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A REMARK ON A CONJECTURE ON THE SYMMETRIC GAUSSIAN PROBLEM

NICOLA FUSCO AND DOMENICO ANGELO LA MANNA

Dipartimento di Matematica ed Applicazioni R. Caccioppoli, Università degli studi di Napoli Federico II, Napoli, Campania, Italy

Corresponding author: Domenico Angelo La Manna, email: domenicoangelo.lamanna@unina.it

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Abstract In this paper, we study the functional given by the integral of the mean curvature of a convex set with Gaussian weight with Gaussian volume constraint. It was conjectured that the ball centred at the origin is the only minimizer of such a functional for certain values of the mass. We prove that this is the case in dimension 2 while in higher dimension the situation is different. In fact, for small values of mass, the ball centred at the origin is a local minimizer, while for larger values the ball is a maximizer among convex sets with a uniform bound on the curvature.

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Let $\gamma(E)$ and $P_\gamma(E)$ be the Gaussian measure and the Gaussian perimeter of a set $E \subset \mathbb{R}^n$, see the definition in the next section. A classical result states that that among all sets of given Gaussian measure the half space is the only one minimizing the Gaussian perimeter [4, 18]. This inequality has also been established in a quantitative form with different approaches (see [1, 6]), both of independent interest. Due to its large number of applications, the nonlocal form of the Gaussian isoperimetric inequality has also been studied. In particular, it is known that the half space is still a minimizer of the fractional Gaussian perimeter if we define it by the means of the Stinga–Torrea extension (see [5, 16]), while this is no longer the case if one defines the fractional perimeter by means of a singular integral with a Gaussian weight (see [7]). On the contrary, if one restricts to the class of symmetric sets, the characterization of perimeter minimizers under a volume constraint is still an open problem. In [3] it was conjectured that these minimizers would be either the ball or its complement. This conjecture has been disproved in [13], see also [10], and in [2] it has been proven that for small values of the volume the minimizer among symmetric sets is given by a strip. Later on, in [11], it was shown that if such minimizer is convex and satisfies some additional condition, then it is a round cylinder. Finally, in a very recent paper Heilman, adapting previous ideas introduced by Colding and Minicozzi in the study of the mean curvature flow proved that if the minimizer of the Symmetric Gaussian Problem (SGP) is of the form $\Omega \times \mathbb{R} \subset \mathbb{R}^{n+1}$, then either Ω or Ω^c is convex [12].

In this paper, we study a problem related to the above SGP, raised in [15, p. 37, Question 1], where it was asked whether the ball centred at the origin maximizes the energy functional

$$\mathscr{H}(E) = \int_{\partial E} H_{\partial E} \mathrm{e}^{-\frac{|x|^2}{2}} \, \mathrm{d}\mathcal{H}^{n-1}$$

among all convex, symmetric sets with a fixed Gaussian volume. Here and in the following, $H_{\partial E}$ denotes the mean curvature of E. In the same paper, see Proposition 3.21, it is shown that a positive answer to the Heilman–Morgan Gaussian isoperimetric conjecture (see [12]) would imply the maximality of the ball centred at the origin for the functional above. The result we prove here seems to hint that also Heilman–Morgan conjecture should be true.

More precisely, we prove the maximality of the ball centred at the origin in two dimensions, even if one replaces $\mathrm{e}^{-\frac{|x|^2}{2}}$ with more general radial weights on the volume and on the perimeter. Furthermore, we show that this maximality property of the disk holds in a stronger form, in the sense that if $\gamma(E) = \gamma(B_r)$, the gap $\mathscr{H}(B_r) - \mathscr{H}(E)$ can be estimated in a quantitative way both from above and below, see Theorem 1. The estimate from above follows by a simple integration by parts, while the one from below follows by a calibration argument.

The situation is completely different in higher dimension. A simple example given in § 3 shows that, already in dimension 3, one can find a symmetric cylinder E such that $\gamma(E) = \gamma(B_r)$ but $\mathscr{H}(E) > \mathscr{H}(B_r)$ for sufficiently small values of r. Even worse, in Theorem 3, we prove that if E is a symmetric set sufficiently close in $W^{2,\infty}$ to the ball B_r with the same Gaussian volume, then $\mathscr{H}(B_r) < \mathscr{H}(E)$ provided that $r^2 < \frac{(n-2)(n-1)}{2n}$. On the other hand, if $r^2 > n-2$, we are able to show that the ball is a local maximizer of $\mathscr{H}$ with respect to all competitors E close in $W^{2,\infty}$, with the same Gaussian volume and not necessarily symmetric. These two local minimality and maximality results are obtained with a more or less standard second order Taylor expansion of the functional $\mathscr{H}$ around the ball. However, a completely different calibration argument allows us to show the maximality property of the ball among all convex sets satisfying a uniform bound on the curvature, see Theorem 4. This latter result suggests that for sufficiently large Gaussian volume, the ball should be indeed a global maximizer, but at the moment this seems to be a difficult open problem.

1. Preliminary definitions

For a measurable set $E \subset \mathbb{R}^n$, we denote by $\gamma(E)$ its Gaussian measure

$$\gamma(E) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_E \mathrm{e}^{-\frac{|x|^2}{2}} \, \mathrm{d}x, \tag{1.1}$$

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normalized so that $\gamma(\mathbb{R}^n) = 1$. We say that E has finite Gaussian perimeter if

$$P_\gamma(E) = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \sup_{\|X\|_{L^\infty(\mathbb{R}^n)} \leqslant 1} \left\{ \int_\Omega \mathrm{div}(\mathrm{e}^{-\frac{|x|^2}{2}} X(x))\,\mathrm{d}x,\ X \in C_c^1(\mathbb{R}^n, \mathbb{R}^n) \right\} < \infty.$$

Note that if E is a bounded open set with Lipschitz boundary, then

$$P_\gamma(E) = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\partial E} \mathrm{e}^{-\frac{|x|^2}{2}}\,\mathrm{d}\mathcal{H}^{n-1},$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure.

Let E be an open set of class C^2 and $X : M \to \mathbb{R}^n$ a C^1 vector field. For any $x \in M$, denoting by $\tau_1, \ldots, \tau_{n-1}$ an orthonormal base for the tangent space T_xM with $x \in M$, the tangential divergence of X is given by

$$\mathrm{div}_\tau X = \sum_{i=1}^{n-1} \langle \nabla_{\tau_i} X, \tau_i \rangle,$$

where $\nabla_{\tau_i} X$ is the derivative of X in the direction τ_i. Note that if we still denote by X a C^1 extension of the vector field in a tubular neighbourhood of ∂E, then

$$\mathrm{div}_\tau X = \mathrm{div}\, X - \langle DX\nu_{\partial E}, \nu_{\partial E} \rangle,$$

where $\nu_{\partial E}$ is the exterior normal to E. We also recall that the mean curvature of ∂E (actually the sum of the principal curvatures) is given by

$$H_{\partial E} = \mathrm{div}_\tau\, \nu_{\partial E}. \tag{1.2}$$

If we extend $\nu_{\partial E}$ in a tubular neighbourhood of ∂E to obtain a vector field ν still of class C^1, with $|\nu| = 1$, then

$$H_{\partial E} = \mathrm{div}\, \nu. \tag{1.3}$$

Observe that with this definition it turns out that if E is locally the subgraph of a $C^2(\mathbb{R}^{n-1})$ function u then

$$H_{\partial E} = -\,\mathrm{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}} \right). \tag{1.4}$$

We recall that if E is a bounded open set of class C^2 and $X \in C^1(\partial E, \mathbb{R}^n)$, the divergence theorem for manifolds states that

$$\int_{\partial E} \mathrm{div}_\tau X\,\mathrm{d}\mathcal{H}^{n-1} = \int_{\partial E} H_{\partial E} \langle X, \nu_{\partial E} \rangle\,\mathrm{d}\mathcal{H}^{n-1}.$$

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In particular, if X is a tangent vector field, it holds

$$\int_{\partial E} \operatorname{div}_\tau X \, \mathrm{d}\mathcal{H}^{n-1} = 0.$$

Note that if E is an open set of class $C^{1,1}$, hence it is locally the subgraph of a $C^{1,1}$ function u, the mean curvature of ∂E can be defined using (1.4). With this definition, the above divergence theorem still holds. Finally, the Laplace–Beltrami operator on ∂E is defined for any $h \in C^2(\partial E)$ as

$$\Delta_{\partial E} h = \operatorname{div}_\tau \nabla h,$$

where ∇h denotes the tangential gradient of h.

Finally, if μ is a non negative Borel measure in $\mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}_+$ is a Borel function and E is a Borel set, we denote by $f\mu \llcorner E$ the measure $\tilde{\mu}$ defined for any Borel set $F \subset \mathbb{R}^n$ as

$$\tilde{\mu}(F) = \int_{F \cap E} f \, \mathrm{d}\mu.$$

2. Two-dimensional case: a two-side estimate of the integral of the curvature in weighted spaces

In this section, we provide an estimate for the weighted integral of the curvature under suitable assumptions on the weight. Let $f : [0,\infty) \to (0,\infty)$ be a C^1 not increasing function and $w : (0,+\infty) \to [0,\infty)$ be defined as

$$w(r) = -\frac{f'(r)}{r}. \tag{2.1}$$

We define the weighted area $|E|_w$ of a set E as

$$|E|_w = \int_E w(|x|) \, \mathrm{d}x.$$

Note that if $f = \mathrm{e}^{-\frac{r^2}{2}}$ then $w = \mathrm{e}^{-\frac{r^2}{2}}$. Hence, the results given in this section apply to the particular case of the Gaussian weight.

We start by proving an isoperimetric type inequality concerning a weighted integral of the curvature. To this aim, here and in the following, we denote by B_r the ball centred at the origin with radius r.

Proposition 1. *Let $r > 0$, $f : [0,\infty) \to (0,\infty)$ a C^1 not increasing function and w be defined as in (2.1). For any closed convex set $E \subset \mathbb{R}^2$ containing the origin of class $C^{1,1}$*

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with $|E|_w = |B_r|_w$ *it holds*

$$\int_{\partial E} H_{\partial E} f(|x|)\,\mathrm{d}\mathcal{H}^1 \leqslant \int_{\partial B_r} H_{\partial B_r} f(|x|)\,\mathrm{d}\mathcal{H}^1. \tag{2.2}$$

If w *is not increasing too, (2.2) holds for any convex set* E *of class* $C^{1,1}$ *with* $|E|_w = |B_r|_w$.

Proof. For a convex set E of class $C^{1,1}$ containing the origin in the interior we denote by $\rho : \mathbb{R} \to (0,\infty)$ a $C^{1,1}$ periodic function such that $\partial E = \{\rho(\theta)(\sin\theta, \cos\theta) : \theta \in [0,2\pi]\}$. Note that, for almost every $\theta \in [0,2\pi]$, the curvature at $\rho(\theta)(\sin\theta,\cos\theta)$ is given by

$$H_{\partial E} = \frac{\rho^2 + 2\rho'^2 - \rho\rho''}{(\rho^2+\rho'^2)^{\frac{3}{2}}}.$$

Thus, we compute

$$|E|_w = \int_0^{2\pi} \mathrm{d}\theta \int_0^{\rho(\theta)} t w(t)\,\mathrm{d}t$$

and

$$\int_{\partial E} H_{\partial E} f(|x|)\,\mathrm{d}\mathcal{H}^1 = \int_0^{2\pi} \frac{\rho^2 + 2\rho'^2 - \rho\rho''}{(\rho^2+\rho'^2)^{\frac{3}{2}}} f(\rho)\sqrt{\rho^2+\rho'^2}\,\mathrm{d}\theta.$$

Since $|E|_w = |B_r|_w$, recalling (2.1) we have

$$\begin{aligned} 2\pi f(0) - \int_0^{2\pi} f(\rho)\,\mathrm{d}\theta &= \int_0^{2\pi}\int_0^{\rho} -f'(t)\,\mathrm{d}t\,\mathrm{d}\theta \\ &= \int_0^{2\pi}\int_0^{\rho} t w(t)\,\mathrm{d}t\,\mathrm{d}\theta = 2\pi\int_0^r t w(t)\,\mathrm{d}t = 2\pi(f(0)-f(r)), \end{aligned}$$

which gives

$$\int_0^{2\pi} f(\rho)\,\mathrm{d}\theta = 2\pi f(r).$$

Hence, integrating by parts,

$$\begin{aligned} \int_{\partial E} H_{\partial E} f(|x|)\,\mathrm{d}\mathcal{H}^1 &= \int_0^{2\pi} \frac{\rho'^2 - \rho\rho''}{\rho^2+\rho'^2} f(\rho)\,\mathrm{d}\theta + \int_0^{2\pi} f(\rho)\,\mathrm{d}\theta \\ &= -\int_0^{2\pi} \frac{\mathrm{d}}{\mathrm{d}\theta}\arctan\left(\frac{\rho'}{\rho}\right) f(\rho)\,\mathrm{d}\theta + \int_0^{2\pi} f(\rho)\,\mathrm{d}\theta \\ &= \int_0^{2\pi} \rho'\arctan\left(\frac{\rho'}{\rho}\right) f'(\rho)\,\mathrm{d}\theta + 2\pi f(r) \\ &\leqslant 2\pi f(r) = \int_{\partial B_r} H_{\partial B_r} f(r)\,\mathrm{d}\mathcal{H}^1, \end{aligned} \tag{2.3}$$

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where in the last inequality we used that $t\arctan t \geqslant 0$ for all $t\in\mathbb{R}$ and $f'(\rho)\leqslant 0$.

When $0\in\partial E$, given $\varepsilon>0$ small, we may translate E to get a set $E_\varepsilon = E + x_\varepsilon$ with $|x_\varepsilon|<\varepsilon$ and such that $0\in \operatorname{int} E_\varepsilon$. Then, the validity of (2.2) for E follows by applying the same inequality to E_ε and then letting $\varepsilon\to 0$.

In the case that the closed set E does not contain the origin let x_0 be the nearest point of $\overline{E}$ to the origin. Hence, since E is convex we have that $\langle x, x_0\rangle \geqslant |x_0|^2$ for all $x\in\overline{E}$, which in turn implies that $|x-x_0|<|x|$ for all $x\in\overline{E}$. Thus, $|E-x_0|_w \geq |E|_w$. Let $s>0$ such that $|B_s|_w = |E-x_0|_w$. Since $|E-x_0|_w\geq|E|_w$, we have that s$\geq$r and using that $E-x_0$ passes through the origin we find

$$\int_{\partial E} H_{\partial E} f(|x|)\,d\mathcal{H}^1 \leqslant \int_{\partial(E-x_0)} H_{\partial E} f(|x|)\,d\mathcal{H}^1 \leqslant 2\pi f(s)\leqslant 2\pi f(r).$$

□

We note that Proposition 1 remains true if we replace the convexity assumption on E with the assumption that E is starshaped with respect to the origin. Next result shows that when E is a convex set containing the origin, the inequality above can be given in a stronger quantitative form. To this aim, given any sufficiently smooth set $E\subset\mathbb{R}^2$, we introduce the following positive quantities

$$\alpha_f(E) = -\int_{\partial E}\left(|x| - \frac{\langle x,\nu\rangle^2}{|x|}\right) f'(|x|)\,\mathrm{d}\mathcal{H}^1 \tag{2.4}$$

and

$$\beta_f(E) = \int_{\partial E}\left(|x| - \frac{\langle x,\nu\rangle^2}{|x|}\right)\left(\frac{f(|x|)-|x|f'(|x|)}{|x|^2}\right)\mathrm{d}\mathcal{H}^1. \tag{2.5}$$

Theorem 1. *Let E be a convex set of class $C^{1,1}$ containing the origin such that $|E|_w = |B_r|_w$. Then*

$$\alpha_f(E)\leqslant \int_{\partial B_r} H_{\partial B_r} f(|x|)\,\mathrm{d}\mathcal{H}^1 - \int_{\partial E} H_{\partial E} f(|x|)\,\mathrm{d}\mathcal{H}^1 \leqslant \beta_f(E). \tag{2.6}$$

Proof. Denote by $\rho:\mathbb{R}\to(0,\infty)$ a $C^{1,1}$ periodic function such that $\partial E = \{\rho(\theta)(\sin\theta,\cos\theta) : \theta\in[0,2\pi]\}$. To prove the first inequality we observe that

$$|x| - \frac{\langle x,\nu\rangle^2}{|x|} \leqslant \frac{|x|^2-\langle x,\nu\rangle^2}{\langle x,\nu\rangle} = \frac{\rho'^2}{\sqrt{\rho^2+\rho'^2}}. \tag{2.7}$$

Using that

$$t\arctan t \geqslant \frac{t^2}{\sqrt{1+t^2}}$$

for all $t \in \mathbb{R}$, using (2.3) and arguing as in the proof of Proposition 1 we get

$$\begin{aligned}\int_{\partial E} H_{\partial E} f(|x|)\,\mathrm{d}\mathcal{H}^1 &= \int_0^{2\pi} \rho \frac{\rho'}{\rho} \arctan\left(\frac{\rho'}{\rho}\right) f'(\rho)\,\mathrm{d}\theta + 2\pi f(r) \\ &\leqslant \int_0^{2\pi} \frac{\rho'^2}{\sqrt{\rho'^2+\rho^2}} f'(\rho)\,\mathrm{d}\theta + 2\pi f(r) \\ &\leqslant \int_{\partial E} \left(|x| - \frac{\langle x,\nu\rangle^2}{|x|}\right) f'(|x|)\,\mathrm{d}\mathcal{H}^1 + \int_{\partial B_r} H_{\partial B_r} f(|x|)\,\mathrm{d}\mathcal{H}^1,\end{aligned}$$

where the last inequality follows from (2.7). To prove the second inequality, we first recall that up to a constant $u(x) = \log|x|$ is the fundamental solution of the Laplacian in two dimensions. As a consequence of this fact we claim that if E contains the origin and $|E|_w = |B_r|_w$

$$\int_{\partial E} \frac{1}{|x|} f(|x|)\,\mathrm{d}\mathcal{H}^1 \geqslant \int_{\partial B_r} \frac{1}{|x|} f(|x|)\,\mathrm{d}\mathcal{H}^1. \tag{2.8}$$

Note that for $\varepsilon > 0$ such that $\overline{B_\varepsilon} \subset E$, using the divergence theorem we have

$$\begin{aligned}\int_{\partial E} \frac{\langle x,\nu\rangle}{|x|^2} f(|x|)\,\mathrm{d}\mathcal{H}^1 &= \int_{\partial(E\setminus B_\varepsilon)} \frac{\langle x,\nu\rangle}{|x|^2} f(|x|)\,\mathrm{d}\mathcal{H}^1 + 2\pi f(\varepsilon) \\ &= \int_{E\setminus B_\varepsilon} \operatorname{div}\left(\frac{x}{|x|^2} f(|x|)\right)\,\mathrm{d}x + 2\pi f(\varepsilon) \\ &= \int_{E\setminus B_\varepsilon} \frac{f'(|x|)}{|x|}\,\mathrm{d}x + 2\pi f(\varepsilon).\end{aligned}$$

Therefore, letting $\varepsilon \to +$ we have

$$\int_{\partial E} \frac{\langle x,\nu\rangle}{|x|^2} f(|x|)\,\mathrm{d}\mathcal{H}^1 = \int_E \frac{f'(|x|)}{|x|}\,\mathrm{d}x + 2\pi f(0).$$

Thus, using the assumption $|E|_w = |B_r|_w$ we obtain

$$\begin{aligned}\int_{\partial E} \frac{1}{|x|} f(|x|)\,\mathrm{d}\mathcal{H}^1 &\geqslant \int_{\partial E} \frac{\langle x,\nu\rangle}{|x|^2} f(|x|)\,\mathrm{d}\mathcal{H}^1 \\ &= 2\pi f(0) + \int_E \frac{f'(|x|)}{|x|}\,\mathrm{d}x = \int_{\partial B_r} \frac{1}{|x|} f(|x|)\,\mathrm{d}\mathcal{H}^1.\end{aligned} \tag{2.9}$$

Note that the above inequality is strict, unless $E = B_r$ and it can be actually written in a quantitative form arguing, see, for instance, [14]. Note that in deriving inequality (2.9) we have used that for $n = 2$, up to a multiplicative constant, $x/|x|^2$ is the gradient of the fundamental solution of Laplace equation, a fact which is no longer true in higher dimension.

To prove the proposition, we now use (2.9) and the divergence theorem on a manifold.

$$\begin{aligned}\int_{\partial E} H_{\partial E} f(|x|)\, \mathrm{d}\mathcal{H}^1 &\geqslant \int_{\partial E} H_{\partial E} \left\langle \frac{x}{|x|}, \nu \right\rangle f(|x|)\, \mathrm{d}\mathcal{H}^1 \\ &= \int_{\partial E} \mathrm{div}_\tau \left(\frac{x}{|x|} f(|x|) \right) \mathrm{d}\mathcal{H}^1 \\ &= \int_{\partial E} \frac{1}{|x|} f(|x|)\, \mathrm{d}\mathcal{H}^1 + \int_{\partial E} \left(|x| - \frac{\langle x, \nu \rangle^2}{|x|} \right) \left(\frac{|x| f'(|x|) - f(|x|)}{|x|^2} \right) \mathrm{d}\mathcal{H}^1 \\ &\geqslant \int_{\partial B_r} \frac{1}{|x|} f(|x|)\, \mathrm{d}\mathcal{H}^1 + \int_{\partial E} \left(|x| - \frac{\langle x, \nu \rangle^2}{|x|} \right) \left(\frac{|x| f'(|x|) - f(|x|)}{|x|^2} \right) \mathrm{d}\mathcal{H}^1 \\ &= \int_{\partial B_r} H_{\partial B_r} f(|x|)\, \mathrm{d}\mathcal{H}^1 + \int_{\partial E} \left(|x| - \frac{\langle x, \nu \rangle^2}{|x|} \right) \left(\frac{|x| f'(|x|) - f(|x|)}{|x|^2} \right) \mathrm{d}\mathcal{H}^1\end{aligned}$$

thus proving the second inequality in (2.6). □

Remark 1. Observe that the above proof shows that inequality (2.8) holds for any set E of finite perimeter containing the origin in the interior. Note, however, that this latter assumption can not be weakened as it is shown by an example in [9].

Note that the above theorem essentially says that one may control the gap $\mathscr{H}(E) - \mathscr{H}(B_r)$ with the oscillation of the normals to E and B_r. To be more precise, let us denote by π the projection of ∂E on ∂B_r. Observe that

$$\frac{1}{2}|x||\nu_{\partial E}(x) - \nu_{\partial B_r}(\pi(x))|^2 \leqslant |x| - \frac{\langle x, \nu \rangle^2}{|x|} \leqslant |x||\nu_{\partial E}(x) - \nu_{\partial B_r}(\pi(x))|^2.$$

Then, it is clear that under the assumption of Theorem 1 and if $E \subset B_R$ for some $R > 0$, then

$$\left| \int_{\partial E} H_{\partial E} f(|x|)\, \mathrm{d}\mathcal{H}^1 - \int_{\partial B_r} H_{\partial B_r} f(|x|)\, \mathrm{d}\mathcal{H}^1 \right| \leqslant C(f, R) \|\nu_{\partial E}(x) - \nu_{\partial B_r}(\pi(x))\|^2_{L^2(\partial E)},$$

where the constant $C(f, f', R)$ depends only on the function f and its derivative and R.

Note that the above Theorem applies in particular to the Gaussian weight $\gamma(r) = \mathrm{e}^{-\frac{r^2}{2}}$. As a consequence of this we get

Corollary 1. *Let $E \subset \mathbb{R}^2$ convex and passing through the origin and $r > 0$ such that $\gamma(E) = \gamma(B_r)$. Then it holds*

$$\alpha_\gamma(E) \leqslant \int_{\partial B_r} H_{\partial B_r} \mathrm{e}^{-\frac{|x|^2}{2}}\, \mathrm{d}\mathcal{H}^1 - \int_{\partial E} H_{\partial E} \mathrm{e}^{-\frac{|x|^2}{2}}\, \mathrm{d}\mathcal{H}^1 \leqslant \beta_\gamma(E).$$

Remark 2. We note that inequalities (2.2) and (2.6) can be immediately extended to any bounded convex set E contained in the plane with not empty interior. To see this, we recall (see [17, § 4.2]) that for such E there exists a *curvature measure* μ_E supported

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on ∂E such that if E_h is a sequence of smooth convex sets converging in the Hausdorff distance to E, then the measures $H_{\partial E_h}\,\mathcal{H}^1 \llcorner \partial E_h$ converge weakly* to μ_E. Using this measure, for instance, (2.2) becomes

$$\int_{\partial E} f(|x|)\,\mathrm{d}\mu_E \leqslant \int_{\partial B_r} H_{\partial B_r} f(|x|)\,\mathrm{d}\mathcal{H}^1,$$

whenever E is such that $|E|_w = |B_r|_w$. A similar extension also holds for (2.6) which becomes

$$\alpha_f(E) \leqslant \int_{\partial B_r} H_{\partial B_r} f(|x|)\,\mathrm{d}\mathcal{H}^1 - \int_{\partial E} f(|x|)\,\mathrm{d}\mu_E \leqslant \beta_f(E), \tag{2.10}$$

where $\alpha_f(E)$ and $\beta_f(E)$ are defined in (2.4) and (2.5).

We conclude this section by proving another consequence of Theorem 1. More precisely, if $E_h \to B_r$ in the Hausdorff distance, then the corresponding weighted curvature integrals converge with a speed controlled by the distance of E_h from B_r. To this aim, given two closed sets $E, F \subset \mathbb{R}^2$, we denote by $d_{\mathcal{H}}(E,F)$ the Hausdorff distance between E and F. We will also use the following lemma, which is the two-dimensional version of a more general statement proved in [8] (see proof of Lemma 3.3).

Lemma 1. *Let $E \subset \mathbb{R}^2$ be a convex body containing the origin and let $\rho : [0, 2\pi] \to (0,2)$ be such that $\partial E = \rho(\theta)(\cos\theta, \sin\theta)$. Then*

$$\|\rho'\|_{L^\infty} \leqslant 2\sqrt{\|\rho-1\|_{L^\infty}}\frac{1+\|\rho-1\|_{L^\infty}}{1-\|\rho-1\|_{L^\infty}}.$$

Theorem 2. *Let $f : [0,\infty) \to (0,\infty)$ a C^1 not increasing function and $E_h \subset \mathbb{R}^2$ a sequence of convex sets converging to B_r in the Hausdorff distance as $h \to \infty$. Then, there exists a constant depending only on r and f such that for h large*

$$\left|\int_{\partial E_h} f(|x|)\,\mathrm{d}\mu_{E_h} - \int_{\partial B_r} f(|x|) H_{\partial B_r}\,\mathrm{d}\mathcal{H}^1\right| \leqslant C\,\mathrm{d}_{\mathcal{H}}(E_h, B_r).$$

Proof. Let w be the function defined as in (2.1) and for all h let r_h be the unique positive number such that $|B_{r_h}|_w = |E_h|_w$. We have

$$\begin{aligned} \Big|\int_{\partial B_r} f(|x|) H_{\partial B_r}\,\mathrm{d}\mathcal{H}^1 - \int_{\partial E_h} f(|x|)\,\mathrm{d}\mu_{E_h}\Big| \\ \leqslant \left|\int_{\partial B_r} f(|x|) H_{\partial B_r}\,\mathrm{d}\mathcal{H}^1 - \int_{\partial B_{r_h}} f(|x|) H_{\partial B_r}\,\mathrm{d}\mathcal{H}^1\right| \\ + \int_{\partial B_{r_h}} H_{\partial B_{r_h}} f(|x|)\,\mathrm{d}\mathcal{H}^1 - \int_{\partial E_h} f(|x|)\,\mathrm{d}\mu_{E_h}. \end{aligned} \tag{2.11}$$

Setting $d_h = d_{\mathcal{H}}(E_h, B_r)$, since for h large $B_{r-d_h} \subset E_h \subset B_{r+d_h}$ we have $r - d_h \leqslant r_h \leqslant r + d_h$, that is $|r - r_h| \leqslant d_h$. Hence, for h large we may estimate the first integral on the right-hand side of (2.11) as follows.

$$\left| \int_{\partial B_r} f(|x|) H_{\partial B_r} \, \mathrm{d}\mathcal{H}^1 - \int_{\partial B_{r_h}} f(|x|) H_{\partial B_r} \, \mathrm{d}\mathcal{H}^1 \right| \leqslant 2\pi |f(r_h) - f(r)| \\ \leqslant 2\pi \max_{[r/2,2r]} |f'| |r_h - r| \leqslant C d_h.$$

To estimate the second integral, we denote by ρ_h the Lipschitz function such that $\partial E_h = \rho_h(\theta)(\cos\theta, \sin\theta)$. Then, we use the second inequality in (2.10), (2.7) and Lemma 1 applied to $\frac{1}{r} E_h$ to get for h large

$$\begin{aligned} \int_{\partial B_{r_h}} & H_{\partial B_{r_h}} f(|x|) \, \mathrm{d}\mathcal{H}^1 - \int_{\partial E_h} f(|x|) \, \mathrm{d}\mu_{E_h} \\ & \leqslant \max_{\rho \in [r/2,2r]} \left(\frac{f(\rho) - \rho f'(\rho)}{\rho^2} \right) \int_{\partial E_h} \left(|x| - \frac{\langle x, \nu_{\partial E_h} \rangle^2}{|x|} \right) \mathrm{d}\mathcal{H}^1 \\ & \leqslant C \int_0^{2\pi} \rho_h'^2 \, \mathrm{d}\theta \leqslant 8\pi C r \|\rho_h - r\|_\infty \left(\frac{r + \|\rho_h - r\|_\infty}{r - \|\rho_h - r\|_\infty} \right)^2 \\ & \leqslant C' d_h. \end{aligned}$$

This last estimates concludes the proof. □

Observe that arguing as in final part of the above proof under the assumption of Theorem 1 if $d_{\mathcal{H}}(E, B_r) < 1$, we have

$$\left| \int_{\partial E} H_{\partial E} f(|x|) \, \mathrm{d}\mathcal{H}^1 - \int_{\partial B_r} H_{\partial B_r} f(|x|) \, \mathrm{d}\mathcal{H}^1 \right| \leqslant C(f) d_{\mathcal{H}}(E, B_r)$$

for some constant depending only on f.

3. Higher dimension

The isoperimetric inequality proved in Proposition 1 is false in higher dimension as shown by the following example.

Example 1. Let $n = 3$, $r > 0$. If r is sufficiently small, there exists a C^∞ convex body E such that $\gamma(E) = \gamma(B_r)$ but

$$\mathscr{H}(E) > \mathscr{H}(B_r).$$

Proof. Denote by $C(t)$ the cylinder in

$$C(s) = \{(x', x_3) \in \mathbb{R}^2 \times \mathbb{R} : |x'| \leqslant s\}.$$

For any $r>0$, let $s(r)$ the unique positive number such that $\gamma(C_{s(r)}) = \gamma(B_r)$. Note that

$$\gamma(C_s) = \int_0^s t\mathrm{e}^{-\frac{t^2}{2}}\,\mathrm{d}t = 1 - \mathrm{e}^{-\frac{s^2}{2}},$$

while

$$(2\pi)^{\frac{1}{2}}\gamma(B_r) = 2\int_0^r t^2\mathrm{e}^{-\frac{t^2}{2}}\,\mathrm{d}t = 2\left(-r\mathrm{e}^{-\frac{r^2}{2}} + \int_0^r \mathrm{e}^{-\frac{t^2}{2}}\,\mathrm{d}t\right).$$

Since $\gamma(B_r) = \gamma(C_s)$ we get

$$\mathrm{e}^{\frac{-s^2}{2}} = 1 - \frac{2}{(2\pi)^{\frac{1}{2}}}\left(-r\mathrm{e}^{-\frac{r^2}{2}} + \int_0^r \mathrm{e}^{-\frac{t^2}{2}}\,\mathrm{d}t\right)$$

Moreover, we also have

$$\mathscr{H}\,(C_s) = (2\pi)^{\frac{3}{2}}\mathrm{e}^{-\frac{s^2}{2}}, \qquad \mathscr{H}\,(B_r) = 8\pi r\mathrm{e}^{-\frac{r^2}{2}}.$$

Hence,

$$\begin{aligned}\mathscr{H}\,(C_s) &= (2\pi)^{\frac{3}{2}} + 4\pi(r\mathrm{e}^{-\frac{r^2}{2}} - \int_0^r \mathrm{e}^{-\frac{t^2}{2}}\,\mathrm{d}t)\\ &= \mathscr{H}\,(B_r) + (2\pi)^{\frac{3}{2}} - 4\pi(r\mathrm{e}^{-\frac{r^2}{2}} + \int_0^r \mathrm{e}^{-\frac{t^2}{2}}\,\mathrm{d}t)\\ &\geqslant \mathscr{H}\,(B_r) + 1,\end{aligned}$$

provided r is sufficiently small. Let $C_{T,s(r)}$ the convex body obtained as the union of the cylinder $C_{s(r)} \cap \{|x_3| < T\}$ with the two half balls of radius $s(r)$ placed on the upper and lower basis of the cylinder. Since

$$\gamma(C_{T,s(r)}) \to \gamma(C_{s(r)})$$

as $T \to \infty$, we conclude that $\mathscr{H}\,(C_{T,s(r)}) > \mathscr{H}\,(B_{r'})$ with r' such that $\gamma(C_{T,s(r)}) = \gamma(B_{r'})$, provided r is small and T is sufficiently large. The C^∞ set E is obtained by approximating $C_{T,s(r)}$ with a sequence of smooth convex sets as in Remark 2. □

Lemma 2. *Let E be a bounded open set of class C^2 starshaped with respect to the origin and let $h : \mathbb{S}^{n-1} \to (0,\infty)$ a C^2 function such that*

$$\partial E = \{y = xh(x),\, x \in \mathbb{S}^{n-1}\}.$$

Then,

$$H_{\partial E}(xh(x)) = \frac{-\frac{1}{h}\Delta_{\mathbb{S}^{n-1}}h + n - 1}{\sqrt{|\nabla h|^2 + h^2}} + \frac{h\frac{1}{2}\langle\nabla|\nabla h|^2, \nabla h\rangle + h^2|\nabla h|^2}{h^2\sqrt{(|\nabla h|^2 + h^2)^3}}. \tag{3.1}$$

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Proof. First, we extend h to $\mathbb{R}^n \setminus \{0\}$ as a homogeneous function of degree 0 still denoted by h. Note that with this definition for any $x \in \mathbb{S}^{n-1}$ the tangential gradient $\nabla_\tau h(x)$ of h at x coincides with the gradient of h at the same point. For $x \in \mathbb{R}^n \setminus 0$, we define the unitary vector

$$n(x) := \frac{xh(x) - \nabla h(x)}{\sqrt{h^2(x) + |\nabla h(x)|^2}}. \tag{3.2}$$

Observe that the exterior normal ν to ∂E at $y = xh(x)$, where $x \in \mathbb{S}^{n-1}$, is given by

$$\nu(y) = n\left(\frac{y}{h(x)}\right) = n(x).$$

Recalling (1.2) we have

$$H_{\partial E}(y) = \operatorname{div} \nu(y) = \frac{\partial \nu_i}{\partial y_i}(y) = \frac{\partial n_i}{\partial x_j}\left(\frac{y}{h(y)}\right)\frac{\partial x_j}{\partial y_i}(y) = \frac{\partial n_i}{\partial x_j}(x)\frac{\partial x_j}{\partial y_i}(y),$$

where we have adopted the standard convention of summation over repeated indexes. Since the derivatives of h are homogeneous of degree -1, we have

$$\begin{aligned}\frac{\partial x_j}{\partial y_i} = \frac{\partial}{\partial y_i}\frac{y_j}{h(y)} &= \frac{\delta_{ij}}{h(y)} - \frac{y_j}{h^2(y)}\frac{\partial h}{\partial y_i}(y)\\ &= \frac{\delta_{ij}}{h(x)} - \frac{x_j}{h^2(x)}\frac{\partial h}{\partial x_i}(x).\end{aligned}$$

Hence,

$$H_{\partial E}(xh(x)) = \frac{1}{h(x)}\operatorname{div} n(x) - \frac{\partial n_i}{\partial x_j}(x)\frac{x_j}{h^2(x)}\frac{\partial h}{\partial x_i}(x).$$

Denoting by div_τ, the tangential divergence on $\mathbb{S}^{n-1}$ we have

$$\operatorname{div}_\tau n(x) = \operatorname{div} n(x) - \frac{\partial n_i(x)}{\partial x_j}x_i x_j.$$

Hence,

$$\begin{aligned}H_{\partial E}(xh(x)) &= \frac{1}{h(x)}\operatorname{div}_\tau n(x) + \frac{1}{h}\frac{\partial n_i}{\partial x_j}x_i x_j - \frac{\partial n_i}{\partial x_j}\frac{x_j}{h^2}\frac{\partial h}{\partial x_i}\\ &= \frac{1}{h(x)}\operatorname{div}_\tau n(x) + \frac{1}{h^2}(h^2 + |\nabla h|^2)^{\frac{1}{2}}\frac{\partial n_i}{\partial x_j}n_i x_j = \frac{1}{h(x)}\operatorname{div}_\tau n(x),\end{aligned}$$

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where in the last equality we used that $n_i \partial_{x_j} n_i = 0$ for every $1 \leqslant j \leqslant n$. Then, since $\langle x, \nabla_\tau h\rangle = 0$, recalling (3.2) we calculate

$$\begin{aligned} H_{\partial E}(xh(x)) =& \frac{1}{h(x)} \operatorname{div}_\tau \left(\frac{xh(x) - \nabla h(x)}{\sqrt{h^2(x) + |\nabla h(x)|^2}} \right) \\ =& \frac{\operatorname{div}_\tau (xh(x) - \nabla_\tau h)}{h(x)\sqrt{h^2 + |\nabla h|^2}} - \frac{\langle xh(x) - \nabla h(x), \nabla(h^2 + |\nabla_\tau h|^2)\rangle}{2h(x)(h(x)^2 + |\nabla h|^2)^{\frac{3}{2}}} \\ =& \frac{(n-1)h - \Delta_{\mathbb{S}^{n-1}} h}{h(x)\sqrt{h^2 + |\nabla h|^2}} + \frac{\langle \nabla h(x), \nabla(h^2 + |\nabla h|^2)\rangle}{2h(x)(h(x)^2 + |\nabla h|^2)^{\frac{3}{2}}} \end{aligned}$$

from which (3.1) immediately follows. □

We conclude by proving that in higher dimension if $r > \sqrt{n-2}$ then the ball B_r is a local maximizer of the integral of the weighted mean curvature with respect to C^2 perturbations. Quite surprisingly, the ball B_r is a local minimizer if r is small enough.

Theorem 3. *For all $r > 0$ there exist $\varepsilon_0(r), C(r) > 0$ with the property that if $u \in W^{2,\infty}(\mathbb{S}^{n-1})$, $\|u\|_{W^{2,\infty}} \leqslant \varepsilon < \varepsilon_0$ and $E = \{trx(1+u(x)),\ x \in \mathbb{S}^{n-1},\ t \in (0,1)\}$ is such that $\gamma(E) = \gamma(B_r)$ then*

$$\mathscr{H}(B_r) - \mathscr{H}(E) \geqslant r^{n-2} \mathrm{e}^{-\frac{r^2}{2}} \left(r^2 - n + 2 - C\varepsilon_0\right) \|u\|_{W^{1,2}(\mathbb{S}^{n-1})}. \tag{3.3}$$

Moreover, if $E = -E$

$$\mathscr{H}(B_r) - \mathscr{H}(E) \leqslant r^{n-2} \mathrm{e}^{-\frac{r^2}{2}} \left(C\varepsilon + r^2 - (n-2)\frac{n-1}{2n}\right) \|u\|_{W^{1,2}(\mathbb{S}^{n-1})}. \tag{3.4}$$

Proof. To prove our statement, we use (3.1) with h replaced by $r(1+u)$, thus getting

$$\begin{aligned} \mathscr{H}(E) =& \int_{\partial E} H_{\partial E} \, \mathrm{d}\mathcal{H}^{n-1} \\ =& \int_{\mathbb{S}^{n-1}} \left[(n-1) - \left(\frac{\Delta u}{1+u} \right) \right] \frac{r^{n-2} \mathrm{e}^{-\frac{r^2}{2}(1+u)^2}}{((1+u))^{2-n}} \, \mathrm{d}\mathcal{H}^{n-1} \\ &+ \int_{\mathbb{S}^{n-1}} \left(\frac{\langle \nabla^2 u \nabla u, \nabla u\rangle + (1+u)|\nabla u|^2}{(1+u)(|\nabla u|^2 + (1+u)^2)} \right) \frac{r^{n-2} \mathrm{e}^{-\frac{r^2}{2}(1+u)^2}}{((1+u))^{2-n}} \, \mathrm{d}\mathcal{H}^{n-1}. \\ =& r^{n-2} \left[(n-1)I - J + K \right]. \end{aligned}$$

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By Taylor expansion and the smallness of u, we get

$$\begin{aligned} I = & \int_{\mathbb{S}^{n-1}} (1+u)^{n-2} \mathrm{e}^{-\frac{r^2}{2}(1+u)^2} \, \mathrm{d}\mathcal{H}^{n-1} \\ = & \mathrm{e}^{-\frac{r^2}{2}} \left(n\omega_n + ((n-2)-r^2) \int_{\mathbb{S}^{n-1}} u \, \mathrm{d}\mathcal{H}^{n-1} \right) \\ & + \mathrm{e}^{-\frac{r^2}{2}} \left(\frac{(n-2)(n-3)-(2n-3)r^2+r^4)}{2} \int_{\mathbb{S}^{n-1}} u^2 \, \mathrm{d}\mathcal{H}^{n-1} \right) + o(\|u\|^2_{L^2(\mathbb{S}^{n-1})}) \end{aligned}$$

and

$$\begin{aligned} & \mathrm{e}^{\frac{r^2}{2}} \int_{\mathbb{S}^{n-1}} (1+u)^{n-3} \Delta u \mathrm{e}^{-\frac{r^2(1+u)^2}{2}} \, \mathrm{d}\mathcal{H}^{n-1} \\ & \qquad = \int_{\mathbb{S}^{n-1}} \Delta u \, \mathrm{d}\mathcal{H}^{n-1} + (n-3-r^2) \int_{\mathbb{S}^{n-1}} u \Delta u \, \mathrm{d}\mathcal{H}^{n-1} \\ & \qquad + \int_{\mathbb{S}^{n-1}} u^2 \Delta u G(u) \, \mathrm{d}\mathcal{H}^{n-1}, \end{aligned}$$

where $G(u)$ contains the remainder in the Taylor expansion. Using that $\int \Delta u \, \mathrm{d}\mathcal{H}^{n-1} = 0$ and integrating by parts the terms involving the Laplace–Beltrami operator we infer

$$\begin{aligned} J = \int_{\mathbb{S}^{n-1}} (1+u)^{n-3} \Delta u \mathrm{e}^{-\frac{r^2(1+u)^2}{2}} = -\mathrm{e}^{-\frac{r^2}{2}} (n-3-r^2) \int_{\mathbb{S}^{n-1}} |\nabla u|^2 \, \mathrm{d}\mathcal{H}^{n-1} \\ + o(\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})}). \end{aligned}$$

The last term is actually easier to treat since we are also assuming the smallness of the Hessian of u. Thus, we have

$$\begin{aligned} K = & \int_{\mathbb{S}^{n-1}} \frac{\langle \nabla^2 u, \nabla u, \nabla u \rangle + (1+u)|\nabla u|^2}{(1+u)^{1-n}((1+u)^2 + |\nabla u|^2)} \mathrm{e}^{-\frac{r^2}{2}(1+u)^2} \, \mathrm{d}\mathcal{H}^{n-1} \\ = & \mathrm{e}^{-\frac{r^2}{2}} (1 + o(\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})})) \int_{\mathbb{S}^{n-1}} \langle \nabla^2 u, \nabla u, \nabla u \rangle + |\nabla u|^2 \, \mathrm{d}\mathcal{H}^{n-1}. \end{aligned}$$

Collecting all the previous equalities, we then get

$$\begin{aligned} \mathscr{H}(E) - \mathscr{H}(B_r) = & r^{n-2} \mathrm{e}^{-\frac{r^2}{2}} (n-1) \left(((n-2)-r^2) \int_{\mathbb{S}^{n-1}} u \, \mathrm{d}\mathcal{H}^{n-1} \right) \\ & + r^{n-2} \mathrm{e}^{-\frac{r^2}{2}} (n-1) \left(\frac{(n-2)(n-3)-(2n-3)r^2+r^4)}{2} \int_{\mathbb{S}^{n-1}} u^2 \, \mathrm{d}\mathcal{H}^{n-1} \right) \\ & + r^{n-2} \mathrm{e}^{-\frac{r^2}{2}} \left((n-2-r^2) \int_{\mathbb{S}^{n-1}} |\nabla u|^2 \, \mathrm{d}\mathcal{H}^{n-1} + \int_{\mathbb{S}^{n-1}} \langle \nabla^2 u, \nabla u, \nabla u \rangle \, \mathrm{d}\mathcal{H}^{n-1} \right) \\ & + o(\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})}). \end{aligned} \tag{3.5}$$

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To estimate the integral of u in the previous equation, we need to exploit the assumption that the Gaussian measures of E and B_r are equal. In fact, since

$$\gamma(B_r) = \gamma(E) = \frac{r^n}{(2\pi)^{n/2}} \int_B (1+u(x))^n \mathrm{e}^{-\frac{r^2|x|^2(1+u(x))^2}{2}} \,\mathrm{d}x, \tag{3.6}$$

we can expand the integral via Taylor formula to find

$$\int_0^1 t^{n-1}\,\mathrm{d}t \int_{\mathbb{S}^{n-1}} \left[(1+u)^n \mathrm{e}^{-\frac{r^2t^2(1+u)^2}{2}} - \mathrm{e}^{-\frac{r^2t^2}{2}}\right] \mathrm{d}\mathcal{H}^{n-1} = 0.$$

Using again Taylor expansion, we then easily get

$$\begin{aligned} 0 &= \int_0^1 t^{n-1}\mathrm{e}^{-\frac{r^2t^2}{2}}\,\mathrm{d}t \int_{\mathbb{S}^{n-1}} \left[(1+u)^n \mathrm{e}^{-r^2t^2(u+u^2/2)} - 1\right] \mathrm{d}\mathcal{H}^{n-1} \\ &= \int_0^1 t^{n-1}\mathrm{e}^{-\frac{r^2t^2}{2}}\,\mathrm{d}t \int_{\mathbb{S}^{n-1}} \left[(n-r^2t^2)u + \left(\frac{n(n-1)}{2} - \frac{(2n+1)r^2t^2}{2} + \frac{r^4t^4}{2}\right)u^2\right] \mathrm{d}\mathcal{H}^{n-1} \\ &\qquad + o(\|u\|^2_{L^2(\mathbb{S}^{n-1})}) \\ &= \int_{\mathbb{S}^{n-1}} \left[(na_n - r^2b_n)u + \left(\frac{n(n-1)a_n}{2} - \frac{(2n+1)r^2b_n}{2} + \frac{r^4c_n}{2}\right)u^2\right] \mathrm{d}\mathcal{H}^{n-1} \\ &\qquad + o(\|u\|^2_{L^2(\mathbb{S}^{n-1})}), \end{aligned} \tag{3.7}$$

where we have set

$$a_n = \int_0^1 t^{n-1}\mathrm{e}^{-\frac{r^2t^2}{2}}\,\mathrm{d}t, \qquad b_n = \int_0^1 t^{n+1}\mathrm{e}^{-\frac{r^2t^2}{2}}\,\mathrm{d}t, \qquad c_n = \int_0^1 t^{n+3}\mathrm{e}^{-\frac{r^2t^2}{2}}\,\mathrm{d}t.$$

A simple integration by parts gives that

$$b_n = \frac{na_n}{r^2} - \frac{\mathrm{e}^{-\frac{r^2}{2}}}{r^2}, \qquad c_n = \frac{n(n+2)a_n}{r^4} - \frac{(n+2)\mathrm{e}^{-\frac{r^2}{2}}}{r^4} - \frac{\mathrm{e}^{-\frac{r^2}{2}}}{r^2}.$$

Thus, inserting the above values of b_n and c_n into (3.7), we arrive at

$$\int_{\mathbb{S}^{n-1}} u\,\mathrm{d}\mathcal{H}^{n-1} = -\frac{n-1-r^2}{2} \int_{\mathbb{S}^{n-1}} u^2\,\mathrm{d}\mathcal{H}^{n-1} + o(\|u\|^2_{L^2(\mathbb{S}^{n-1})}). \tag{3.8}$$

Hence, (3.5) combined with (3.8) gives

$$\begin{aligned}\mathscr{H}(E) - \mathscr{H}(B_r) = & - r^{n-2}\mathrm{e}^{-\frac{r^2}{2}}(n-1)(n-2)\int_{\mathbb{S}^{n-1}} u^2 \,\mathrm{d}\mathcal{H}^{n-1} \\ & + (n-2-r^2)r^{n-2}\mathrm{e}^{-\frac{r^2}{2}}\int_{\mathbb{S}^{n-1}} |\nabla u|^2 \,\mathrm{d}\mathcal{H}^{n-1} \\ & + r^{n-2}\mathrm{e}^{-\frac{r^2}{2}}\int_{\mathbb{S}^{n-1}} \langle \nabla^2 u \nabla u, \nabla u\rangle \,\mathrm{d}\mathcal{H}^{n-1} + o(\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})}).\end{aligned} \tag{3.9}$$

From this equality, we immediately conclude the proof of (3.3).

To prove the second inequality for any integer $k \geqslant 0$, we denote by $y_{k,i}$, $i = 1, \dots, G(n,k)$ the spherical harmonics of order k, i.e., the restrictions to $\mathbb{S}^{n-1}$ of the homogeneous harmonic polynomials of degree k, normalized so that $||y_{k,i}||_{L^2(\mathbb{S}^{n-1})} = 1$. The functions $y_{k,i}$ are eigenfunctions of the Laplace–Beltrami operator on $\mathbb{S}^{n-1}$ and for all k and i

$$-\Delta_{\mathbb{S}^{n-1}} y_{k,i} = k(k+n-2)y_{k,i}\,.$$

Therefore, if we write

$$u = \sum_{k=0}^{\infty}\sum_{i=1}^{G(n,k)} a_{k,i}y_{k,i}, \quad \text{where} \quad a_{k,i} = \int_{\mathbb{S}^{n-1}} u y_{k,i}\,\mathrm{d}\mathcal{H}^{n-1},$$

we have

$$||u||^2_{L^2(\mathbb{S}^{n-1})} = \sum_{k=0}^{\infty}\sum_{i=1}^{G(n,k)} a^2_{k,i}, \qquad ||D_\tau u||^2_{L^2(\mathbb{S}^{n-1})} = \sum_{k=1}^{\infty} k(k+n-2)\sum_{i=1}^{G(n,k)} a^2_{k,i}\,. \tag{3.10}$$

Note that (3.7) implies

$$|a_0|^2 = \left|\int_{\mathbb{S}^{n-1}} u\, d\mathcal{H}^{n-1}\right|^2 = o(\|u\|_{L^2(\mathbb{S}^{n-1})}).$$

Since $E = -E$, we also have that u is an even function; hence, $a_{2k+1,i} = 0$ for all $k \in \mathbb{N}$ and $i \in \{1, \dots, G(2k+1, n)\}$. Hence, we can write

$$\|\nabla u\|^2 \geqslant 2n\|u\|_{L^2(\mathbb{S}^{n-1})} - o(\|u\|^2_{L^2(\mathbb{S}^{n-1})}),$$

which finally gives

$$\begin{aligned}\int_{\partial E} H_{\partial E}\mathrm{e}^{-\frac{|x|^2}{2}}\,\mathrm{d}\mathcal{H}^{n-1} - & \int_{\partial B_r} H_{\partial B_r}\mathrm{e}^{-\frac{|x|^2}{2}}\,\mathrm{d}\mathcal{H}^{n-1} \\ & \geqslant r^{n-2}\mathrm{e}^{-r^2}\left((n-2)\frac{n+1}{2n} - r^2 - \varepsilon_0\right)\|\nabla u\|_{L^2(\mathbb{S}^{n-1})}.\end{aligned}$$

□

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The next result is a local maximality results under weaker assumptions. To this aim, we introduce the function

$$\psi(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} \mathrm{e}^{-\frac{t^2}{2}}\,\mathrm{d}t,$$

which is the value of the Gaussian volume of the half space $H_s = \{x \in \mathbb{R}^n : x_1 \leqslant s\}$.

Theorem 4. *Let $n \geqslant 3$, $M > 0$ and $m \geqslant \max\{\psi(2M), \psi(\sqrt{n-2})\}$. For any C^2 convex set containing the origin with $\gamma(E) = \gamma(B_r) = m$ and $\|H_{\partial E}\|_{L^\infty} \leqslant M$ it holds*

$$\mathscr{H}(E) \leqslant \mathscr{H}(B_r). \tag{3.11}$$

Moreover, if $m \geqslant \psi(\sqrt{n-2})$

$$\int_{\partial E} \frac{\langle x, \nu\rangle}{|x|} H_{\partial E} \mathrm{e}^{-\frac{|x|^2}{2}} \leqslant \int_{\partial B_r} \frac{\langle x, \nu\rangle}{|x|} H_{\partial B_r} \mathrm{e}^{-\frac{|x|^2}{2}} \tag{3.12}$$

for any convex set E containing the origin with $\mathscr{H}(E) < \infty$ and $\gamma(E) = \gamma(B_r) = m$.

Proof. Let E as in the statement and let r_E the radius of the largest ball centred at the origin and contained in E, i.e.

$$r_E = \sup\{r : B_r \subset E\}. \tag{3.13}$$

Let $x \in \partial B_{r_E} \cap \partial E$ and let H be the halfspace containing the origin and such that the hyperplane ∂H is tangent to E at x. Since by convexity $E \subset H$ we have $\psi(r_E) = \gamma(H) \geqslant m$, hence $r_E \geqslant \psi^{-1}(m)$. Therefore, our assumption on m implies $r_E \geqslant \max\{2M, \sqrt{2(n-2)}\}$. Now, using the divergence theorem on mainfolds, we infer

$$\begin{aligned}\mathscr{H}(E) &= \int_{\partial E} H_{\partial E} \frac{\langle x, \nu\rangle}{|x|} \mathrm{e}^{-\frac{|x|^2}{2}}\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\partial E} H_{\partial E} \left(1 - \frac{\langle x, \nu\rangle}{|x|}\right) \mathrm{e}^{-\frac{|x|^2}{2}}\,\mathrm{d}\mathcal{H}^{n-1} \\ &= \int_{\partial E} \mathrm{div}_\tau \left(\frac{x}{|x|} \mathrm{e}^{-\frac{|x|^2}{2}}\right) \mathrm{d}\mathcal{H}^{n-1} + \int_{\partial E} H_{\partial E} \left(1 - \frac{\langle x, \nu\rangle}{|x|}\right) \mathrm{e}^{-\frac{|x|^2}{2}}\,\mathrm{d}\mathcal{H}^{n-1}.\end{aligned}$$

We compute the tangential divergence to find

$$\mathrm{div}_\tau \left(\frac{x}{|x|} \mathrm{e}^{-\frac{|x|^2}{2}}\right) = \frac{n-1}{|x|} \mathrm{e}^{-\frac{|x|^2}{2}} - \left(1 - \frac{\langle x, \nu\rangle^2}{|x|^2}\right) \left(\frac{1}{|x|} + |x|\right) \mathrm{e}^{-\frac{|x|^2}{2}}.$$

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This gives

$$\begin{aligned}\mathscr{H}(E) &= \int_{\partial E} \frac{n-1}{|x|} \mathrm{e}^{-\frac{|x|^2}{2}} \, \mathrm{d}\mathcal{H}^{n-1} \\ &\quad + \int_{\partial E} \left(1 - \frac{\langle x, \nu\rangle}{|x|}\right) \left(H_{\partial E} - \left(\frac{1}{|x|} + |x|\right) \left(1 + \frac{\langle x, \nu\rangle}{|x|}\right)\right) \mathrm{e}^{-\frac{|x|^2}{2}} \, \mathrm{d}\mathcal{H}^{n-1} \\ &= \int_{\partial E} \frac{n-1}{|x|^2} \langle x, \nu\rangle \mathrm{e}^{-\frac{|x|^2}{2}} \, \mathrm{d}\mathcal{H}^{n-1} \\ &\quad + \int_{\partial E} \left(1 - \frac{\langle x, \nu\rangle}{|x|}\right) \left(H_{\partial E} + \frac{n-1}{|x|} - \left(\frac{1}{|x|} + |x|\right) \left(1 + \frac{\langle x, \nu\rangle}{|x|}\right)\right) \mathrm{e}^{-\frac{|x|^2}{2}} \, \mathrm{d}\mathcal{H}^{n-1}.\end{aligned} \tag{3.14}$$

Note that, differently from the two-dimensional case, the integral quantity $\int_{\partial E} \frac{n-1}{|x|^2} \langle x, \nu\rangle \mathrm{e}^{-\frac{|x|^2}{2}} \, \mathrm{d}\mathcal{H}^{n-1}$ is maximized by the ball centred at the origin with the same Gaussian volume of E. Indeed, using the divergence theorem

$$\begin{aligned}\int_{\partial E} \frac{1}{|x|^2} \langle x, \nu\rangle \mathrm{e}^{-\frac{|x|^2}{2}} \, \mathrm{d}\mathcal{H}^{n-1} &= \int_E \operatorname{div}\left(\frac{x}{|x|^2} \mathrm{e}^{-\frac{|x|^2}{2}} \, \mathrm{d}x\right) \\ &= \int_E \frac{n-2}{|x|^2} \mathrm{e}^{-\frac{|x|^2}{2}} \, \mathrm{d}x - (2\pi)^{\frac{n}{2}} \gamma(E) \\ &\leqslant \int_{B_r} \frac{n-2}{|x|^2} \mathrm{e}^{-\frac{|x|^2}{2}} \, \mathrm{d}x - (2\pi)^{\frac{n}{2}} \gamma(B_r) \\ &= \int_{\partial B_r} \frac{1}{|x|} \mathrm{e}^{-\frac{|x|^2}{2}} \, \mathrm{d}\mathcal{H}^{n-1} = \frac{1}{n-1} \mathscr{H}(B_r).\end{aligned} \tag{3.15}$$

Since E is a convex set containing the origin, we have that $\langle x, \nu\rangle \geqslant 0$ for all $x \in \partial E$. This fact together with (3.14) and (3.15) leads to

$$\mathscr{H}(E) \leqslant \mathscr{H}(B_r) + \int_{\partial E} \left(1 - \frac{\langle x, \nu\rangle}{|x|}\right) \left(H_{\partial E} + \frac{n-2}{|x|} - |x|\right) \mathrm{e}^{-\frac{|x|^2}{2}} \, \mathrm{d}\mathcal{H}^{n-1}. \tag{3.16}$$

Since $H_{\partial E}(x) \leqslant M \leqslant r_E/2$, $r_E \leqslant |x|$ for all $x \in \partial E$ the assumption $r_E \geqslant 2\sqrt{n-2}$ implies that the integrand on the right-hand side is negative, which, in turn, gives (3.11). Inequality (3.12) is also a consequence of (3.16). Indeed, (3.16) implies that if $r \geqslant \psi(\sqrt{n-2})$

$$\int_{\partial E} \frac{\langle x, \nu\rangle}{|x|} H_{\partial E} \mathrm{e}^{-\frac{|x|^2}{2}} \, \mathrm{d}\mathcal{H}^{n-1} \leqslant \mathscr{H}(B_r) = \int_{\partial B_r} \frac{\langle x, \nu\rangle}{|x|} H_{\partial B_r} \mathrm{e}^{-\frac{|x|^2}{2}} \, \mathrm{d}\mathcal{H}^{n-1}.$$

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