 algebraic number (that is, the largest eigenvalue λ of f_*). By Herman [28], see also [33, 34], $\hat{\phi}$ is bi-Lipschitz conjugate to R_λ. As a result, there exists a constant $C_2 > 0$ such that, for any small segment I that is contained in a leaf of $\mathcal{F}^{cs}(x)$,

$$\frac{1}{C_2} \le \frac{|I|}{|\mathrm{Hol}_t(I)|} \le C_2. \tag{4.2}$$

Here, $|\cdot|$ is the length function and $\mathrm{Hol}_t : \mathcal{F}^{cs}(x) \to \mathcal{F}^{cs}(\phi_t(x))$ is the holonomy map induced by ϕ_t satisfying

$$\mathrm{Hol}_t(x) = \phi_t(x) \quad \text{and} \quad \mathrm{Hol}_t(I) \subset \mathcal{F}^{cs}(\phi_t(x))$$

for any segment $I \subset \mathcal{F}^{cs}(x)$ of length small enough. We remark that the constant C_2 is independent of I and t.

Now, fix two distinct hyperbolic periodic points p, q. Then the Lyapunov exponents of p, q along E^{cs} satisfy $\lambda^{cs}(p), \lambda^{cs}(q) < 0$. In particular, we have that both $\mathcal{F}^{cs}(p)$ and $\mathcal{F}^{cs}(q)$ are contained in the stable manifolds of p and q, respectively. Denote $\pi > 0$ to be the common period of p, and q: $f^\pi(p) = p$ and $f^\pi(q) = q$.

Let x be an intersecting point of $\mathcal{F}^u(q)$ with the local stable manifold $\mathcal{F}^{cs}_{\mathrm{loc}}(p)$. Then there exists $t \in \mathbb{R}$ such that $x = \phi_t(q) \in \mathcal{F}^{cs}(p)$. Moreover, we can define the holonomy map

$$\mathrm{Hol}_t : \mathcal{F}^{cs}(q) \to \mathcal{F}^{cs}(x) = \mathcal{F}^{cs}(p) \quad \text{with } \mathrm{Hol}_t(q) = x.$$

Take another point $y \in \mathcal{F}^{cs}_{\mathrm{loc}}(q)$ and a segment $J \subset \mathcal{F}^{cs}(q)$ with two endpoints q, y. Then there exists a unique point $z = \mathrm{Hol}_t(y) \in \mathcal{F}^{cs}(x) = \mathcal{F}^{cs}(p)$. Since we can take y close to q and $|J|$ is small, (4.2) implies $|\mathrm{Hol}_t(J)|$ is small, and $\mathrm{Hol}_t(J) \subset \mathcal{F}^{cs}(p)$ with endpoints $x = \mathrm{Hol}_t(q)$ and $z = \mathrm{Hol}_t(y)$.

Since f is C^r-smooth with $r \ge 2$, E^{cs} is Hölder continuous. This implies both $\mathcal{F}^{cs}_{\mathrm{loc}}(p)$ and $\mathcal{F}^{cs}_{\mathrm{loc}}(q)$ are $C^{1+\text{Hölder}}$-smooth sub-manifolds. Since f is uniformly contracting in $\mathcal{F}^{cs}_{\mathrm{loc}}(p)$ and $\mathcal{F}^{cs}_{\mathrm{loc}}(q)$, the distortion control argument shows that there exists $K > 0$ such that, for every $n \ge 0$,

$$\frac{1}{K} \le \frac{|f^{\pi n}(J)|}{\exp(\lambda^{cs}(q)\pi n)} \le K \quad \text{and} \quad \frac{1}{K} \le \frac{|f^{\pi n} \circ \mathrm{Hol}_t(J)|}{\exp(\lambda^{cs}(p)\pi n)} \le K.$$

However, since both $\mathcal{F}^{cs}$ and $\mathcal{F}^u$ are f-invariant, the holonomy map Hol_t is commuting with f, and thus, for every $n > 0$, there exists $t_n \in \mathbb{R}$ such that

$$f^{\pi n} \circ \mathrm{Hol}_t(J) = \mathrm{Hol}_{t_n} \circ f^{\pi n}(J) \quad \text{and} \quad \frac{1}{C_2} \le \frac{|f^{\pi n}(J)|}{|\mathrm{Hol}_{t_n} \circ f^{\pi n}(J)|} = \frac{|f^{\pi n}(J)|}{|f^{\pi n} \circ \mathrm{Hol}_t(J)|} \le C_2.$$

https://doi.org/10.1017/etds.2024.82 Published online by Cambridge University Press

This implies

$$\frac{1}{KC_2} \leq \frac{\exp(\lambda^{cs}(p)\pi n)}{\exp(\lambda^{cs}(q)\pi n)} \leq KC_2 \quad \text{for all } n > 0.$$

Thus, we must have $\lambda^{cs}(p) = \lambda^{cs}(q)$ for every pair of hyperbolic periodic points p and q. □

From this claim and (4.1), we know that

$$\lambda^{cs}(p) \leq -\log \lambda < 0 \tag{4.3}$$

for every hyperbolic periodic point p of f.

Since $f : \mathbb{T}^2 \to \mathbb{T}^2$ is topologically conjugate to $A : \mathbb{T}^2 \to \mathbb{T}^2$, it also satisfies the specification property in [40]. If there exists some periodic point $p \in \mathrm{Per}(f)$ satisfying $\lambda^{cs}(p) = 0$, then by the specification property, there exist hyperbolic periodic points of f with Lyapunov exponents arbitrarily close to zero along E^{cs}. This is absurd since $\lambda^{cs}(p) \leq -\log \lambda$ for every hyperbolic periodic point p. Thus, every periodic point p of f is hyperbolic with $\lambda^{cs}(p) \leq -\log \lambda$.

This implies f is Anosov and $\lambda^{cs}(\mu) \leq -\log \lambda$ for every ergodic measure μ of f. If we consider the Sinai–Ruelle–Bowen measure μ^- of f^{-1}, its Lyapunov exponent along E^{cs} is equal to its measure entropy $h(\mu^-, f^{-1})$, which is smaller than $\log \lambda$. So we have

$$-\lambda^{cs}(\mu^-) = \lambda^u(\mu^-, f^{-1}) = h(\mu^-, f^{-1}) \leq h_{\mathrm{top}}(f^{-1}) = \log \lambda.$$

This implies $\lambda^{cs}(p) = \lambda^{cs}(\mu^-) \geq -\log \lambda$. Combined with (4.3) and Lemma 2.1, we have

$$\lambda^{cs}(p) = -\log \lambda \quad \text{and} \quad \lambda^u(p) = \log \lambda \quad \text{for all } p \in \mathrm{Per}(f).$$

Finally, the work of de la Llave [13, 14] (e.g. [14, Theorem 1.1]) indicates that the conjugacy h is $C^{r-\epsilon}$-smooth when all periodic points of f have the same Lyapunov exponents to A. □

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. We have shown that by the $C^{r-\epsilon}$-smooth conjugacy,

$$A = h \circ f \circ h^{-1} = h \circ \rho(1, 0) \circ h^{-1}.$$

Now, we can define a $C^{r-\epsilon}$-smooth flow on $\mathbb{T}^2$:

$$\psi_t = h \circ \phi_t \circ h^{-1} = h \circ \rho(0, t) \circ h^{-1} \quad \text{satisfying } A \circ \psi_t = \psi_{\lambda t} \circ A.$$

Moreover, we have shown that $h(\mathcal{F}^u) = L^u$ which maps the orbit of ϕ_t to the linear unstable foliation of A. Thus, the orbit of ψ_t is L^u. To prove Theorem 1.1, we only need to show that ψ_t has constant velocity.

Denote

$$\mathcal{Z}(x) = \frac{d}{dt}|_{t=0}\psi_t(x).$$

Let p be a fixed point of A and $x \in L^u(p)$ with $\psi_t(p) = x$ for some $t \in \mathbb{R}$. Then, we have

$$A^n \circ \psi_t(p) = \psi_{\lambda^n t} \circ A(p) \quad \text{and} \quad DA^n \circ D\psi_t(\mathcal{Z}(p)) = D\psi_{\lambda^n t} \circ DA^n(\mathcal{Z}(p)).$$

This implies that $DA^n(\mathcal{Z}(x)) = D\psi_{\lambda^n t}(\lambda^n \cdot \mathcal{Z}(p))$. By taking the norm, we have

$$\lambda^n \cdot \|\mathcal{Z}(x)\| = \lambda^n \cdot \|\mathcal{Z}(\psi_{\lambda^n t}(p))\| \quad \text{for all } n \in \mathbb{Z}.$$

Letting $n \to -\infty$, we have

$$\psi_{\lambda^n t}(p) \to p \quad \text{and} \quad \|\mathcal{Z}(\psi_{\lambda^n t}(p))\| \to \|\mathcal{Z}(p)\|.$$

This implies that $\|\mathcal{Z}(x)\| = \|\mathcal{Z}(p)\|$ for every $x \in L^u(p)$.

Since $L^u(p)$ is dense in $\mathbb{T}^2$, we have $\|\mathcal{Z}(x)\| = \|\mathcal{Z}(p)\| \triangleq a$ for every $x \in \mathbb{T}^2$. Thus, ψ_t is the linear flow with constant velocity. This proves Theorem 1.1, that ρ is $C^{r-\epsilon}$ conjugate to the affine action $\{A, v_{at}\}$. □

Acknowledgements. The authors are grateful to the anonymous referee for careful reading and many useful suggestions. C.D. was supported by the Nankai Zhide Foundation and ‘the Fundamental Research Funds for the Central Universities’ Nos. 100-63233106 and 100-63243066. Y.S. was supported by the National Key R&D Program of China (2021YFA1001900), the NSFC (12071007, 12090015) and the Institutional Research Fund of Sichuan University (2023SCUNL101).

REFERENCES

[1] M. Asaoka. Local rigidity of homogeneous actions of parabolic subgroups of rank-one Lie groups. *J. Mod. Dyn.* **9** (2015), 191–201.
[2] M. Asaoka. Local rigidity problem of smooth group actions. *Sugaku Expositions* **30**(2) (2017), 207–233.
[3] Y. Benoist, P. Foulon and F. Labourie. Flots d’Anosov à distributions de Liapounov différentiables, I. Hyperbolic behaviour of dynamical systems (Paris, 1990). *Ann. Inst. H. Poincaré Phys. Théor.* **53**(4) (1990), 395–412.
[4] Y. Benoist, P. Foulon and F. Labourie. Flots d’Anosov à distributions stable et instable différentiables. *J. Amer. Math. Soc.* **5**(1) (1992), 33–74.
[5] Y. Benoist and F. Labourie. Sur les difféomorphismes d’Anosov affines à feuilletages stable et instable différentiables. *Invent. Math.* **111**(2) (1993), 285–308.
[6] D. Berend. Multi-invariant sets on tori. *Trans. Amer. Math. Soc.* **280**(2) (1983), 509–532.
[7] C. Bonatti, I. Monteverde, A. Navas and C. Rivas. Rigidity for C^1 actions on the interval arising from hyperbolicity I: solvable groups. *Math. Z.* **286**(3–4) (2017), 919–949.
[8] L. Burslem and A. Wilkinson. Global rigidity of solvable group actions on S^1. *Geom. Topol.* **8**(2) (2004), 877–924.
[9] S. Crovisier and R. Potrie. *Introduction to Partially Hyperbolic Dynamics*. School on Dynamical Systems, ICTP, Trieste, 2015.
[10] D. Damjanović and A. Katok. Local rigidity of partially hyperbolic actions I. KAM method and $\mathbb{Z}^k$ actions on the torus. *Ann. of Math. (2)* **172**(3) (2010), 1805–1858.
[11] D. Damjanović and A. Katok. Local rigidity of partially hyperbolic actions. II. The geometric method and restrictions of Weyl chamber flows on $SL(n, \mathbb{R})/\Gamma$. *Int. Math. Res. Not. IMRN* **2011**(19) 2011, 4405–4430.
[12] D. Damjanović, R. Spatzier, K. Vinhage and D. Xu. Anosov actions: classification and the Zimmer program. *Preprint*, 2023, arXiv:2211.08195.
[13] R. de la Llave. Invariants for smooth conjugacy of hyperbolic dynamical systems II. *Comm. Math. Phys.* **109** (1987), 368–378.
[14] R. de la Llave. Smooth conjugacy and SRB measures for uniformly and non-uniformly hyperbolic systems. *Comm. Math. Phys.* **150** (1992), 289–320.

[15] J. DeWitt and A. Gogolev. Dominated splitting from constant periodic data and global rigidity of Anosov automorphisms. *Geom. Funct. Anal.* **34**(5) (2024), 1370–1398.

[16] M. Einsiedler, A. Katok and E. Lindenstrauss. Invariant measures and the set of exceptions to Littlewood's conjecture. *Ann. of Math. (2)*, **164**(2) (2006), 513–560.

[17] M. Einsiedler and E. Lindenstrauss. On measures invariant under tori on quotients of semisimple groups. *Ann. of Math. (2)* **181**(3) (2015), 993–1031.

[18] D. Fisher. Local rigidity of group actions: past, present, future. *Dynamics, Ergodic Theory, and Geometry (Mathematical Sciences Research Institute Publications, 54)*. Ed. D. Fisher. Cambridge University Press, Cambridge, 2007, pp. 45–97.

[19] L. Flaminio and A. Katok. Rigidity of symplectic Anosov diffeomorphisms on low-dimensional tori. *Ergod. Th. & Dynam. Sys.* **11**(3) (1991), 427–441.

[20] J. Franks. Anosov diffeomorphisms. *Global Analysis (Proceedings of Symposia in Pure Mathematics, XIV–XVI)*. American Mathematical Society, Providence, RI, 1970, pp. 61–93.

[21] H. Furstenberg. Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation. *Math. Systems Theory* **1** (1967), 1–49.

[22] E. Ghys. Rigidité différentiable des groupes fuchsiens. *Publ. Math. Inst. Hautes Études Sci.* **78** (1993), 163–185.

[23] P. Giulietti and C. Liverani. Parabolic dynamics and anisotropic Banach spaces. *J. Eur. Math. Soc. (JEMS)* **21**(9) (2019), 2793–2858.

[24] A. Gogolev. Bootstrap for local rigidity of Anosov automorphisms of the 3-torus. *Comm. Math. Phys.* **352**(2) (2017), 439–455.

[25] A. Gogolev and Y. Shi. Joint integrability and spectral rigidity for Anosov diffeomorphisms. *Proc. Lond. Math. Soc. (3)* **127**(6) (2023), 1693–1748.

[26] R. Gu. Smooth stable foliations of Anosov diffeomorphisms. *Preprint*, 2023, arXiv:2310.19088, *C. R. Math.*, accepted.

[27] G. Hector and U. Hirsch. *Introduction to the Geometry of Foliations. Part A. Foliations on Compact Surfaces, Fundamentals for Arbitrary Codimension, and Holonomy*, 2nd edn. Friedr. Vieweg & Sohn, Braunschweig, 1986.

[28] M. R. Herman. Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations. *Publ. Math. Inst. Hautes Études Sci.* **49**(1) (1979), 5–233.

[29] S. Hurtado and J. Xue. Global rigidity of some abelian-by-cyclic group actions on T^2. *Geom. Topol.* **25**(6) (2021), 3133–3178.

[30] J.-L. Journé, A regularity lemma for functions of several variables. *Rev. Mat. Iberoam.* **4**(2) (1988), 187–193.

[31] B. Kalinin. Livšic theorem for matrix cocycles. *Ann. of Math. (2)* **173**(2) (2011), 1025–1042.

[32] B. Kalinin, A. Katok and F. Rodriguez Hertz. Nonuniform measure rigidity. *Ann. of Math. (2)* **174**(1) (2011), 361–400.

[33] Y. Katznelson and D. Ornstein. The differentiability of the conjugation of certain diffeomorphisms of the circle. *Ergod. Th. & Dynam. Sys.* **9**(4) (1989), 643–680.

[34] K. Khanin and A. Teplinsky. Herman's theory revisited. *Invent. Math.* **178**(2) (2009), 333–344.

[35] Q. Liu. Local rigidity of certain solvable group actions on tori. *Discrete Contin. Dyn. Syst.* **41**(2) (2021), 553–567.

[36] A. A. Pinto and D. A. Rand. Rigidity of hyperbolic sets on surfaces. *J. Lond. Math. Soc. (2)* **71**(2) (2005), 481–502.

[37] R. Potrie. Partial hyperbolicity and attracting regions in 3-dimensional manifolds. *Preprint*, 2012, arXiv:1207.1822; *Doctoral Thesis*, Universidad de la Republica, Montevideo, Uruguay and Université Sorbonne Paris Nord, 2012.

[38] M. Ratner. Invariant measures and orbit closures for unipotent actions on homogeneous spaces. *Geom. Funct. Anal.* **4**(2) (1994), 236–257.

[39] C. Robinson. *Dynamical Systems. Stability, Symbolic Dynamics, and Chaos (Studies in Advanced Mathematics)*. CRC Press, Boca Raton, FL, 1999.

[40] K. Sigmund. Generic properties of invariant measures for Axiom A diffeomorphisms. *Invent. Math.* **11** (1970), 99–109.

[41] S. Smale. Differentiable dynamical systems. *Bull. Amer. Math. Soc. (N.S.)* **73** (1967), 747–817.

[42] R. Spatzier and K. Vinhage. Cartan actions of higher rank abelian groups and their classification. *J. Amer. Math. Soc.* **37**(3) (2024), 731–859.

[43] Z. J. Wang. Local rigidity of higher rank non-abelian action on torus. *Ergod. Th. & Dynam. Sys.* **39**(6) (2019), 1668–1709.

[44] A. Wilkinson and J. Xue. Rigidity of some abelian-by-cyclic solvable group actions on T^n. *Comm. Math. Phys.* **376** (2020) 1223–1259.

https://doi.org/10.1017/etds.2024.82 Published online by Cambridge University Press