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The regularity series of a convergence space II

D.C. Kent and G.D. Richardson

This study is a continuation of an earlier paper on the regularity series of a convergence space. The notions of a R-Hausdorff series and the T_3-modification of a convergence space are introduced, and their relationship with the regularity series is studied. The concept of a symmetric space is shown to be useful in studying T_3-compactifications. Several examples are given; one being a Hausdorff convergence space with an arbitrarily large regularity series.

Introduction

This work is a continuation of the study of the regularity series (or R-series) of a convergence space which was initiated by the authors in [10]. The notation and terminology of [10] will be used without further reference.

In Section 1, we define the R-Hausdorff series (or RH-series), the R-Hausdorff modification, and the T_3-modification of a space X, and study the behavior of these phenomena and their relationship to the R-series. Section 2 is devoted to a study of symmetric spaces. It is shown that, for all practical purposes, a compact symmetric space is topological. Section 3 deals with the problem of extending a compactification of a space X to compactifications of the regular,

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symmetric, and T_3-modifications of X. In Section 4, the behavior of the R-series and RH-series of a space relative to products and certain quotient maps is given further study. Section 5 gives an example to show that the length of the R-series of a Hausdorff convergence space can be arbitrarily large. Examples are also given in this section which show that Theorems 6.3 and 7.4 of [10] cannot be significantly improved. The concluding section utilizes concepts developed earlier in the paper to simplify the proof of theorems obtained by Ramaley and Wyler in [7].

It should be noted that Theorems 3.2 and 4.1 of [10] are incorrect as stated. Corrected statements of these theorems and several others derived from them (Theorems 3.5, 4.2, 4.4, and Corollary 4.3 of [10]) are given in Sections 3 and 4 of this paper, along with counterexamples to the original statements.

1. The *RH*-series

Let X be a space, and let $\{r_\alpha X : 0 \le \alpha \le l_R(X)\}$ denote the R-series of X, which terminates with X_r, the regular modification of X. Recall that X is *R-Hausdorff* if X_r is Hausdorff; a regular Hausdorff space is said to be T_3.

On an arbitrary space X, a relation $\sim$ is defined as follows: $x \sim y$ iff $\dot{x} \to y$ in X_r. It is easy to verify that this is an equivalence relation; let $sX = \{[x] : x \in X\}$ be the set of equivalence classes, and let $\phi : X \to sX$ be the natural map. We assign to sX the quotient convergence structure determined by X and ϕ. In other words, $F \to [y]$ in sX iff there is $G \to x$ in X such that $\phi(G) = F$, and $x \in [y]$. Furthermore, let X_s be the set sX equipped with the quotient convergence structure determined by $\phi : X_r \to sX$. We begin by establishing some basic properties of the space X_s.

LEMMA 1.1. *Let A and B be subsets of a space X such that $\phi(A) = \phi(B)$. Then $A \subseteq \mathrm{cl}_{X_r} B$.*

Proof. Let $x \in A$. Then $[x] \in \phi(B)$, and so there is

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$y \in [x] \cap B$. Since $\dot{y} \to x$ in X_r , $x \in \mathrm{cl}_{X_r} B$. //

LEMMA 1.2. *Let* X *be a space, and* F *a filter on* X *such that* $\phi(F) \to [x]$ *in* X_s . *Then there is* $y \in [x]$ *such that* $F \to y$ *in* X_r . *Thus the map* $\phi : X_r \to X_s$ *is perfect.*

Proof. Since $\phi : X_r \to X_s$ is a convergence quotient map, there is $y \in [x]$ and $G \to y$ in X_r such that $\phi(G) = \phi(F)$. By Lemma 1.1, $F \geq \mathrm{cl}_{X_r} G$, and $F \to y$ since X_r is regular. //

It is established in [4] that a perfect map preserves regularity; hence, by Lemma 1.2, X_s is regular. Moreover, it follows from Lemma 1.2 that X_s is T_1 ; consequently X_s is a T_3 space. Since sX is finer than X_s , it follows that sX is R-Hausdorff.

PROPOSITION 1.3. *For any space* X , X_s *is* T_3 *and* sX *is* *R-Hausdorff.*

For each ordinal number α , define $s_\alpha X$ to be $r_\alpha(sX)$: the ordinal sequence $\{s_\alpha X : 0 \leq \alpha \leq l_R(sX)\}$ will be called the *R-Hausdorff series* (abbreviated RH-series) for X . Note that $X_s \leq (sX)_r$, and so all terms in the RH-series are R-Hausdorff. We will refer to sX as the *R-Hausdorff modification of* X , and to X_s as the T_3*-modification of* X .

THEOREM 1.4. *Let* X *be a space.*

(a) $\phi : r_\alpha X \to s_\alpha X$ *is continuous for all ordinals* α .

(b) $X_s = (sX)_r$.

(c) $l_R(sX) \leq l_R(X)$.

Proof. *(a)* follows immediately from Proposition 3.1 of [10].

(b) Let α be the larger of the ordinals $l_R(X)$ and $l_R(sX)$. Then $\phi : r_\alpha X \to s_\alpha X$ is continuous by *(a)*. Since $r_\alpha X = X_r$ and $s_\alpha X = (sX)_r$,

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we have that $X_s \geq (sX)_r$, since $\phi : X_r \to X_s$ is a convergence quotient map. The reverse inequality is noted above, and so $X_s = (sX)_r$ is the terminal element in the *RH*-series.

(c) Let $\alpha = l_R(X)$. Then $\phi : X_r \to s_\alpha X$ is continuous by *(a)*. But $\phi : X_r \to X_s$ is a convergence quotient map, and so $s_\alpha X \leq X_s$. On the other hand, $X_s \leq s_\beta X$ for all ordinals β , and so $s_\alpha X = X_s$. Hence $l_R(sX) \leq \alpha$. //

We will use $l_{RH}(X)$ rather than $l_R(sX)$ to designate the length of the *RH*-series.

PROPOSITION 1.5. *If* X *is locally compact, then* $l_{RH}(X) \leq 1$.

Proof. Since local compactness is preserved by convergence quotient maps, sX is locally compact and R-Hausdorff. The assertion thus follows from Theorem 2.5 of [*10*]. //

The straightforward proof of the next proposition will be omitted.

PROPOSITION 1.6. *Let* X *and* Y *be spaces,* $f : X \to Y$ *a continuous function, and define* $\overline{f} : sX \to sY$ *by* $\overline{f}([x]) = [f(x)]$. *Then* $\overline{f}$ *is a well-defined function, the diagram that follows is commutative, and the functions involved are all continuous:*

$$\begin{tikzcd}
X \arrow[r, "\phi_X"] \arrow[d, "f"'] & sX \arrow[r, "\mathrm{id}"] \arrow[d, "\overline{f}"'] & s_\alpha X \arrow[r, "\mathrm{id}"] \arrow[d, "\overline{f}"'] & X_s \arrow[d, "\overline{f}"'] \\
Y \arrow[r, "\phi_Y"] & sY \arrow[r, "\mathrm{id}"] & s_\alpha Y \arrow[r, "\mathrm{id}"] & Y_s .
\end{tikzcd}$$

The term *convergence group* is used here as in [*10*]; in particular, we consider only abelian groups and use the additive notation.

Given a convergence group X , it is shown in Theorem 3.4 of [*10*] that $r_\alpha X$ is a convergence group for all ordinals α ; in particular, X_r is a regular convergence group. The equivalence class $[0]$ is a subgroup of X , since $x, y \in [0]$ implies $\dot{0}$ converges to x and y , and hence to

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$x - y$, so that $x\text{-}y \in [0]$. Note that sX has the same underlying set as the factor group $X/[0]$. It is easy to check that the group operations on $X/[0]$ are continuous relative to the quotient convergence structure on sX . These observations, along with Theorem 3.4 of [10] prove the following.

THEOREM 1.7. *If* X *is a convergence group, then* $s_\alpha X$ *is a convergence group for each ordinal* α .

Let $C(X, Y)$ be the set of all continuous functions from a space X into a space Y . The continuous convergence structure on $C(X, Y)$ is the coarsest relative to which the evaluation map $e : C(X, Y) \times X \to Y$, defined by $e(f, x) = f(x)$, is continuous. The function space with the continuous convergence structure is denoted by $C_c(X, Y)$. Convergence in $C_c(X, Y)$ can be characterized as follows: if Φ is a filter on $C_c(X, Y)$, then $\Phi \to f$ in $C_c(X, Y)$ iff $\Phi(F) \to f(x)$ in Y whenever $F \to x$ in X . Function spaces with the continuous convergence structure have been studied by a number of authors; see, for example, [1] and [2].

THEOREM 1.8. *Let* X *be a space and* Y *a regular space. Then, for all ordinals* α , $C_c(X, Y) = C_c(r_\alpha X, Y)$. *If* Y *is also Hausdorff, then* $C_c(X, Y) = C_c(s_\alpha X, Y)$ *for all* α .

PROOF. First note that the sets $C(X, Y)$ and $C(r_\alpha X, Y)$ are equal, since $f : X \to Y$ is continuous iff $f : X_r \to X$ is continuous. Also $C_c(X, Y) \le C_c(r_\alpha X, Y)$ is clear. Let $\Phi \to f$ in $C_c(X, Y)$, and let $F \to x$ in $r_\alpha X$. Then there is $\beta < \alpha$ and $n \in N$ such that $F \ge \mathrm{cl}^n_{r_\beta X} G$, where $G \to x$ in X . Then $\Phi(F) \ge \Phi\left(\mathrm{cl}^n_{r_\beta X} G\right) \ge \Phi\left(\mathrm{cl}^n_{X_r} G\right) \ge \mathrm{cl}^n_Y \Phi(G)$, and the latter filter converges to $f(x)$ in Y since Y is regular. This establishes that $\Phi \to f$ in $C_c(r_\alpha X, Y)$.

To prove the second equality, it suffices, in view of the first equality, to show that $C_c(X, Y) = C_c(sX, Y)$. If $\dot{x} \to y$ in X_r , then $f \in C_c(X, Y)$ implies $f(x) = f(y)$. It follows that $\overline{f} : sX \to Y$, where

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$\bar{f}([x]) = f(x)$, is well defined. By identifying $f \in C(X, Y)$ with $\bar{f}$ in $C(sX, Y)$, we obtain set equality. It is also easy to verify that, under this identification, the continuous convergence structures on the two sets coincide. //

2. Symmetric spaces

A *symmetric space* X is defined to be a regular space with the property that $F \to x$ whenever $F \to y$ and $\dot{y} \to x$. Examples of symmetric spaces include T_3 spaces, regular topological spaces, regular convergence groups, and regular uniform convergence spaces. We shall see that compact symmetric spaces are, for all practical purposes, topological.

Some additional terminology concerning mappings will be needed for this and later sections. Let f be a continuous map from a space X onto a space Y . The map is *open* if, whenever F is an ultrafilter, $F \to y$ in Y , and $x \in f^{-1}(y)$, there is a filter G on X such that $G \to x$ and $f(G) = F$. We will say that f is *strongly open* if the preceding condition is satisfied when F is an arbitrary filter (not necessarily an ultrafilter). Finally, X is said to have the *initial structure* determined by f and Y if $G \to x$ in X whenever $f(G) \to f(x)$ in Y . Observe that if X has the initial structure determined by f and Y , then f is strongly open and perfect; indeed, f is strongly perfect in the sense defined by the statement of Lemma 1.2.

We omit the simple proof of the next assertion.

LEMMA 2.1. *Let f map X onto Y . If Y is symmetric (respectively, regular, locally compact, compact, topological), and X has the initial structure determined by Y , then X is symmetric (respectively, regular, locally compact, compact, topological).*

PROPOSITION 2.2. *The following statements about a regular space X are equivalent:*

(a) X is symmetric;

(b) X has the initial structure determined by $\phi : X \to X_s$;

(c) $\phi : X \to X_s$ is strongly open.

Proof. First note that when X is regular, $X_s = sX$. By Lemma 2.1, *(b)* ⇒ *(a)*. To establish that *(a)* ⇒ *(c)*, let $F \to [x]$ in X_s, and let $y \in [x]$. By definition of X_s, there is $z \in [x]$ and a filter $H \to z$ in X such that $\phi(H) = F$. But symmetry implies that $H \to y$ in X, and so ϕ is strongly open. To show that *(c)* ⇒ *(b)*, let G be a filter on X such that $\phi(G) \to [x]$. Then there is $H \to y \in [x]$ such that $\phi(H) = \phi(G)$. But $G \geq \mathrm{cl}_X H$ follows by Lemma 1.1, and $G \to y$ in X since X is regular. //

It is shown in [4] that the image of a topological space under an open map is topological. This observation, along with Lemma 2.1 and Proposition 2.2 implies:

COROLLARY 2.3. *If X is a regular topological space, then X_s is a T_3 topological space. If X is symmetric and X_s is topological (respectively, compact, locally compact) then X is topological (respectively, compact, locally compact).*

Recall the notation λX for the topological modification of X, and $l_D(X)$ for the length of the decomposition series of X. The next theorem generalizes a well-known result (see [9]) concerning compact T_3 spaces.

THEOREM 2.4. *(a) If X is a compact regular space, then $l_D(X) \leq 2$.*

(b) If X is a compact symmetric space, then λX is a compact regular topological space, X and λX have the same ultrafilter convergence, and $l_D(X) \leq 1$.

Proof. *(a)* Let $A \subseteq X$, and $x \in \mathrm{cl}_X^3 A$. Then there is an ultrafilter $F \to x$ such that $\mathrm{cl}_X^2 A \in F$. By Lemma 2.1 of [5], there is an ultrafilter G containing A such that $F \geq \mathrm{cl}_X^2 G$. Since X is compact, there is $y \in X$ such that $G \to y$; since X is regular, $F \to y$. Therefore $[x] = [y]$, since $\dot{x} \geq \mathrm{cl}_X F$, and so $y \in \mathrm{cl}_X A$ implies $x \in \mathrm{cl}_X^2 A$. Thus we have shown that the closure operator cl_X^2 is

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idempotent.

(b) If X is symmetric, then an argument very similar to that of the preceding paragraph can be used to show that cl_X is idempotent; thus the pretopological modification $\pi X = \lambda X$. Let F be an ultrafilter which is finer than $V_X(x)$, the neighborhood filter at x. Then $\dot{x} \geq \mathrm{cl}_X F$. Since X is compact, there is $y \in X$ such that $F \to y$. But then $\dot{x} \to y$, $\dot{y} \to x$, and $F \to x$ by symmetry. Thus X and λX have the same ultrafilter convergence, and the proof is complete. //

THEOREM 2.5. *Let X be a regular convergence group.*

(a) A filter F is Cauchy in X iff $\phi(F)$ is Cauchy in X_s.

(b) X is complete iff X_s is complete.

(c) X is totally bounded iff X_s is totally bounded.

(d) X is a topological group iff X_s is a topological group.

Proof. We will prove only *(a)*. Recall that F is Cauchy iff $F - F \to 0$. If $F - F \to 0$ in X, then $\phi(F-F) = \phi(F) - \phi(F) \to [0]$ in X_s by continuity of ϕ. Conversely, if $\phi(F-F) = \phi(F) - \phi(F) \to [0]$ in X_s, then we use the fact that X is symmetric along with Proposition 2.2 to establish that $F - F \geq \phi^{-1}\big(\phi(F)-\phi(F)\big) \to 0$ in X. //

For any space X, let σX be the set X with the initial structure determined by ϕ and X_s. It is easy to show that $F \to x$ in σX iff there is $y \in [x]$ such that $F \to y$ in X_r. The space σX is the finest symmetric space coarser than X, and will be called the *symmetric modification of* X. Some basic properties of σX are stated without proof.

PROPOSITION 2.6. *Let X be a space.*

(1) *For each subset A of X,* $\mathrm{cl}_{X_r} A \subseteq \mathrm{cl}_{\sigma X} A \subseteq \mathrm{cl}^2_{X_r} A$.

(2) $\lambda X_r = \lambda(\sigma X)$.

(3) *If $f : X \to Y$ is continuous, then $f : \sigma X \to \sigma Y$ is continuous.*

(4) $X_s = (\sigma X)_s = s(\sigma X) = \sigma(X_s)$.

(5) *If* X *is R-Hausdorff, then* $X_r = \sigma X$.

3. Compactifications

In this section we examine the interrelationships between compactifications of X, X_r , and X_s . *The assumption is made throughout this section that all compactifications are strict.*

As noted in the introduction, Theorems 3.2 and 3.5 of [10] are incorrect as stated; Example 5.6 of [10] is a counterexample to Theorem 3.2. Both of these theorems are correct if the additional assumption is made that the compactification or completion is R-Hausdorff. It seems to be rather difficult to obtain a satisfactory version of Theorem 3.2 of [10] when the compactification is not R-Hausdorff; a partial result in this direction is obtained in Theorem 3.2 below.

Let (Y, f) be a compactification of a space X . Then $\overline{f} : s_\alpha X \to s_\alpha Y$ is continuous for all ordinals α by Proposition 1.6. However $\overline{f}$ need not be injective, and, when injective, need not be an embedding. The following lemma is obvious.

LEMMA 3.1. *Let* (Y, f) *be a compactification of* X .

(a) If $\overline{f}$ *is injective, then for each* $x \in X$, $f([x]) = [f(x)] \cap f(X)$.

(b) If (Y_r, f) *is a compactification of* X_r *, then* $\overline{f}$ *is injective.*

THEOREM 3.2. *Let* X *be a locally compact Hausdorff space and let* (Y, f) *be a Hausdorff compactification of* X . *Then* (Y_r, f) *is a compactification of* X_r *iff* $\overline{f}$ *is injective.*

Proof. Assume that $\overline{f}$ is injective. By Theorem 2.4 *(a)*, the closure operator $\mathrm{cl}^2_{Y_r}$ is indempotent. Let A be a compact subset of $f(X) \subseteq Y$ and let $B \subseteq A$. If $y \in \mathrm{cl}^2_{Y_r} B$, then there is an ultrafilter

$G \to y$ in Y_r such that $\mathrm{cl}_{Y_r} B \in G$. By Lemma 2.1 of [5], there is an ultrafilter F such that $B \in F$ and $G \geq \mathrm{cl}_{Y_r} F$. Since A is compact, there is $z \in A$ such that $F \to z$ in Y. Also $G \to z$ in Y_r, and so $y \in [z]$. Thus we have shown that

$$\mathrm{cl}^2_{Y_r} B \subseteq \mathrm{cl}_Y B \cup \{y \in Y : y \in [z] \text{ for some } z \in \mathrm{cl}_Y B\} ,$$

and it is easy to see that this inclusion is actually an equality. Since $\mathrm{cl}_Y B \subseteq f(X)$, this result, along with Lemma 3.1 *(a)*, implies that $f^{-1}\left(\mathrm{cl}^2_{Y_r} B\right) = \mathrm{cl}^2_{X_r} f^{-1}(B)$.

Let $F \to f(x)$ in Y_r, where $f(X) \in F$ and $x \in X$. Then there is $G \to f(x)$ in Y such that $F \geq \mathrm{cl}^2_{Y_r} G$. By the strictness condition, we can assume that $f(X) \in G$. Thus the results of the preceding paragraph imply that $f^{-1}\left(\mathrm{cl}^2_{Y_r} G\right) = \mathrm{cl}^2_{X_r} f^{-1}(G)$. Since (Y, f) is a compactification of X, $f^{-1}(G) \to x$ in X, and so $\mathrm{cl}^2_{X_r} f^{-1}(G) \to x$ in X_r. Thus $f^{-1}(F) \geq f^{-1}\left(\mathrm{cl}^2_{Y_r} G\right)$ implies that $f^{-1}(F) \to x$ in X_r, and it follows that (Y_r, f) is a compactification of X_r. //

EXAMPLE 3.3. Let X be an infinite set, $x \in X$, and F a free ultrafilter on X; let X have the finest convergence structure such that each free ultrafilter other than F converges to x. The space X is easily seen to be locally compact, regular, and Hausdorff. Let $Y = X \cup \{a\}$, where $a \notin X$ be a one-point compactification of X, where $F \to a$ in Y. This compactification of X is Hausdorff, but not R-Hausdorff. One can easily verify that $\bar{f}$ is injective, and so Y_r is a regular compactification of $X_r = X$. But Y_r is not a Hausdorff space, and indeed it can be shown that X has no regular Hausdorff compact compactification. Thus Y_s is not a compactification of $X_s = X$, even though $\bar{f}$ is injective and continuous. //

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For the remainder of this section, we will investigate the circumstances under which (Y, f) a compactification of X implies that $(Y_s, \overline{f})$ is a T_3 compactification of X_s.

THEOREM 3.4. *Let (Y, f) be a regular compactification of X. Then $(Y_s, \overline{f})$ is a T_3 compactification of X iff the following condition is satisfied:*

if $F \to y$ in Y, where $f(X) \in F$ and $[y] \cap f(X) \neq \emptyset$, then $F \to f(x) \in [y]$ in Y for some $x \in X$.

Proof. We refer to the diagram of Proposition 1.6. Assume that $(Y_s, \overline{f})$ is a T_3 compactification of X_s, and let $F \to y$ in Y, where $f(X) \in F$ and $f(x) \in [y]$. Since $\overline{f}(X_s) \in \phi_y F$, then $\overline{f}^{-1}(\phi_Y F) \to [x]$ in X_s. Let G be a filter on X such that $\phi_X G = \overline{f}^{-1}(\phi_Y G)$ and $G \to x_1 \in [x]$. Then $\overline{f}(\phi_X G) = \phi_Y F$, and by commutativity of the diagram, $\phi_Y(fG) = \phi_Y F$. By Lemma 1.1, $F \geq \mathrm{cl}_Y f(G) \to f(x_1) \in [y]$ in Y.

Conversely, assume the given condition: if $[f(x_1)] = [f(x_2)]$, then $f(\dot{x}_1) \to f(x_2)$ in Y, and so $\dot{x}_1 \to x_2$ in X, since $f : X \to Y$ is an embedding. Thus $[x_1] = [x_2]$, and $\overline{f} : X_s \to Y_x$ is injective. It remains to show that $\overline{f}^{-1} : \overline{f}(X_s) \to X_s$ is continuous. Let $F \to [f(x)]$ in Y_s, where $\overline{f}(X_s) \in F$. Then there exists a filter G on Y such that $G \to y \in [f(x)]$ in Y, where $\phi_Y G = F$. Since $\overline{f}(X_s) \in F$, there is a filter H on Y with $f(X) \in H$ and $\phi_Y H = F$. Also, $\phi_Y G = \phi_Y H$, and so $H \geq \mathrm{cl}_Y G$ by Lemma 1.1. Thus $H \to y$ in Y, and $f(X) \in H$. By the hypothesis, there is $x_1 \in X$ such that $H \to f(x_1) \in [y]$ in Y. Thus $f^{-1}(H) \to x_1$ in X, and so $\phi_X(f^{-1}H) \to [x]$ in X_s. Since $\overline{f}^{-1}(F) = \phi_X(f^{-1}H)$, $\overline{f}$ is an embedding. //

THEOREM 3.5. *Let (Y, f) be a compactification of a space X. Then $(Y_s, \overline{f})$ is a T_3 compactification of X_s iff $(\sigma Y, f)$ is a* *symmetric compactification of* σX .

Proof. Assume that $(Y_s, \overline{f})$ is a compactification of X_s , and consider the diagram:

$$\begin{array}{ccc} \sigma X & \xrightarrow{f} & \sigma Y \\ {\scriptstyle\phi_X}\downarrow & & \downarrow{\scriptstyle\phi_Y} \\ X_s & \xrightarrow{\overline{f}} & Y_s \end{array} \quad .$$

From the fact that (Y, f) is a compactification of X , it follows that f is a continuous injection in the previous diagram (see Proposition 2.6). If $G \to y \in f(X)$ in σY , where $f(X) \in G$, then $\phi_Y(G) \to [y]$ in Y_s . If $x = f^{-1}(y)$, then $\overline{f}^{-1}(\phi_Y G) = \phi_X f^{-1}(G) \to [x]$ in X_s , since $\overline{f}$ is an embedding. But this implies that $f^{-1}(G) \to x$ in σX , since σX has the initial structure determined by ϕ_X and X_s .

Conversely, assume that $(\sigma Y, f)$ is a compactification of σX . Then $(\sigma Y)_s = Y_s$ and $(\sigma X)_s = X_s$ by Proposition 2.6. Thus, we can apply Theorem 3.4 with σX and σY playing the roles of X and Y . Since σY is symmetric, the condition specified in Theorem 3.4 is clearly satisfied, and so $(Y_s, \overline{f})$ is a compactification of X_s . //

COROLLARY 3.6. *Let* X *be a locally compact, Hausdorff convergence group, and let* (Y, f) *be a Hausdorff compactification of* X *, where* Y *is also a convergence group and* f *is injective. Then* (Y_r, f) *is a regular convergence group compactification of* X_r *, and* (Y_s, f) *is a* T_3 *convergence group compactification of* X_s .

Proof. This assertion follows from Theorem 1.7, Theorem 3.2, Theorem 3.5, and the fact that regular convergence groups are symmetric. //

A space which has a T_3 compactification is said to be *Tychonoff*. In [9], it is shown that a space X is Tychonoff iff X is a regular space which has the same ultrafilter convergence as a completely regular,

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Hausdorff topological space.

THEOREM 3.7. *For any space* X , X_s *is Tychonoff iff* σX *has a symmetric compactification.*

Proof. If (Y, f) is a symmetric compactification of σX , then the argument used in the proof of Theorem 3.5 establishes that $\left(Y_s, \bar{f}\right)$ is a T_3-compactification of X_s . Conversely, assume that (Z, g) is a T_3-compactification of X_s . Let $W = Z - g\left(X_s\right)$, let $Y = X \cup W$, and define $\Psi : Y \to Z$ by $\Psi(y) = g([y])$ if $y \in X$, and $\Psi(y) = y$ if $y \in Z$. Let Y have the initial structure determined by Ψ . By this construction, Y is symmetric and compact, and it can be shown that $\left(Y, \mathrm{id}_X\right)$ is a compactification of σX ; we omit the details. //

4. Quotients and products

We begin this section by correcting an error in Theorem 4.1 of [10]. The assertion that $f^{-1}\left(\mathrm{cl}^n_{r_{\beta Y}} G\right) = \mathrm{cl}^n_{r_{\beta Y}} f^{-1}G$, which occurs about half way through the proof of Theorem 4.1, is invalid without an additional assumption. The following lemma, which is easy to prove, provides what is needed.

LEMMA 4.1. *If* $f : X \to Y$ *is an open map, then, for each* $A \subseteq Y$ *and* $n \in N$, $\mathrm{cl}^n_X f^{-1}(A) = f^{-1}\left(\mathrm{cl}^n_Y A\right)$.

We now state the corrected version of this theorem.

THEOREM 4.2. *If* $f : X \to Y$ *is open and proper, then* $f : r_\alpha X \to r_\alpha Y$ *is open and proper for all ordinals* α .

Proof. Assume that $f : r_\beta X \to r_\beta Y$ is open and proper for all ordinals $\beta < \alpha$. With the help of Lemma 4.1, the proof of Theorem 4.1 of [10] is now a valid argument to establish that $f : r_\alpha X \to r_\alpha Y$ is proper. It remains to show that the same map is open. Let F be an ultrafilter which converges to y in $r_\alpha Y$. Let $x \in f^{-1}(y)$. Then there is an ultrafilter $G \to y$ in Y , $\beta < \alpha$, and $n \in N$ such that $F \geq \mathrm{cl}^n_{r_{\beta Y}} G$.

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Since $f : X \to Y$ is open, there is an ultrafilter H on X such that $f(H) = G$ and $H \to x$ in X. Then $f\left(\mathrm{cl}^n_{r_{\beta X}} H\right) = \mathrm{cl}^n_{r_{\beta Y}} G \leq F$; this follows from the induction hypothesis and the fact that proper maps are closure-preserving. Thus there is an ultrafilter $K \geq \left(\mathrm{cl}^n_{r_{\beta X}} H\right) \vee f^{-1}(F)$. Hence $f(K) = F$ and $K \to x$ in $r_\alpha X$, which completes the proof. //

Note that Corollaries 4.2 and 4.3 and Theorem 4.4 of [10] are all derived from Theorem 4.1. All of these results are valid if one adds the additional assumption that the map f is open.

If $f : X \to Y$ has one of the two properties (open and proper) but not the other, then it need not be true that $f : r_\alpha X \to r_\alpha Y$ has the same property. We give a counterexample below for the "proper" case; examples for the "open" case are at hand, but are omitted for brevity.

EXAMPLE 4.3. Let A, B , and C be countably infinite disjoint sets, let $X = A \cup B \cup C$, and let $Y = A \cup B$. Let x_0 be a fixed point in Y , and let F be a free ultrafilter on X containing A . For each infinite subset D of A and for each point z in B , choose a free ultrafilter $G_{D,z}$ on X which contains D . Because of the large supply of free ultrafilters which contain D , this selection can be made in such a way that $G_{D,z} \neq G_{E,y}$ when $D \neq E$ or $z \neq y$. Let X be equipped with the finest convergence structure which satisfies the following conditions:

(1) $F \to x_0$;

(2) $G_{D,z} \to z$.

Let Y be a subspace of X . Since C and B are countable sets, there exists an injective function g mapping C onto B . We now define $f : X \to Y$ as follows:

(1) if $x \in A \cup B$, then $f(x) = x$;

(2) if $x \in C$, then $f(x) = g(x)$.

By this construction, $f : X \to Y$ is a perfect map. Note that $\mathrm{cl}_X F = F \cap \dot{B} \cap \dot{x}_0$, where $\dot{B}$ is the filter of all oversets of B . Thus each free ultrafilter H on Y containing B, r_1Y-converges to x_0. But no free ultrafilter on X which contains C can r_1X-converge, and consequently $f : r_1X \to r_1Y$ is not proper. //

It is interesting to note that the map $f : X \to Y$ is not only perfect, but is also a retract. It follows from Theorem 4.5 of [10] that $f : r_1X \to r_1Y$ is a retract even though it is not proper.

PROPOSITION 4.4. *Let* $f : X \to Y$ *and* $\phi_Y : Y \to sY$ *be open (respectively, proper) maps. Then* $\overline{f} : sX \to sY$ *is open (respectively, proper).*

Proof. We refer to the diagram of Proposition 1.6. The proof will be given for the "open" case; the other case is similar.

Let F be an ultrafilter on sY converging to $[y]$, and let $[x] \in f^{-1}([y])$. Let $x_1 \in [x]$ and $y_1 = f(x_1)$. Since ϕ_Y is open, there is an ultrafilter H on Y such that $H \to y_1$ in Y and $\phi_Y(H) = F$. Since f is open, there is an ultrafilter G on X such that $G \to x_1$ in X and $f(G) = H$. Thus $\phi_X(G) \to [x]$ in sX, and $\overline{f}(\phi_X(G)) = F$, since the diagram is commutative. //

COROLLARY 4.5. *If* $f : X \to Y$ *is open and proper and* $\phi_Y : Y \to sY$ *is open and proper, then* $f : s_\alpha X \to s_\alpha Y$ *is open and proper for all ordinals* α.

COROLLARY 4.6. *If* $f : X \to Y$ *is open and proper and* Y *is symmetric, then* $f : s_\alpha X \to s_\alpha Y$ *is open and proper for all ordinals* α.

We will conclude this section by briefly considering the behavior of the *RH*-series relative to products. Let $X = \pi\{X_i : i \in I\}$ be a product space, where I denotes a finite index set. One can easily verify that under the natural identification the sets sX and $\pi\{sX_i : i \in I\}$ coincide.

THEOREM 4.7. *Let* $X = \pi\{X_i : i \in I\}$ *be a product space, where* I

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is a finite index set.

(a) $sX = \pi\{sX_i : i \in I\}$.

(b) $s_\alpha X = \pi\{s_\alpha X_i : i \in I\}$.

(c) $l_{RH}(X) = \sup\{l_{RH}(X_i) : i \in I\}$.

The proof of *(a)* is routine and statements *(b)* and *(c)* follow immediately from *(a)*, and the results of Section 6 of [10]. //

5. Examples

In [10], there are spaces constructed whose R-series have length at least 3 . In our first example, we construct a Hausdorff space X^* such that $l_R(X) \geq \sigma$, where σ is any preassigned ordinal. An interesting feature of the space X^* is that it differs from a locally compact, regular, Hausdorff space only in the convergence of one filter and its refinements.

EXAMPLE 5.1. Let σ be an uncountable cardinal ordinal (that is, the least ordinal of a given cardinality), and let $\{X^\alpha : 0 \leq \alpha < \sigma\}$ be a disjoint collection of denumerable sets. Let $X = \cup\{X^\alpha : 0 \leq \alpha < \sigma\}$. Partition each X^α into sets $\left\{X_n^\alpha : 0 \leq n < \omega\right\}$, where each X_n^α is denumerable. Also, let the elements of X^α be indexed as follows: $X^\alpha = \left\{x_n^\alpha : 0 \leq n < \omega\right\}$. No relationship is assumed between the sets X_n^α and the elements x_n^α . Assign to X the finest convergence structure subject to the following conditions:

(1) each free ultrafilter which contains X_n^α , X-converges to $x_n^{\alpha+1}$, for all $\alpha < \sigma$;

(2) for each $n < \omega$, let $T_n = \left\{x_n^\alpha : 0 \leq \alpha < \sigma\right\}$, and let T_n have the well order determined by the superscripts of its elements.

A filter F which contains T_n converges to y in T_n iff the restriction of F to T_n converges to y in the order topology on T_n

The space X resembles closely the space (S, r) of Example 2.10 of [5]. Like the latter space, X is locally compact, regular, and Hausdorff.

We now define a space X^* on the same underlying set as follows: X^* has the finest convergence structure which satisfies the conditions imposed on X along with the condition that the filter A generated by the collection $\left\{X^0-X_n^0 : n < \omega\right\}$ converges to an arbitrary but prechosen point $a \in X^1$.

If $A = X^0 - X_n^0$ is a subbasic member of A, then $\mathrm{cl}_{X^*}A = A \cup \left(X^1-\left\{x_n^1\right\}\right)$ and $X^n \subseteq \mathrm{cl}_{X^*}^n A$ for $n \geq 2$. From this, we easily deduce the following results:

(1) $\dot{y} \to a$ in r_1X^* for all $y \in \bigcup\{X^\alpha : 2 \leq \alpha \leq \omega\}$;

(2) $\dot{a} \to y$ in r_2X^* for all $y \in \bigcup\{X^\alpha : 2 \leq \alpha \leq \omega\}$;

(3) $\dot{a} \not\to y$ in r_2X^* for all $y \in X^\alpha$ if $\alpha > \omega$.

Similarly, one can show that $\dot{a} \to y$ in r_4X^* for each $y \in X^{2\omega}$, but $\dot{a}$ does not r_4X^*-converge to any point in X^α if $\alpha > 2\omega$. Note that for a point $z \in X^\alpha$, $r_\beta X^*$-convergence of filters to z does not differ from X^*-convergence of filters to z until a critical ordinal α is reached when $\dot{a} \to z$ in $r_\alpha X^*$. If β is the largest ordinal such that $\dot{a}$ $r_\alpha X^*$-converges to points in X^β, then $\beta + \omega$ will be the largest ordinal such that $\dot{a}$ $r_{\alpha+2}X^*$-converges to points in $X^{\beta+\omega}$. Thus it is clear that $l_RX^* \geq \sigma$. //

It is shown in Theorem 7.4 of [10] that if X is first countable, then $l_R(X) \leq \omega_1 + 1$. The next example shows that a first countable space can achieve an R-series length of precisely $\omega_1 + 1$.

EXAMPLE 5.2. Let X^* be the space constructed in Example 5.1 and let $\sigma = \omega_1$. In the definition of convergence in X^* , three types of non-trivial filters constitute a base for the convergence structure. All three types have countable filter bases if $\sigma = \omega_1$. Thus X^* is first countable.

Let $B = X^* - (X^0 \cup X^1)$, and let $\mathcal{B}$ be the filter consisting of all oversets of B . Then $B \subseteq \mathrm{cl}_{r_{\omega_1}X^*}\{a\}$, and so $\mathcal{B} \to a$ in $r_{\omega_1+1}X^*$. Suppose $\mathcal{B} \to a$ in $r_{\omega_1}X^*$. Then there is an $\alpha < \omega_1$ and $n < \omega$ such that $\mathcal{B} \geq \mathrm{cl}^n_{r_\alpha X^*}\mathcal{G}$, where $\mathcal{G} \to a$ in X^* . But then $X^0 \cup \{a\} \in \mathcal{G}$, and it is clear that $B \nsubseteq \mathrm{cl}^n_{r_\alpha X^*}(X^0 \cup \{a\})$ if $\alpha < \omega_1$. Therefore it is impossible for $\mathcal{B}$ to $r_{\omega_1}X^*$-converge to a , and so $l_R X^* \geq \omega_1 + 1$. But $l_R X^* \leq \omega_1 + 1$ by Theorem 7.4 of *10* , and so $l_R X^* = \omega_1 + 1$. //

In Theorem 6.3 of *10* , it is shown that if $X = \prod \{X_i : i \in I\}$ is a product space, where I is a finite index set, then $r_\alpha X = \prod \{r_\alpha X_i : i \in I\}$, for all ordinals α . Our next example shows that this conclusion does not extend to products over an infinite index set.

EXAMPLE 5.3. Let N denote the set of all non-negative integers. Define on the set N the finest convergence structure such that $\dot{n} \to n + 1$. Let $X = N^N$ be the product space; let x denote the point in X whose coordinates are all 0 , and let $y = (0, 1, 2, \ldots)$. Since $\mathrm{cl}^n_N\{0\} = \{0, 1, 2, \ldots, n\}$, then $\dot{n} \to 0$ in $r_1 N$ for all $n \in N$. Thus $\dot{y} \to x$ in $(r_1 N)^N$.

Now suppose that $\dot{y} \to x$ in $r_1 X$; then there is $m \in N$ and

$F \to x$ in X such that $\dot{y} \geq \mathrm{cl}_X^m F$. This means that $\dot{k} \geq \mathrm{cl}_N^m F_k$ for each $k \in N$, where F_k is the projection of F onto the k-th copy of N . Since the only filter which converges to 0 in N is $\dot{0}$, it follows that $F_k = \dot{0}$, and so $\dot{k} \geq \mathrm{cl}_N^m \dot{0}$, for each $k \in N$. But this is impossible if $k > m$, since $\mathrm{cl}_N^m \{0\} = \{0, 1, 2, \ldots, m\}$. This contradiction establishes that $\dot{y} \not\to x$ in $r_1 X$, and so $r_1 X \neq (r_1 N)^N$. //

6. Application

The notions of a compactification and a completion were studied by Ramaley and Wyler in [7] from the categorical point of view. We give an alternate proof to Theorem 7.1 of [7] by making use of the T_3-modification of a space discussed in Section 1.

THEOREM 6.1 ([7]). *The class of compact T_3 spaces is a reflective subcategory of the category of all convergence spaces.*

Proof. Let (Y, j) denote the compactification given in [8] for a space X . It is not necessary for X to be Hausdorff as was assumed in [8]. It was shown in [8] that j is a universal mapping relative to compact T_3 spaces and continuous mappings.

Let f be a continuous function from X into a compact T_3 space Z . Then there exists a continuous function $g : Y \to Z$ such that $g \circ j = f$. Let ϕ denote the natural mapping from Y into Y_s . Then by commutativity of the diagram in Proposition 1.6, $\bar{g} : Y_s \to Z$ is a continuous function such that $\bar{g} \circ \phi = g$. Hence $f = \bar{g} \circ \phi \circ j$, and so $\phi \circ j$ is a universal mapping. //

The notion of a Cauchy space was defined in [6], and previously by several other authors. The reader is asked to refer to [6] for basic definitions not given here. Let (X, C) denote a Cauchy space; then the relation $F \sim G$ iff $F \cap G \in C$ defines an equivalence relation on C . Let $T = \{[F] : F \in C\}$ denote the quotient set, and let $j : X \to T$ be defined by $j(x) = [\dot{x}]$. Further, let $\mathcal{D}$ be defined to be the Cauchy structure $\{G : G \geq jF \cap [\dot{F}], F \in C\}$ on T .

A function f from a Cauchy space (X, C) into a Cauchy space $(Y, \mathcal{D})$ is called Cauchy-continuous if $f(F) \in \mathcal{D}$ whenever $F \in C$. The above mapping $j : (X, C) \to (T, \mathcal{D})$ is Cauchy continuous. In fact, $(T, \mathcal{D})$ is the Cauchy space associated with the uniform convergence space completion given by Wyler in [11]. Hence it follows that j is a universal mapping relative to complete Hausdorff Cauchy spaces and Cauchy-continuous mappings.

Ramaley and Wyler, in Theorem 5.3 of [7], have shown that a class of T_3 complete Cauchy spaces is a reflective class of Cauchy spaces satisfying conditions specified there. We show the following version by using the same technique as in the above theorem.

THEOREM 6.2. *The class of* T_3 *complete Cauchy spaces is reflective in the class of all Cauchy spaces.*

Proof. Let (X, C) be a Cauchy space and let $f : (X, C) \to (R, E)$ be a Cauchy continuous mapping into the T_3 complete Cauchy space (R, E) . As mentioned above, since j is a universal mapping, then there exists a Cauchy-continuous mapping $g : (T, \mathcal{D}) \to (R, E)$ such that $f = g \circ j$.

Let T_s denote the T_3-modification of the convergence space induced by $(T, \mathcal{D})$ on T . Let $\mathcal{D}_s$ denote the Cauchy structure associated with the convergence structure of T_s . Then $\left(T_s, \mathcal{D}_s\right)$ is a T_3 complete Cauchy space. From Proposition 1.6, $\bar{g} : \left(T_s, \mathcal{D}_s\right) \to (R, E)$ is Cauchy-continuous, since both spaces are complete, and $\bar{g} \circ \phi = g$. Hence $f = \bar{g} \circ \phi \circ j$, so $\phi \circ j$ is a universal mapping. //

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Department of Mathematics,
Washington State University,
Pullman,
Washington,
USA;

Department of Mathematics,
East Carolina University,
Greenville,
North Carolina,
USA.