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Abstract

We prove that there exists at least one positive Einstein metric on $\mathbb{HP}^{m+1}\sharp\overline{\mathbb{HP}}^{m+1}$ for $m \geq 2$. Based on the existence of the first Einstein metric, we give a criterion to check the existence of a second Einstein metric on $\mathbb{HP}^{m+1}\sharp\overline{\mathbb{HP}}^{m+1}$. We also investigate the existence of cohomogeneity-one positive Einstein metrics on $\mathbb{S}^{4m+4}$ and prove the existence of a non-standard Einstein metric on $\mathbb{S}^8$.

1. Introduction

A Riemannian manifold (M, g) is *Einstein* if its Ricci curvature is a constant multiple of g:

$$\mathrm{Ric}(g) = \Lambda g.$$

The metric g is then called an *Einstein metric* and Λ is the *Einstein constant*. Depending on the sign of Λ, we call g a positive Einstein ($\Lambda > 0$) metric, a negative Einstein ($\Lambda < 0$) metric or a Ricci-flat ($\Lambda = 0$) metric. A positive Einstein manifold is compact by Myers' theorem [Mye41].

In this paper we investigate the existence of positive Einstein metrics of *cohomogeneity one*. A Riemannian manifold (M, g) is of cohomogeneity one if a Lie group G acts isometrically on M such that the principal orbit G/K is of codimension one. The first example of an inhomogeneous positive Einstein metric was constructed in [Pag78]. The metric is defined on $\mathbb{CP}^2\sharp\overline{\mathbb{CP}}^2$ and is of cohomogeneity one. The result was later generalized in [Ber82, KS86, Sak86, PP87, KS88, WW98]. A common feature shared by positive Einstein metrics constructed in this series of works is that the principal orbits are principal $U(1)$ bundles over either a Fano manifold or a product of Fano manifolds. From this perspective, one can view the Einstein metric on $\mathbb{HP}^2\sharp\overline{\mathbb{HP}}^2$ in [Böh98] as another type of generalization to the Page metric, whose principal orbit is a principal $Sp(1)$ bundle over $\mathbb{HP}^1$.

A natural question arises whether there exists a positive Einstein metric of cohomogeneity one on $\mathbb{HP}^{m+1}\sharp\overline{\mathbb{HP}}^{m+1}$ with $m \geq 2$, where the principal orbit is the total space of the quaternionic Hopf fibration formed by the following group triple:

$$(\mathsf{K}, \mathsf{H}, \mathsf{G}) = (Sp(m)\Delta Sp(1), Sp(m)Sp(1)Sp(1), Sp(m+1)Sp(1)). \tag{1.1}$$

The condition of being G-invariant reduces the Einstein equations to an ordinary differential equation (ODE) system defined on the one-dimensional orbit space. The solution takes the form

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$g = dt^2 + g(t)$, where $g(t)$ is a G-invariant metric on $\mathbb{S}^{4m+3}$ for each t. One looks for a $g(t)$ that is defined on a closed interval $[0, T]$ with an initial condition and a terminal condition. If $g(t)$ collapses to the quaternionic Kähler metric on the singular orbit $\mathbb{HP}^m$ at $t = 0$ and $t = T$, then g defines a positive Einstein metric on the connected sum $\mathbb{HP}^{m+1}\sharp\overline{\mathbb{HP}}^{m+1}$, or equivalently, an $\mathbb{S}^4$ bundle over $\mathbb{HP}^m$. It has been conjectured that such an Einstein metric exists on $\mathbb{HP}^{m+1}\sharp\overline{\mathbb{HP}}^{m+1}$ for all $m \geq 2$, as indicated by numerical evidence provided in [PP86, Böh98, DHW13].

Some well-known Einstein metrics are realized as integral curves to the cohomogeneity-one Einstein equation. For example, the standard sphere metric, the sine cone over Jensen's sphere and the quaternionic Kähler metric on $\mathbb{HP}^{m+1}$ are represented by integral curves to the cohomogeneity-one system. Furthermore, the cone solution is an attractor to the system. It was realized in [Böh98] that the winding of integral curves around the cone solution plays an important role in the existence problem described above. To investigate the winding, one studies a quantity (denoted by $\sharp C_w(\bar{h})$ in [Böh98]) that is assigned to each local solution that does not globally define a complete Einstein metric on $\mathbb{HP}^{m+1}\sharp\overline{\mathbb{HP}}^{m+1}$. From the point of view of geometry, the quantity records the number of times that the principal orbit becomes isoparametric while its mean curvature remains positive. In general, an estimate for $\sharp C_w(\bar{h})$ can be obtained from the linearization along the cone solution. For $m = 1$, the estimate is good enough to prove the global existence. This is not the case, however, if $m \geq 2$. For higher-dimensional cases, it is from the global analysis of the system that we obtain a further estimate for $\sharp C_w(\bar{h})$ and we prove the following existence theorem.

Theorem 1.1. *On each $\mathbb{HP}^{m+1}\sharp\overline{\mathbb{HP}}^{m+1}$ with $m \geq 2$, there exists at least one positive Einstein metric with $\mathsf{G}/\mathsf{K} = \mathbb{S}^{4m+3}$ as its principal orbit.*

Numerical studies in [Böh98, DHW13] indicate that there exists another Einstein metric on $\mathbb{HP}^{m+1}\sharp\overline{\mathbb{HP}}^{m+1}$ with $m \geq 2$. Based on Theorem 1.1, an estimate for $\sharp C_w(\bar{h})$ in a limiting subsystem (essentially obtained from the linearization along the cone solution) helps us propose a criterion to check the existence of the second Einstein metric. Let n be the dimension of G/K. Such a criterion only depends on n (or m).

Theorem 1.2. *Let θ_Ψ be the solution to the following initial value problem:*

$$\frac{d\theta}{d\eta} = \frac{n-1}{2n}\tanh\left(\frac{2\eta}{n}\right)\sin(2\theta) + \frac{2}{n}\sqrt{\frac{(2m+1)(2m+2)(2m+3)}{(2m+3)^2+2m}}, \quad \theta(0) = 0. \tag{1.2}$$

Let $\Omega = \lim_{\eta\to\infty}\theta_\Psi$. For $m \geq 2$, there exist at least two positive Einstein metrics on $\mathbb{HP}^{m+1}\sharp\overline{\mathbb{HP}}^{m+1}$ if $\Omega < 3\pi/4$.

The upper bound for Ω in Theorem 1.2 is not sharp. Although it is difficult to solve the initial value problem (1.2) explicitly, one can use the Runge–Kutta fourth-order algorithm to approximate Ω. Since the right-hand side of (1.2) does not vanish at $\eta = 0$, the initial Runge–Kutta step is well defined. Our numerical study shows that $\Omega < 3\pi/4$ for integers $m \in [2, 100]$.

We also look into the case where G/K completely collapses at two ends of a compact manifold. In that case, the cohomogeneity-one space is $\mathbb{S}^{4m+4}$. No new Einstein metric is found on $\mathbb{S}^{4m+4}$ for $m \geq 2$. For $m = 1$, however, we obtained a non-standard positive Einstein metric on $\mathbb{S}^8$. Such a metric is inhomogeneous by the classification in [Zil82].

Theorem 1.3. *There exists a non-standard $Sp(2)Sp(1)$-invariant positive Einstein metric $\hat{g}_{\mathbb{S}^8}$ on $\mathbb{S}^8$.*

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It is worth mentioning that all new solutions found are symmetric. Metrics that are represented by these solutions all have a totally geodesic principal orbit.

This paper is structured as follows. In § 2 we present the dynamical system for positive Einstein metrics of cohomogeneity one with G/K as the principal orbit. Then we apply a coordinate change that makes the cohomogeneity-one Ricci-flat system serve as a limiting subsystem. Initial conditions and terminal conditions are transformed into critical points of the new system. The new system admits $\mathbb{Z}_2$-symmetry. By a sign change, one can transform initial conditions into terminal conditions. Hence, the problem of finding globally defined positive Einstein metrics boils down to finding heteroclines that join two different critical points.

In § 3 we compute linearizations of the critical points mentioned above and obtain two one-parameter families of locally defined positive Einstein metrics. One family is defined on a tubular neighborhood around $\mathbb{HP}^m$, represented by a one-parameter family of integral curves γ_{s_1}. The other family is defined on a neighborhood of a point in $\mathbb{S}^{4m+4}$, represented by another one-parameter family of integral curves ζ_{s_2}.

In § 4 we make a little modification to the quantity $\sharp C_w(\bar{h})$ in [Böh98] and assign it to both γ_{s_1} and ζ_{s_2} (hence denoted by $\sharp C(\gamma_{s_1})$ and $\sharp C(\zeta_{s_2})$). We construct a compact set to obtain an estimate for $\sharp C(\gamma_{s_1})$ of some local solutions. Then we apply Lemma 4.4 in [Böh98] and prove Theorem 1.1.

In § 5 we apply another coordinate change that allows us to obtain more information on $\sharp C(\gamma_{s_1})$ and $\sharp C(\zeta_{s_2})$, which is encoded in the initial value problem (1.2) in Theorem 1.2. We also prove Theorem 1.3.

Visual summaries of Theorem 1.1–1.3 are presented at the end of this paper.

2. Cohomogeneity-one system

Consider the group triple $(\mathsf{K},\mathsf{H},\mathsf{G})$ in (1.1). The isotropy representation $\mathfrak{g}/\mathfrak{k}$ consists of two inequivalent irreducible summands $\mathfrak{p}_1=\mathfrak{h}/\mathfrak{k}$ and $\mathfrak{p}_2=\mathfrak{g}/\mathfrak{h}$. Let the standard sphere metric $g_{\mathbb{S}^{4m+3}}$ on $\mathsf{G}/\mathsf{K}=\mathbb{S}^{4m+3}$ be the background metric. As any G-invariant metric on G/K is determined by its restriction to one tangent space $\mathfrak{g}/\mathfrak{k}$, the metric takes the form

$$f_1^2 g_{\mathbb{S}^{4m+3}}|_{\mathfrak{p}_1}+f_2^2 g_{\mathbb{S}^{4m+3}}|_{\mathfrak{p}_2}.$$

Let f_1 and f_2 be functions that are defined on the one-dimensional orbit space. We consider Einstein equations for the cohomogeneity-one metric

$$g:=dt^2+f_1^2 g_{\mathbb{S}^{4m+3}}|_{\mathfrak{p}_1}+f_2^2 g_{\mathbb{S}^{4m+3}}|_{\mathfrak{p}_2}.$$

By [EW00], the metric g is an Einstein metric on $(t_*-\epsilon,t_*+\epsilon)\times\mathsf{G}/\mathsf{K}$ if (f_1,f_2) is a solution to

$$\begin{aligned}\frac{\ddot{f}_1}{f_1}-\left(\frac{\dot{f}_1}{f_1}\right)^2&=-\left(3\frac{\dot{f}_1}{f_1}+4m\frac{\dot{f}_2}{f_2}\right)\frac{\dot{f}_1}{f_1}+2\frac{1}{f_1^2}+4m\frac{f_1^2}{f_2^4}-\Lambda,\\ \frac{\ddot{f}_2}{f_2}-\left(\frac{\dot{f}_2}{f_2}\right)^2&=-\left(3\frac{\dot{f}_1}{f_1}+4m\frac{\dot{f}_2}{f_2}\right)\frac{\dot{f}_2}{f_2}+(4m+8)\frac{1}{f_2^2}-6\frac{f_1^2}{f_2^4}-\Lambda,\end{aligned}\tag{2.1}$$

with a conservation law

$$3\left(\frac{\dot{f}_1}{f_1}\right)^2+4m\left(\frac{\dot{f}_2}{f_2}\right)^2-\left(d_1\frac{\dot{f}_1}{f_1}+d_2\frac{\dot{f}_2}{f_2}\right)^2+6\frac{1}{f_1^2}+4m(4m+8)\frac{1}{f_2^2}-12m\frac{f_1^2}{f_2^4}-(n-1)\Lambda=0.\tag{2.2}$$

To fix homothety, we set $\Lambda=n$ in this paper. We leave Λ in the equations for readers to trace the Einstein constant.

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Remark 2.1. If we replace the principal orbit G/K by $\mathbb{S}^{4m+3}=[Sp(m+1)U(1)]/[Sp(m)\Delta U(1)]$, then the isotropy representation $\mathfrak{g}/\mathfrak{k}$ consists of three inequivalent irreducible summands. The principal orbit can collapse either as $\mathbb{HP}^m$ or $\mathbb{CP}^{2m+1}$, depending on the choice of intermediate group. For such a principal orbit, the dynamical system of cohomogeneity-one Einstein metrics involves three functions and has (2.1) as its subsystem. A numerical solution in [HYI03] indicates the existence of a positive Einstein metric where G/K collapses to $\mathbb{HP}^m$ at one end and $\mathbb{CP}^{2m+1}$ at the other end.

We consider (2.1) and (2.2) with the following two initial conditions. By [EW00], for the metric g to extend smoothly to the singular orbit $\mathbb{HP}^m$, we have

$$\lim_{t\to 0}(f_1,f_2,\dot f_1,\dot f_2)=(0,f,1,0) \tag{2.3}$$

for some $f>0$. On the other hand, for g to extend smoothly to a point where G/K fully collapses, one considers

$$\lim_{t\to 0}(f_1,f_2,\dot f_1,\dot f_2)=(0,0,1,1). \tag{2.4}$$

By Myers' theorem, any solution obtained from (2.1) that represents an Einstein metric on $\mathbb{HP}^{m+1}\sharp\overline{\mathbb{HP}}^{m+1}$ must be defined on $[0,T]$ for some finite $T>0$. Specifically, one looks for solutions with the initial condition (2.3) and the terminal condition

$$\lim_{t\to T}(f_1,f_2,\dot f_1,\dot f_2)=(0,\bar f,-1,0) \tag{2.5}$$

for some $\bar f>0$. Similarly, to construct an Einstein metric on $\mathbb{S}^{4m+4}$, one looks for solutions with the initial condition (2.4) and the terminal condition

$$\lim_{t\to T}(f_1,f_2,\dot f_1,\dot f_2)=(0,0,-1,-1). \tag{2.6}$$

Remark 2.2. In [Koi81], one takes a non-collapsed principal orbit G/K as the initial data. Specifically, consider

$$(f_1,f_2,\dot f_1,\dot f_2)=(\bar f_1,\bar f_2,\bar h_1,\bar h_2)$$

for some positive $\bar f_i$. To construct a positive Einstein metric, one looks for a solution that extends backward and forward smoothly to either $\mathbb{HP}^m$ or a point on $\mathbb{S}^{4m+4}$ in finite time.

Inspired by the coordinate change in [DW09] and a personal communication from Wei Yuan, we introduce a coordinate change that transforms (2.1) into a polynomial ODE system. Let L be the shape operator of principal orbit. Define

$$X_1:=\frac{\dot f_1/f_1}{\sqrt{(\mathrm{tr}L)^2+n\Lambda}},\quad X_2:=\frac{\dot f_2/f_2}{\sqrt{(\mathrm{tr}L)^2+n\Lambda}},\quad Y:=\frac{1/f_1}{\sqrt{(\mathrm{tr}L)^2+n\Lambda}},\quad Z:=\frac{f_1/f_2^2}{\sqrt{(\mathrm{tr}L)^2+n\Lambda}}.$$

Also, define

$$H:=3X_1+4mX_2,\quad G:=3X_1^2+4mX_2^2,$$
$$R_1:=2Y^2+4mZ^2,\quad R_2:=(4m+8)YZ-6Z^2.$$

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Consider $d\eta = \sqrt{\mathrm{tr}(L)^2 + n\Lambda}dt$. Let $'$ denote taking the derivative with respect to η. Then (2.1) becomes

$$\begin{bmatrix} X_1 \\ X_2 \\ Y \\ Z \end{bmatrix}' = V(X_1, X_2, Y, Z) = \begin{bmatrix} X_1 H\left(G + \frac{1}{n}(1-H^2) - 1\right) + R_1 - \frac{1}{n}(1-H^2) \\ X_2 H\left(G + \frac{1}{n}(1-H^2) - 1\right) + R_2 - \frac{1}{n}(1-H^2) \\ Y\left(H\left(G + \frac{1}{n}(1-H^2)\right) - X_1\right) \\ Z\left(H\left(G + \frac{1}{n}(1-H^2)\right) + X_1 - 2X_2\right) \end{bmatrix}. \tag{2.7}$$

The conservation law (2.2) becomes

$$\mathcal{C}_{\Lambda\geq 0} : G + \frac{1}{n}(1-H^2) + 6Y^2 + 4m(4m+8)YZ - 12mZ^2 = 1. \tag{2.8}$$

Or equivalently,

$$\mathcal{C}_{\Lambda\geq 0} : \frac{12m}{n}(X_1 - X_2)^2 + 6Y^2 + 4m(4m+8)YZ - 12mZ^2 = 1 - \frac{1}{n}. \tag{2.9}$$

We can retrieve the original system by

$$t = \int_{\eta_*}^{\eta} \sqrt{\frac{1-H^2}{n\Lambda}} d\tilde{\eta}, \quad f_1 = \frac{1}{Y}\sqrt{\frac{1-H^2}{n\Lambda}}, \quad f_2 = \frac{1}{\sqrt{YZ}}\sqrt{\frac{1-H^2}{n\Lambda}}. \tag{2.10}$$

It is clear that $H^2 \leq 1$ by the definition of H and the X_i. However, such a piece of information can be obtained from the new system alone without (2.1) and (2.2). Note that

$$\begin{aligned} H' &= \langle \nabla H, V\rangle \\ &= H^2\left(G + \frac{1}{n}(1-H^2) - 1\right) + 6Y^2 + 4m(4m+8)YZ - 12mZ^2 - (1-H^2) \\ &= H^2\left(G + \frac{1}{n}(1-H^2) - 1\right) + 1 - G - \frac{1}{n}(1-H^2) - (1-H^2) \quad \text{by (2.8)} \\ &= (H^2-1)\left(G + \frac{1}{n}(1-H^2)\right) = (H^2-1)\left(\frac{1}{n} + \frac{12m}{n}(X_1 - X_2)^2\right). \end{aligned} \tag{2.11}$$

Therefore, the algebraic surface in $\mathbb{R}^4$ with boundary

$$\mathcal{E} := \mathcal{C}_{\Lambda\geq 0} \cap \{Y, Z \geq 0\} \cap \{H^2 \leq 1\}$$

is invariant. Moreover, $\mathcal{E} \cap \{H = \pm 1\}$ are two invariant sets of lower dimension. The $\mathbb{Z}_2$-symmetry on the sign of (X_1, X_2) gives a one-to-one correspondence between integral curves on $\mathcal{E} \cap \{H = 1\}$ and those on $\mathcal{E} \cap \{H = -1\}$.

Remark 2.3. The restricted system of (2.7) on $\mathcal{E} \cap \{H = 1\}$ is in fact (2.1) with $\Lambda = 0$ under the coordinate change $d\eta = (\mathrm{tr}L)dt$. The dynamical system is essentially the same as the one that appears in [Win17]. An integral curve on the subsystem is known for representing a complete Ricci-flat metric defined on the non-compact manifold $\mathbb{HP}^{m+1}\backslash\{*\}$ [Böh98]. The Ricci-flat metric on $\mathbb{HP}^{m+1}\backslash\{*\}$ is the limit cone for locally defined positive Einstein metrics on the tubular neighborhood around $\mathbb{HP}^m$.

Remark 2.4. If an integral curve to (2.7) enters $\mathcal{E}\cap\{H<1\}$ and is defined on $\mathbb{R}$, then from (2.11) it must cross $\mathcal{E}\cap\{H=0\}$ transversally. The crossing point corresponds to the *turning point* in [Böh98]. For any integral curve to (2.7) that has a turning point, we choose the η_* in (2.10) so that $t_* := \int_{\eta_*}^{0}\sqrt{(1-H^2)/n\Lambda}\,d\tilde{\eta}$ is the value at which $\mathrm{tr}L$ vanishes. By our choice of η_*, the integral curve crosses $\mathcal{E}\cap\{H=0\}$ at $\eta=0$. There are cohomogeneity-one Einstein systems with additional geometric structure, such as the one considered in [FH17], where every trajectory has a turning point.

Remark 2.5. From (2.9), the inequality $6Y^2+4m(4m+8)YZ-12mZ^2\leq 1-1/n$ is always valid. Therefore, the set $\mathcal{E}\cap\{Z-\rho Y\leq 0\}$ is compact for any fixed $\rho\in[0,(m(4m+8)+\sqrt{m^2(4m+8)^2+18m})/6m)$. If the maximal interval of existence of an integral curve to (2.7) is $(-\infty,\bar{\eta})$ for some $\bar{\eta}\in\mathbb{R}$, it must escape $\mathcal{E}\cap\{Z-\rho Y\leq 0\}$. The crossing point corresponds to the *W-intersection point* in [Böh98]. In Proposition 3.3 and Definition 4.9, we introduce an invariant set $\mathcal{W}$ and a modified definition for the W-intersection point, which fixes $\rho=1$ in the original definition in [Böh98].

3. Linearization at critical points

The local existence of positive Einstein metrics around the singular orbit $\mathbb{HP}^m$ is well established in [Böh98]. We interpret the result using the new coordinate. For $m\geq 2$, the vector field V has in total 10 critical points (12 critical points for $m=1$) on $\mathcal{E}$. As indicated by their superscripts, these critical points lie on either $\mathcal{E}\cap\{H=1\}$ or $\mathcal{E}\cap\{H=-1\}$.

- $p_0^{\pm}=\left(\pm\frac{1}{3},0,\frac{1}{3},0\right)$
  These points represent the initial condition (2.3) and the terminal condition (2.5). Integral curves that emanate from p_0^+ and enter $\mathcal{E}\cap\{H<1\}$ represent positive Einstein metrics defined on a tubular neighborhood around $\mathbb{HP}^m$. A complete Einstein metric on $\mathbb{HP}^{m+1}\sharp\overline{\mathbb{HP}}^{m+1}$ is represented by a heterocline that joins $p_0^{\pm}$.
- $p_1^{\pm}=(\pm(1/n),\pm(1/n),1/n,1/n)$
  These points represent the initial condition (2.4) and the terminal condition (2.6). Integral curves that emanate from p_0^+ and enter $\mathcal{E}\cap\{H<1\}$ represent positive Einstein metrics defined on a tubular neighborhood around a point on $\mathbb{S}^{4m+4}$. The standard sphere metric is represented by a straight line that joins $p_1^{\pm}$. It is also worth mentioning that the quaternionic Kähler metric on $\mathbb{HP}^{m+1}$ (respectively, $\overline{\mathbb{HP}}^{m+1}$) is represented by an integral curve that joins p_0^+ and p_1^- (respectively, p_0^- and p_1^+).
- $p_2^{\pm}=(\pm(1/n),\pm(1/n),(2m+3)z_0,z_0)$, $z_0=(1/n)\sqrt{(2m+1)/(2m+(2m+3)^2)}$
  These points represent the initial condition and the terminal condition where the principal orbit collapses as Jensen's sphere [Jen73]. There is only one integral curve that emanates from p_2^+ and it represents the singular sine metric cone with its base as Jensen's sphere [Jen73]. It is also worth mentioning that p_2^+ is a sink for the Ricci-flat subsystem of (2.7) restricted on $\mathcal{E}\cap\{H=1\}$, representing the asymptotically conical limit.
- $q_1^{\pm}=(\pm((3+2\sqrt{12m^2+6m})/3n),\pm((4m-2\sqrt{12m^2+6m})/4mn),0,0)$,
  $q_2^{\pm}=(\pm((3-2\sqrt{12m^2+6m})/3n),\pm((4m+2\sqrt{12m^2+6m})/4mn),0,0)$
  These critical points are in general 'bad' points for our study. Integral curves that converge to q_1^- or q_2^- represent metrics with blown-up $\dot{f}_1$ and $\dot{f}_2$. Straightforward computations show that for $m\geq 2$, critical points q_1^+ and q_2^+ are sources for (2.7) on (2.8). By the $\mathbb{Z}_2$ symmetry, critical points q_1^- and q_2^- are sinks for $m\geq 2$.
- $q_3^{\pm}=(\mp\frac{1}{3},\pm(2/4m),0,\sqrt{3-2m}/6m)$

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These critical points are also 'bad' points as $q_1^\pm$ and $q_2^\pm$. They only exist for $m = 1$. Integral curves that converge to q_3^- represent metrics with blown-up $\dot{f}_1$. For $m = 1$, critical points q_1^+ and q_3^+ are sources and q_1^- and q_3^- are sinks; q_2^+ and q_2^- are saddles.

Proposition 3.1. *The list above exhausts all critical points on $\mathcal{E}$.*

Proof. By (2.11), it is clear that critical points on $\mathcal{E}$ must lie on $\{H^2 = 1\}$. The list is complete by considering the vanishing of the Y-entry and Z-entry. □

For any m, linearizations at $q_i^\pm$ show that the phase space $\mathcal{E}$ is 'filled' with integral curves that emanate from q_i^+ or those that converge to q_i^-. Hence, most integral curves that emanate from p_0^+ or p_1^+ are anticipated to converge to one of these q_i^-. In the following, we give a detailed analysis of the linearizations at p_0^+ and p_1^+ and integral curves that emanate from these critical points.

The linearization at p_0^+ is

$$\begin{bmatrix} -\frac{8m-6}{3n} & -\frac{32m^2-24m}{9n} & \frac{4}{3} & 0 \\ \frac{6}{n} & \frac{16m-6}{3n} & 0 & \frac{4m+8}{3} \\ \frac{8m}{3n} & \frac{16m^2-12m}{9n} & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{3} \end{bmatrix}.$$

Eigenvalues and eigenvectors are

$$\lambda_1 = \lambda_2 = \frac{2}{3}, \quad \lambda_3 = -\frac{2}{3}, \quad \lambda_4 = \frac{8m}{3n};$$

$$v_1 = \begin{bmatrix} -(8m^2+18m+18) \\ -9 \\ -(8m^2+18m) \\ 9 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -4m(m+2) \\ 3(m+2) \\ -2m(m+2) \\ 3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -4m \\ 3 \\ 2m \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} -2(4m-3) \\ 18 \\ 4m-3 \\ 0 \end{bmatrix}.$$

The first three eigenvectors are tangent to $\mathcal{E}$. Consider linearized solutions of the form

$$p_0^+ + e^{(2/3)\eta} v_1 + s_1 e^{(2/3)\eta} v_2 \tag{3.1}$$

for some $s_1 \in \mathbb{R}$. By the Hartman–Grobman theorem, there is a one-to-one correspondence between each choice of $s_1 \in \mathbb{R}$ and an actual solution curve that emanates from p_0^+ and leaves $\mathcal{E} \cap \{H = 1\}$ initially. Hence, we use γ_{s_1} to denote an actual solution that approaches the linearized solution (3.1) near p_0^+. Moreover, by the unstable version of Theorem 4.5 in Chapter 13 of [CL55], there is some $\epsilon > 0$ such that

$$\gamma_{s_1} = p_0^+ + e^{(2/3)\eta} v_1 + s_1 e^{(2/3)\eta} v_2 + O(e^{(2/3+\epsilon)\eta}).$$

From the linearization at p_0^+ and (2.10), the parameter s_1 is related to the initial condition f in (2.3) as follows.

$$f = \lim_{\eta \to -\infty} \left(\frac{1}{\sqrt{YZ}} \frac{1-H^2}{n\Lambda} \right)(\gamma_{s_1}) = \sqrt{\frac{6m+18}{n} \frac{1}{3+s_1}}. \tag{3.2}$$

We set $s_1 > -3$ so that f is positive. From another perspective, in order to have γ_{s_1} be in $\mathcal{E}$, we only consider γ_{s_1} with $s_1 > -3$ so that Z is positive initially along the integral curve.

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Note that v_2 is tangent to $\mathcal{E} \cap \{H = 1\}$. Therefore, it makes sense to let γ_∞ denote the integral curve that lies in $\mathcal{E} \cap \{H = 1\}$ such that

$$\gamma_\infty \sim p_0^+ + 0 \cdot e^{(2/3)\eta} v_2 + 1 \cdot e^{(2/3)\eta} v_2 \tag{3.3}$$

near p_0^+. The integral curve γ_∞ represents the Ricci-flat metric on $\mathbb{HP}^{m+1} \backslash \{*\}$ constructed in [Böh98]. For $m = 1$, the metric is the Spin(7) metric in [BS89] and [GPP90]. Furthermore, as shown in Proposition 6.3 in [Chi21], the integral curve γ_∞ lies on the one-dimensional invariant set

$$\mathcal{B}_{\mathrm{Spin}(7)} := \mathcal{E} \cap \{Y - 2Z - X_1 = 0\} \cap \{3Z - X_2 = 0\} \tag{3.4}$$

and joins p_0^+ and p_2^+.

Remark 3.2. The defining equations in (3.4) are equivalent to the cohomogeneity-one Spin(7) condition on $\mathbb{HP}^2 \backslash \{*\}$. Specifically, we have the dynamical system

$$\begin{aligned} \frac{\dot{f}_1}{f_1} &= \frac{1}{f_1} - 2\frac{f_1}{f_2^2}, \\ \frac{\dot{f}_2}{f_2} &= 3\frac{f_1}{f_2^2}. \end{aligned} \tag{3.5}$$

Similar to the initial value problem in Remark 3.6, the initial condition can be obtained from the coordinate of p_0^+ and the limit $\lim_{\eta \to -\infty}(X_2/\sqrt{YZ})(\gamma_\infty(\eta))$. Since a Ricci-flat metric is homothety invariant, the extra freedom allows us to set $f_2(0)$ as any positive number. Solving the initial value problem with

$$(f_1(0), \dot{f}_1(0), \dot{f}_2(0), \dot{f}_2(0))) = (0, 1, \beta, 0), \quad \beta > 0,$$

yields the homothetic family of Spin(7) metrics in [BS89] and [GPP90].

The linearization at p_1^+ is

$$\begin{bmatrix} -\dfrac{16m^2+8m-6}{n^2} & \dfrac{16m(m+1)}{n^2} & \dfrac{4}{n} & \dfrac{8m}{n} \\ \dfrac{12m+12}{n^2} & -\dfrac{4m+6}{n^2} & \dfrac{4m+8}{n} & \dfrac{4m-4}{n} \\ -\dfrac{4m}{n^2} & \dfrac{4m}{n^2} & 0 & 0 \\ \dfrac{4m+6}{n^2} & -\dfrac{4m+6}{n^2} & 0 & 0 \end{bmatrix}.$$

Eigenvalues and eigenvectors are

$$\mu_1 = \mu_2 = \frac{2}{n}, \quad \mu_3 = 0, \quad \mu_4 = -\frac{4(m+1)}{n};$$

$$w_1 = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad w_2 = \begin{bmatrix} -4m \\ 3 \\ 2m \\ -(2m+3) \end{bmatrix}, \quad w_3 = \begin{bmatrix} -(n-1) \\ -(n-1) \\ 1 \\ 1 \end{bmatrix}, \quad w_4 = \begin{bmatrix} -8m(m+1) \\ 6(m+1) \\ -2m \\ 2m+3 \end{bmatrix}.$$

The first three eigenvectors are tangent to $\mathcal{E}$. Hence, there exists a one-parameter family of integral curves ζ_{s_2} that emanate from p_1^+ and

$$\zeta_{s_2} = p_1^+ + e^{(2/n)\eta} w_1 + s_2 e^{(2/n)\eta} w_2 + O(e^{(2/3+\epsilon)\eta}). \tag{3.6}$$

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The initial condition (2.4) has a degree of freedom in the second-order derivative. Specifically, the parameter s_2 is related to the limit $\lim_{t\to 0}(\ddot{f}_2/f_2)(f_1/\ddot{f}_1)$. From (2.1), (2.10) and the linearization at p_1^+, we have

$$\lim_{t\to 0}\frac{\ddot{f}_2}{f_2}\frac{f_1}{\ddot{f}_1}=\lim_{\eta\to-\infty}\frac{X_2^2-HX_2+R_2-\frac{1-H^2}{n}}{X_1^2-HX_1+R_1-\frac{1-H^2}{n}}=\frac{n}{4m(1+6ms_2)}-\frac{9}{12m}. \tag{3.7}$$

Although it is clear that ζ_{s_2} is in $\mathcal{E}$ for any $s_2\in\mathbb{R}$, we mainly consider $s_2\geq 0$ in this paper. We have the following proposition for ζ_{s_2} with $s_2<0$.

PROPOSITION 3.3. *Each ζ_{s_2} with $s_2<0$ either does not converge to any critical point in $\mathcal{E}$ or converges to q_2^- (q_2^- or q_3^- if $m=1$).*

Proof. From the linearized solution, it is clear that each ζ_{s_2} with $s_2<0$ is initially in

$$\mathcal{W}:=\mathcal{E}\cap\{Z-Y\geq 0\}\cap\{X_1-X_2\geq 0\}. \tag{3.8}$$

As $\mathcal{W}$ includes all points $(\sqrt{(n-1)/12m}\cosh(\lambda),0,0,\sqrt{(n-1)/12nm}\sinh(\lambda))$ with $\lambda\geq 0$, the set is non-compact. Furthermore, since

$$\left\langle\nabla\left(\frac{Y}{Z}\right),V\right\rangle=-2\frac{Y}{Z}(X_1-X_2)\leq 0 \tag{3.9}$$

and

$$\langle\nabla(X_1-X_2),V\rangle=(X_1-X_2)H\left(G+\frac{1}{n}(1-H^2)-1\right)+2(Z-Y)((2m+3)Z-Y), \tag{3.10}$$

the set $\mathcal{W}$ is invariant. Note that the second term in (3.10) is non-negative in $\mathcal{W}\cap\{X_1-X_2=0\}$.

By (3.9), it is clear that the function Y/Z monotonically decreases from 1 along each ζ_{s_2} with $s_2<0$. If the function Y/Z converges to some positive number, the function X_1-X_2 would converge to zero. From (2.9), we know that both Y and Z converge to some positive numbers with $Y/Z<1$. Hence, the right-hand side of (3.10) is eventually positive as X_1-X_2 converges to zero, a contradiction. Hence, the function Y/Z converges to zero. Therefore, if a ζ_{s_2} with $s_2<0$ converges to a critical point in $\mathcal{E}$, it must be q_2^- (q_2^- or q_3^- if $m=1$). □

Remark 3.4. For $s_2=0$, it is clear that ζ_0 lies on the one-dimensional invariant set

$$\mathcal{E}\cap\{X_1=X_2\}\cap\left\{Y=Z=\frac{1}{n}\right\} \tag{3.11}$$

and joins $p_1^\pm$. The integral curve represents the standard sphere metric $g_{\mathbb{S}^{4m+4}}$ on $\mathbb{S}^{4m+4}$. Specifically, the defining equations in (3.11) give the initial value problem

$$\begin{gathered}\frac{\dot{f}_1}{f_1}=\frac{\dot{f}_2}{f_2},\quad f_1^2+(\dot{f}_1)^2=1,\\ f_1(0)=f_2(0)=0,\quad \dot{f}_1(0)=\dot{f}_2(0)=1,\end{gathered} \tag{3.12}$$

in the original coordinates. The solution is exactly the standard sphere metric

$$g_{\mathbb{S}^{4m+4}}=dt^2+\sin^2(t)g_{\mathbb{S}^{4m+3}}.$$

We define ζ_∞ to be the integral curve that emanates from p_1^+ and lies in $\mathcal{C}_{\Lambda\geq 0}\cap\{H=1\}$. We have

$$\zeta_\infty\sim p_1^++e^{(2/n)\eta}w_2. \tag{3.13}$$

As studied in [Chi21], the integral curve ζ_∞ is known to be defined on $\mathbb{R}$; it joins p_1^+ and p_2^+ and represents a complete non-trivial Ricci-flat metric defined on $\mathbb{R}^{4m+4}$.

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As shown in the following proposition, there exists an integral curve that joins p_0^+ and p_1^-, and it represents the standard quaternionic Kähler metric on $\mathbb{HP}^{m+1}$. By the $\mathbb{Z}_2$-symmetry of (2.7) on the sign of (X_1, X_2), we know that there also exists an integral curve that emanates from p_1^+ and tends to p_0^-, and it represents the standard quaternionic Kähler metric on $\overline{\mathbb{HP}}^{m+1}$.

Proposition 3.5. *The integral curve γ_0 lies on the one-dimensional invariant set*

$$\mathcal{B}_{QK} := \mathcal{E} \cap \{X_1 - X_2 + Z - Y = 0\} \cap \{X_2 + Z = 0\}.$$

The integral curve $\zeta_{1/(2m+6)}$ lies on the one-dimensional invariant set

$$\bar{\mathcal{B}}_{QK} := \mathcal{E} \cap \{X_2 - X_1 + Z - Y = 0\} \cap \{X_2 - Z = 0\}.$$

Proof. As $X_1 - X_2 = Y - Z$ and $X_2 = -Z$ on $\mathcal{B}_{QK}$, we can eliminate X_1 and X_2 in (2.9). Hence,

$$\frac{12m}{n}(Y-Z)^2 + 6Y^2 + 4m(4m+8)YZ - 12mZ^2 = 1 - \frac{1}{n} \tag{3.14}$$

holds on $\mathcal{B}_{QK}$. Therefore,

$$\begin{aligned}
&\langle \nabla(X_2+Z), V\rangle|_{\mathcal{B}_{QK}}\\
&\quad= (X_2+Z)H\left(G + \frac{1}{n}(1-H^2)\right) - \frac{1}{n}(1-H^2) - X_2H + (4m+8)YZ - 6Z^2 + Z(X_1 - 2X_2)\\
&\quad= -\frac{1}{n}(1-H^2) - X_2H + (4m+8)YZ - 6Z^2 + Z(X_1-2X_2)\\
&\quad= \frac{1}{n-1}\left(\frac{12m}{n}(Z-Y)^2 + 6Y^2 + 4m(4m+8)YZ - 12mZ^2 - \frac{n-1}{n}\right)\\
&\qquad \text{eliminating } X_1 \text{ and } X_2 \text{ by the definition of } \mathcal{B}_{QK}\\
&\quad= 0 \quad \text{by (3.14)}.
\end{aligned} \tag{3.15}$$

On the other hand, we have

$$\begin{aligned}
&\langle \nabla(X_1 - X_2 + Z - Y), V\rangle|_{\mathcal{B}_{QK}}\\
&\quad= (X_1 - X_2 + Z - Y)\left(H\left(G + \frac{1}{n}(1-H^2) - 1\right) + 4Z - 2Y\right) + (n-1)(X_2+Z)(Z-Y)\\
&\quad= 0.
\end{aligned} \tag{3.16}$$

Therefore, the set $\mathcal{B}_{QK}$ is indeed invariant.

Since $X_1 = Y - 2Z$ and $X_2 = -Z$ on $\mathcal{B}_{QK}$, one can realize $\mathcal{B}_{QK}$ as a hyperbola (3.14). Note that p_0^+ and p_1^- are the only critical points in $\mathcal{B}_{QK}$ and they are in the same connected component in (3.14). Therefore, there is an integral curve that joins p_0^+ and p_1^- and lies on $\mathcal{B}_{QK}$. Hence, the integral curve must be some γ_{s_1}. Let $v(\eta)$ be the normalized velocity of the linearized solution that uniquely corresponds to γ_0. It is clear that $\lim_{\eta\to-\infty} v(\eta) = v_1/\|v_1\|$ is tangent to $\mathcal{B}_{QK}$ at p_0^+. Hence, we know that γ_0 lies on $\mathcal{B}_{QK}$. By the $\mathbb{Z}_2$-symmetry on the sign of (X_1, X_2), we know that $\zeta_{1/(2m+6)}$ lies on the invariant set

$$\bar{\mathcal{B}}_{QK} := \mathcal{C}_{\Lambda\geq 0} \cap \{X_2 - X_1 + Z - Y = 0\} \cap \{X_2 - Z = 0\}$$

and joins p_1^+ and p_0^-. □

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Remark 3.6. The defining equations for $\mathcal{B}_{QK}$ are equivalent to the dynamical system

$$\begin{aligned}\frac{\dot f_1}{f_1} &= -2\frac{f_1}{f_2^2} + \frac{1}{f_1},\\ \frac{\dot f_2}{f_2} &= -\frac{f_1}{f_2^2}.\end{aligned} \tag{3.17}$$

While $f_1(0) = 0$ and $\dot f_1(0) = 1$ can be obtained from the coordinate of p_0^+, the initial conditions $f_2(0)$ and $\dot f_2(0)$ are obtained from v_1. Specifically, from (3.1) we have

$$f_2(0) = \lim_{\eta\to-\infty}\left(\frac{\sqrt{1-H^2}}{n\sqrt{YZ}}\right)(\gamma_0(\eta)) = \sqrt{\frac{4(m+3)}{n}}, \quad \dot f_2(0) = \lim_{\eta\to-\infty}\left(\frac{X_2}{\sqrt{YZ}}\right)(\gamma_0(\eta)) = 0.$$

Solving the initial value problem, the standard quaternionic Kähler metric on $\mathbb{HP}^{m+1}$ is

$$g = dt^2 + \frac{m+3}{n}\sin^2\left(2\sqrt{\frac{n}{4(m+3)}}t\right)g_{\mathbb{S}^{4m+3}}|_{\mathfrak{p}_1} + \frac{4(m+3)}{n}\cos^2\left(\sqrt{\frac{n}{4(m+3)}}t\right)g_{\mathbb{S}^{4m+3}}|_{\mathfrak{p}_2}.$$

Finally, we consider the linearization at p_2^+. We have

$$\begin{bmatrix} -\frac{16m^2+8m-6}{n^2} & \frac{16m(m+1)}{n^2} & (8m+12)z_0 & 8mz_0\\ \frac{12m+12}{n^2} & -\frac{4m+6}{n^2} & 4(m+2)z_0 & 4(2m+1)(m+3)z_0\\ -\frac{4m(2m+3)z_0}{n} & \frac{4m(2m+3)z_0}{n} & 0 & 0\\ \frac{(4m+6)z_0}{n} & -\frac{(4m+6)z_0}{n} & 0 & 0\end{bmatrix}.$$

Eigenvalues and eigenvectors are

$$\begin{aligned}\delta_1 &= -\frac{2m+1-\sqrt{(2m+1)^2-8(2m+3)(m+1)n^2z_0^2}}{n},\\ \delta_2 &= -\frac{2m+1+\sqrt{(2m+1)^2-8(2m+3)(m+1)n^2z_0^2}}{n},\\ \delta_3 &= \frac{2}{n}, \quad \delta_4 = 0;\end{aligned}$$

$$u_1 = \begin{bmatrix}\frac{2mn}{2m+3}\delta_1\\ -\frac{3n}{2(2m+3)}\delta_1\\ -2mnz_0\\ nz_0\end{bmatrix}, \quad u_2 = \begin{bmatrix}\frac{2mn}{2m+3}\delta_2\\ -\frac{3n}{2(2m+3)}\delta_2\\ -2mnz_0\\ nz_0\end{bmatrix},$$

$$u_3 = \begin{bmatrix}-1\\-1\\0\\0\end{bmatrix}, \quad u_4 = \begin{bmatrix}-2n((2m+3)^2+2m)z_0\\ -2n((2m+3)^2+2m)z_0\\ 2m+3\\ 1\end{bmatrix}.$$

The first three eigenvectors are tangent to $\mathcal{E}$. Furthermore, the first two eigenvectors are tangent to $\mathcal{E}\cap\{H=1\}$ and $\delta_2 < \delta_1 < 0$. For $m \geq 1$, the critical point p_2^+ is a stable node for the restricted

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system on $\mathcal{E}\cap\{H=1\}$. Let Φ be the only integral curve that emanates from p_2^+. It converges to p_2^- and lies on the one-dimensional invariant set

$$\mathcal{E}\cap\{X_1=X_2\}\cap\{Y=(2m+3)Z=(2m+3)z_0\}. \tag{3.18}$$

Remark 3.7. The initial value problem from the defining equations (3.18) is similar to the one from (3.11). Specifically, we have

$$\begin{aligned} f_1^2=\frac{1}{2m+3}f_2^2,\quad \frac{(2m+1)(2m+3)^2}{2m+(2m+3)^2}(f_1^2+(\dot f_1)^2)=1,\\ f_1(0)=f_2(0)=0,\quad \dot f_1(0)=\frac{1}{\sqrt{2m+3}}\dot f_2(0)=\frac{1}{2m+3}\sqrt{\frac{2m+(2m+3)^2}{2m+1}}. \end{aligned} \tag{3.19}$$

Hence, Φ represents the sine cone over Jensen's sphere

$$\begin{aligned} g&=dt^2+\frac{2m+(2m+3)^2}{(2m+1)(2m+3)^2}\sin^2(t)g_{\mathbb{S}^{4m+3}}|_{\mathfrak{p}_1}+\frac{2m+(2m+3)^2}{(2m+1)(2m+3)}\sin^2(t)g_{\mathbb{S}^{4m+3}}|_{\mathfrak{p}_2}\\ &=dt^2+\sin^2(t)g_{\text{Jensen}}. \end{aligned}$$

4. Existence of the first Einstein metric

We prove the existence of a heterocline that joins $p_0^\pm$ in this section. The technique is to construct a compact set $\mathcal{S}$ such that a γ_{s_1} that enters the set can only escape through points in $\mathcal{E}\cap\{H\geq 0\}\cap\{X_1-X_2=0\}$. Then we apply Lemma 4.4 in [Böh98] to complete the proof.

We define the compact set $\mathcal{S}$ as follows. Define polynomials

$$\begin{aligned} A&:=YX_2-\frac{3}{m}Z\left(X_1+\frac{2m}{3}X_2\right),\\ B&:=\frac{1-H^2}{n}-\frac{2n^2(2m+3)(m-1)}{m(2m+1)(8m+3)}YZ,\\ P&:=X_1\left(R_2-\frac{1}{n}(1-H^2)\right)-X_2\left(R_1-\frac{1}{n}(1-H^2)\right)-2X_2\left(X_1+\frac{2m}{3}X_2\right)(X_1-X_2),\\ Q&:=-4X_2Y^2-(4m+8)(4mX_2+2X_1)YZ+(2X_1+(4m+2)X_2)\frac{1-H^2}{n}\\ &\quad+4X_2\left(X_1+\frac{2m}{3}X_2\right)\left(H+\frac{2m}{3}X_2\right). \end{aligned} \tag{4.1}$$

Define

$$\mathcal{S}:=\mathcal{E}\cap\{X_1-X_2\geq 0\}\cap\{X_2\geq 0\}\cap\{A\geq 0\}\cap\{B\geq 0\}\cap\{P\geq 0\}.$$

The following proposition lists some basic properties of $\mathcal{S}$.

Proposition 4.1. *The set $\mathcal{S}$ has the following properties.*

(i) *For $m\geq 1$, the set $\mathcal{S}\cap\{X_2=0\}$ is a union of $\{p_0^+\}$ and a one-dimensional curve $\Gamma:=\mathcal{S}\cap\{X_1=X_2=0\}$. For $m\geq 2$, the set Γ is bounded.*
(ii) *The variable Y is positive in $\mathcal{S}$ for $m\geq 1$.*
(iii) *For $m\geq 1$, the set $\mathcal{S}\cap\{Z=0\}$ is $\{p_0^+,(0,0,\sqrt{(n-1)/6n},0)\}$.*
(iv) *The set $\mathcal{S}$ is compact for $m\geq 2$.*

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Proof. Since $A \geq 0$ in $\mathcal{S}$, a point in $\mathcal{S}$ with vanishing X_2-coordinate must have $ZX_1 = 0$. If we further assume $Z = 0$ at that point, then from $P \geq 0$ we have $X_1(1 - H^2) = X_1(1 - 9X_1^2) \leq 0$. Hence, we obtain $X_1 = \frac{1}{3}$ and that point must be p_0^+. We obtain Γ if $X_1 = 0$ is further assumed.

It is obvious that $\Gamma = \mathcal{E} \cap \{X_1 = X_2 = 0\}$ for $m = 1$ and the set is non-compact. We prove that Γ is bounded for $m \geq 2$. From $B \geq 0$ and (2.9), we have

$$\begin{aligned} 6Y^2 + 4m(4m+8)YZ - 12mZ^2 = \frac{n-1}{n} \geq (n-1)\frac{2n^2(2m+3)(m-1)}{m(2m+1)(8m+3)}YZ \\ \Leftrightarrow 6Y^2 + \left(4m(4m+8) - (n-1)\frac{2n^2(2m+3)(m-1)}{m(2m+1)(8m+3)}\right)YZ - 12mZ^2 \geq 0. \end{aligned} \tag{4.2}$$

For $m \geq 2$, the inequality above implies $mY - Z \geq 0$. Then from (2.9) we have $(n-1)/n \geq 6Y^2 + (4m+32)Z^2$. The first claim is clear.

Suppose there is a point in $\mathcal{S}$ with vanishing Y-coordinate. From $A \geq 0$ we know that $-(3/m)Z(X_1 + (2m/3)X_2) = 0$ at that point. If $Z \neq 0$, then $X_1 = X_2 = Y = 0$ at that point, which is impossible from (2.9). If $X_1 + (2m/3)X_2 \neq 0$, then $Y = Z = 0$ at that point. Then from (2.9), we have $X_1 - X_2 \neq 0$. From $P \geq 0$ we know that $-(1/n)(1 - H^2) - 2X_2(X_1 + (2m/3)X_2) \geq 0$ at that point. The point has to be $(\frac{1}{3}, 0, 0, 0)$, which does not lie on $\mathcal{E}$. The above discussion proves the second claim.

Since $P \geq 0$ on a point with vanishing Z-coordinate in $\mathcal{S}$, we have

$$(X_2 - X_1)\frac{1}{n}(1 - H^2) - 2X_2Y^2 - 2X_2\left(X_1 + \frac{2m}{3}X_2\right)(X_1 - X_2) \geq 0.$$

By the definition of $\mathcal{S}$, each term in the above inequality is non-positive. Since $Y > 0$ from the second claim, the variable X_2 must vanish and the third claim is clear.

Finally, from $A \geq 0$ and $X_1 - X_2 \geq 0$ in $\mathcal{S}$, we know that $X_2(Y - ((2m+3)/m)Z) \geq 0$ in $\mathcal{S}$. If $X_2 \neq 0$, then $mY \geq (2m+3)Z$ and the boundedness of all variables is obtained from (2.9). If $X_2 = 0$, then the boundedness comes from the first claim. Hence, $\mathcal{S}$ is a compact set. $\square$

The case $m = 1$ is very special. In the following proposition, we show that for $m = 1$, the defining inequalities $A \geq 0$ and $P \geq 0$ must be equalities. The set $\mathcal{S}$ is closely related to the integral curves γ_∞ in Remark 3.2 and Φ in Remark 3.7.

Proposition 4.2. *For $m = 1$, the set $\mathcal{S}$ is the union*

$$\{p_0^+, p_2^+\} \cup \gamma_\infty \cup \Gamma \cup (\Phi \cap \{X_1, X_2 > 0\}).$$

Proof. Consider $\mathcal{S}$ with $m = 1$. If $X_2 = 0$ is imposed, we obtain either the point p_0^+ or Γ from Proposition 4.1(a). Hence, we assume $X_2 > 0$ in the following. From $A \geq 0$ it is clear that $X_2(Y - 5Z) \geq YX_2 - 3Z(X_1 + \frac{2}{3}X_2) \geq 0$ and hence $Y - 5Z \geq 0$.

One can easily verify that

$$P = (X_1 - X_2)\big(6YZ - 2X_2\big(X_1 + \tfrac{2}{3}X_2\big)\big) - \tfrac{1}{7}(X_1 - X_2)(1 - H^2) - 2(Y - Z)A. \tag{4.3}$$

If $X_1 = X_2 = X > 0$, the first two terms in (4.3) vanish while the last term is non-positive. Then we must have $A = 0$, from which we deduce $Y - 5Z = 0$. By (3.18) we obtain the line segment $\Phi \cap \{X_1, X_2 > 0\}$.

For $X_1 \neq X_2$, we rewrite (4.3) as follows:

$$P = (X_1 - X_2)\big(6YZ - 2X_2\big(X_1 + \tfrac{2}{3}X_2\big) - \tfrac{10}{147}(1 - H^2)\big) - \tfrac{11}{147}(X_1 - X_2)(1 - H^2) - 2(Y - Z)A. \tag{4.4}$$

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The last two terms in (4.4) are non-positive. Hence, from $P \geq 0$ we have

$$\begin{aligned} 0 &\leq 6YZ - 2X_2\big(X_1 + \tfrac{2}{3}X_2\big) - \tfrac{10}{147}(1 - H^2) \\ &= 6YZ - 2X_2\big(X_1 + \tfrac{2}{3}X_2\big) - \tfrac{10}{147}\big(\tfrac{7}{6}\big(6Y^2 + 48YZ - 12Z^2 + \tfrac{12}{7}(X_1 - X_2)^2\big) - H^2\big) \\ &\quad \text{by (2.9)} \\ &= \tfrac{1}{21}(5X_1 + 4X_2 + 5Y + 2Z)(X_1 - X_2 - Y + 5Z) \\ &\quad + \tfrac{1}{21}(5X_1 + 4X_2 - 5Y - 2Z)(X_1 - X_2 + Y - 5Z). \end{aligned} \tag{4.5}$$

Suppose $X_1 - X_2 - Y + 5Z \leq 0$; then the first term in the last line of (4.5) is non-positive. As the summation above is non-negative, we know that the second term in the last line of (4.5) must be non-negative. As $X_1 - X_2 + Y - 5Z \geq 0$, we must have $\frac{1}{5}(5X_1 + 4X_2 - 2Z) \geq Y$. From $A \geq 0$ we have

$$\frac{X_2}{5}(5X_1 + 4X_2 - 2Z) \geq 3Z\left(X_1 + \frac{2}{3}X_2\right) \Leftrightarrow (5X_1 + 4X_2)(X_2 - 3Z) \geq 0 \Leftrightarrow X_2 - 3Z \geq 0.$$

Then we claim that the first term in (4.3) is non-positive since

$$\begin{aligned} 6YZ - 2X_2\big(X_1 + \tfrac{2}{3}X_2\big) &\leq \tfrac{6}{5}(5X_1 + 4X_2 - 2Z)Z - 2X_2\big(X_1 + \tfrac{2}{3}X_2\big) \\ &\leq \tfrac{2}{5}\big(5X_1 + 4X_2 - \tfrac{2}{3}X_2\big)X_2 - 2X_2\big(X_1 + \tfrac{2}{3}X_2\big) \\ &= 0. \end{aligned} \tag{4.6}$$

But $P \geq 0$. Hence, assumptions $X_1 \neq X_2$ and $X_1 - X_2 - Y + 5Z \leq 0$ lead to the vanishing of A and P.

Suppose $X_1 - X_2 - Y + 5Z \geq 0$. Then $A \geq 0$ implies

$$(X_1 - X_2 + 5Z)X_2 \geq 3Z\big(X_1 + \tfrac{2}{3}X_2\big) \Leftrightarrow X_2 \geq 3Z.$$

Then we claim that the first term in (4.3) is also non-positive since

$$\begin{aligned} 6YZ - 2X_2\big(X_1 + \tfrac{2}{3}X_2\big) &\leq 6(X_1 - X_2 + 5Z)Z - 2X_2\big(X_1 + \tfrac{2}{3}X_2\big) \\ &\leq 2\big(X_1 - X_2 + \tfrac{5}{3}X_2\big)X_2 - 2X_2\big(X_1 + \tfrac{2}{3}X_2\big) \\ &= 0. \end{aligned} \tag{4.7}$$

Hence, the assumption $X_1 \neq X_2$ and $X_1 - X_2 - Y + 5Z \geq 0$ also leads to the vanishing of A and P.

Therefore, points in $\mathcal{S}$ with $X_1 \neq X_2$ must have vanished A and P, which leads to the equalities

$$H = 1, \quad 3Z = X_2, \quad 3Y = 3X_1 + 2X_2.$$

Note that the last two equations above are equivalent to the defining equations in (3.4) for $m = 1$. We obtain γ_∞ and critical points p_0^+ and p_2^+. □

Remark 4.3. For $m = 1$, there is another heterocline χ that also lies on the invariant set $\mathcal{B}_{\mathrm{Spin}(7)}$. The integral curve joins q_3^+ and p_2^+. Note that from the proof to Proposition 4.2, it is clear that $X_1 - X_2$ and $Y - 5Z$ are positive along γ_∞. These two polynomials are negative along χ and hence the integral curve is not in $\mathcal{S}$.

With Proposition 4.2 established, we can take $m = 1$ as our 'initial case' for further analysis of cases with $m > 1$. In particular, we prove the following technical proposition.

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Proposition 4.4. *For $m \geq 1$, the inequality $Q \geq 0$ holds on the set $\mathcal{S} \cap \{P = 0\}$. For $m = 1$, the inequality reaches equality only at $\{p_0^+, p_2^+\} \cup \gamma_\infty \cup \Gamma$. For $m \geq 2$, the inequality reaches equality only at p_0^+, a point on Φ, and points in Γ.*

Proof. We consider $\mathcal{S} \cap \{P = 0\}$ as a union of slices

$$\mathcal{S} \cap \{P = 0\} = \bigcup_{\kappa \in [0,1]} \mathcal{L}_\kappa, \quad \mathcal{L}_\kappa := \mathcal{S} \cap \{P = 0\} \cap \{X_2 - \kappa X_1 = 0\}.$$

Note that each $\mathcal{L}_\kappa$ contains Γ. As $X_1 \geq X_2 \geq 0$ in $\mathcal{S}$, the function Q vanishes once X_1 does. We assume $X_1 > 0$ in the following discussion.

For $\mathcal{L}_0$, we have

$$Q|_{\mathcal{L}_0} = -2(4m + 8)X_1 Y Z + 2X_1 \frac{1 - H^2}{n}.$$

From $A \geq 0$ we have $Z = 0$. Then $Q|_{\mathcal{L}_0} = 2X_1((1 - H^2)/n)$, and $P = 0$ becomes $-X_1((1 - H^2)/n) = 0$. Hence, $\mathcal{L}_0 = \Gamma \cup \{p_0^+\}$ and $Q|_{\mathcal{L}_0} = 0$.

If $\kappa = 1$, let $X := X_1 = X_2 > 0$. Then $A \geq 0$ and $P = 0$ respectively become

$$X\left(Y - \frac{2m+3}{m} Z\right) \geq 0, \quad X(Y - (2m+3)Z)(Y - Z) = 0.$$

For $X > 0$ we must have $Y = (2m + 3)Z$. Then from (2.9) we find that $Y = (2m + 3)Z = (2m + 3)z_0$. Hence, $\mathcal{L}_1$ is a union of Γ and a part of the invariant set (3.18). More specifically, from $B \geq 0$ we have

$$\frac{1}{n} - nX^2 - \frac{2n^2(2m+3)(m-1)}{m(2m+1)(8m+3)} YZ = \frac{1}{n} - nX^2 - \frac{2n^2(2m+3)^2(m-1)}{m(2m+1)(8m+3)} z_0^2 \geq 0, \tag{4.8}$$

and it follows that

$$\mathcal{L}_1 = \Gamma \cup \left(\Phi \cap \left\{0 < X \leq \frac{1}{n}\sqrt{\frac{36m^3 + 90m^2 + 117m + 54}{m(8m+3)(4m^2 + 14m + 9)}} \leq \frac{1}{n}\right\}\right) \tag{4.9}$$

for $m \geq 2$. For $m = 1$, we have

$$\mathcal{L}_1 = \Gamma \cup \left(\Phi \cap \left\{0 < X \leq \frac{1}{n}\right\}\right) \cup \{p_2^+\}. \tag{4.10}$$

And $Q|_{\mathcal{L}_1}$ becomes

$$\begin{aligned} Q|_{\mathcal{L}_1} & \\ &= X\left(-4Y^2 - (4m+8)(4m+2)YZ + (n+1)\left(\frac{1}{n} - nX^2\right) + 4\left(1 + \frac{2m}{3}\right)\left(n + \frac{2m}{3}\right)X^2\right) \\ &= X\left(-4(2m+3)^2 z_0^2 - (4m+8)(4m+2)(2m+3)z_0^2 + \frac{n+1}{n} - \frac{4}{9}m(8m+3)X^2\right) \\ &= \frac{4}{9}m(8m+3)X\left(\frac{1}{n^2}\frac{36m^3 + 90m^2 + 117m + 54}{m(8m+3)(4m^2+14m+9)} - X^2\right) \\ &\geq 0. \end{aligned} \tag{4.11}$$

Hence, for $m \geq 1$, the function $Q|_{\mathcal{L}_1} \geq 0$ and it only vanishes on Γ and boundary points of $\mathcal{L}_1 \cap \Phi$. Note that for $m = 1$, one of the boundary points of $\mathcal{L}_1 \cap \Phi$ is p_2^+.

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For each $\mathcal{L}_\kappa$ with $\kappa \in (0,1)$, replace X_2 with $X_1\kappa$. Then (2.9) and $P=0$ respectively become

$$\begin{aligned}
&\frac{n-1}{n}-\frac{12m}{n}(1-\kappa)^2X_1^2=6Y^2+4m(4m+8)YZ-12mZ^2,\\
&\frac{\kappa-1}{n}-\frac{\kappa-1}{3n}(32\kappa^2m^2-12\kappa^2m+48m\kappa-18\kappa+27)X_1^2\\
&\quad=2\kappa Y^2-(4m+8)YZ+(4m\kappa+6)Z^2.
\end{aligned}\tag{4.12}$$

With the equations above, we can write the constant 1 as a homogeneous polynomial $One(Y,Z)$ of degree 2. Multiplying the constant term in B by $One(Y,Z)$, the function B restricted to each $\mathcal{L}_\kappa\cap\{P=0\}$ then becomes a homogeneous polynomial in Y and Z. Factor out X_1 in Q and multiply the constant term in Q/X_1 by $One(Y,Z)$. We see that Q restricted on each $\mathcal{L}_\kappa\cap\{P=0\}$ is a homogeneous polynomial in X_1, Y and Z. In summary, we have

$$\begin{aligned}
A|_{\mathcal{L}_\kappa\cap\{P=0\}}&=X_1\left(Y\kappa-\frac{3}{m}Z\left(1+\frac{2m}{3}\kappa\right)\right),\\
B|_{\mathcal{L}_\kappa\cap\{P=0\}}&=b_2Z^2+b_1YZ+b_0Y^2,\\
Q|_{\mathcal{L}_\kappa\cap\{P=0\}}&=X_1(q_2Z^2+q_1YZ+q_0Y^2).
\end{aligned}\tag{4.13}$$

The coefficients b_i and q_i in (4.13) are rational functions in m and κ. Explicit formulas for each coefficient are presented in the Appendix for the sake of simplicity. As Y is positive in $\mathcal{S}$ from Proposition 4.1, we factor out Y and consider $A|_{\mathcal{L}_\kappa\cap\{P=0\}}=X_1Y\tilde{A}_{m,\kappa}(Z/Y)$, $B|_{\mathcal{L}_\kappa\cap\{P=0\}}=Y^2\tilde{B}_{m,\kappa}(Z/Y)$ and $Q|_{\mathcal{L}_\kappa\cap\{P=0\}}=X_1Y^2\tilde{Q}_{m,\kappa}(Z/Y)$, where

$$\tilde{A}_{m,\kappa}(x)=\kappa-\left(\frac{3}{m}+2\kappa\right)x,\quad \tilde{B}_{m,\kappa}(x)=b_2x^2+b_1x+b_0,\quad \tilde{Q}_{m,\kappa}(x)=q_2x^2+q_1x+q_0.\tag{4.14}$$

We have

$$q_2=-\frac{4(2\kappa m+3)(32\kappa^3m^3+\kappa^2m^2(96-68\kappa)+\kappa m(90-84\kappa)+27(1-\kappa))}{3(1-\kappa)(2\kappa^2m(8m-5)+6\kappa(4m-1)+9)}\leq 0\tag{4.15}$$

for any $(m,\kappa)\in[1,\infty)\times(0,1)$. Therefore, the restricted function $Q|_{\mathcal{L}_\kappa\cap\{P=0\}}$ is non-negative if it is so on the boundary of $\mathcal{L}_\kappa\cap\{P=0\}$. The upper bound and the lower bound of Z/Y on each slice are respectively provided by $A|_{\mathcal{L}_\kappa\cap\{P=0\}}\geq 0$ and $B|_{\mathcal{L}_\kappa\cap\{P=0\}}\geq 0$. Specifically, from $A|_{\mathcal{L}_\kappa\cap\{P=0\}}\geq 0$ we have $Z/Y\leq m\kappa/(3+2m\kappa)$. As shown in (A.1), we have $b_2<0$. By Proposition A.1 in the Appendix, the smaller real root $\sigma(m,\kappa)$ of $\tilde{B}_{m,\kappa}$ is in the interval $(0,m\kappa/(3+2m\kappa))$. Hence, from $B|_{\mathcal{L}_\kappa\cap\{P=0\}}\geq 0$ we have $Z/Y\geq\sigma(m,\kappa)$. By the arbitrariness of κ, it is clear that the minimizing point of Q on $\mathcal{S}\cap\{P=0\}$ lies on $\mathcal{S}\cap\{P=0\}\cap\{A=0\}$ or $\mathcal{S}\cap\{P=0\}\cap\{B=0\}$.

We have

$$\begin{aligned}
Q|_{\mathcal{L}_\kappa\cap\{A=0\}}&=X_1Y^2\tilde{Q}_{m,\kappa}\left(\frac{m\kappa}{3+2m\kappa}\right)\\
&=X_1Y^2\frac{4(m+3)(m-1)(4\kappa^3m^2(8m-1)+4\kappa^2m(8m-3)+18\kappa m+9(1-\kappa))\kappa^2}{3(2\kappa m+3)(1-\kappa)(2\kappa^2m(8m-5)+6\kappa(4m-1)+9)}\\
&\geq 0.
\end{aligned}\tag{4.16}$$

Therefore, for $m\geq 2$, the function Q is positive on $\mathcal{L}_\kappa\cap\{A=0\}$ for any $\kappa\in(0,1)$. For $m=1$, the function Q vanishes at $\mathcal{S}\cap\{P=0\}\cap\{A=0\}$.

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Proving the non-negativity of $Q|_{\mathcal{L}_\kappa\cap\{B=0\}\cap\{P=0\}}$ is a bit more computationally involved. From Proposition 4.2, we know that for $m=1$, the polynomials A, B and P identically vanish at γ_∞. Therefore, an explicit formula for the root $\sigma(1,\kappa)$ can be obtained from $A=0$. We have

$$\sigma(1,\kappa)=\frac{Z}{Y}=\frac{X_2}{3\left(X_1+\frac{2}{3}X_2\right)}=\frac{\kappa}{2\kappa+3}.$$

Define the function $F(m,\kappa):=\tilde{Q}_{m,\kappa}(\sigma(m,\kappa))$. To show that $Q|_{\mathcal{L}_\kappa\cap\{P=0\}\cap\{B=0\}}\geq 0$, it suffices to show that $F(m,\kappa)\geq 0$ for any $(m,\kappa)\in[1,\infty)\times(0,1)$. Note that the vanishing of $F(m,\kappa)$ means $\sigma(m,\kappa)$ being also a root of $\tilde{Q}_{m,\kappa}$. From the computation (4.16) we have

$$F(1,\kappa):=\tilde{Q}_{1,\kappa}(\sigma(1,\kappa))=\tilde{Q}_{1,\kappa}\left(\frac{\kappa}{3+2\kappa}\right)=0$$

for any $\kappa\in(0,1)$. Furthermore, by implicit derivative, we have

$$\left(\frac{\partial\sigma}{\partial m}\right)(1,\kappa)=\left(-\frac{\frac{\partial b_2}{\partial m}\sigma^2+\frac{\partial b_1}{\partial m}\sigma+\frac{\partial b_0}{\partial m}}{2b_2\sigma+b_1}\right)(1,\kappa)=-\frac{(92\kappa^4+1202\kappa^3+1458\kappa^2-367\kappa-537)\kappa}{66(3+4\kappa)(\kappa+1)(3+2\kappa)^2},$$

and it follows that

$$\begin{aligned}\left(\frac{\partial F}{\partial m}\right)(1,\kappa)&=\left(\frac{\partial q_2}{\partial m}\sigma^2+\frac{\partial q_1}{\partial m}\sigma+\frac{\partial q_0}{\partial m}+2q_2\sigma\frac{\partial\sigma}{\partial m}+q_1\frac{\partial\sigma}{\partial m}\right)(1,\kappa)\\&=\frac{4(\kappa-1)(184\kappa^2-244\kappa-339)\kappa^2}{99(3+4\kappa)(\kappa+1)(2\kappa+3)}>0.\end{aligned}$$

Hence, $F\geq 0$ on a neighborhood around $\{(1,\kappa)\mid\kappa\in(0,1)\}\subset[1,\infty)\times(0,1)$. In other words, for an m that is slightly larger than 1, the root $\sigma(m,\kappa)$ of $\tilde{B}_{m,\kappa}$ is strictly between the two real roots of $\tilde{Q}_{m,\kappa}$. Hence, proving $F\geq 0$ on $[1,\infty)\times(0,1)$ is equivalent to showing that $\sigma(m,\kappa)$ *stays* between the two real roots of $\tilde{Q}_{m,\kappa}$ for varying (m,κ). This idea leads us to consider the resultant $r(\tilde{Q}_{m,\kappa},\tilde{B}_{m,\kappa})$ for the two polynomials. We have

$$\begin{aligned}r(\tilde{Q}_{m,\kappa},\tilde{B}_{m,\kappa})&=-\frac{64\kappa^2(m-1)(2\kappa m+3)}{(8m+3)^2(2m+1)^2m^2(1-\kappa)(2\kappa^2m(8m-5)+6\kappa(4m-1)+9)^2}\tilde{r},\\ \tilde{r}&=262144\kappa^4m^{10}+(516096\kappa^4+679936\kappa^3)m^9\\&\quad+(373760\kappa^4+1233920\kappa^3+675840\kappa^2)m^8\\&\quad+(-275904\kappa^4+1151040\kappa^3+1143744\kappa^2+308160\kappa)m^7\\&\quad+(-926496\kappa^4+248832\kappa^3+1432512\kappa^2+507456\kappa+54432)m^6\\&\quad+(-800496\kappa^4-1256256\kappa^3+1472688\kappa^2+857520\kappa+92016)m^5\\&\quad+(-281880\kappa^4-1525392\kappa^3+266328\kappa^2+1353024\kappa+199584)m^4\\&\quad+(-33048\kappa^4-644436\kappa^3-672624\kappa^2+957420\kappa+375192)m^3\\&\quad+(-90396\kappa^3-433026\kappa^2+186624\kappa+333882)m^2\\&\quad+(-74358\kappa^2-59049\kappa+133407)m+19683(1-\kappa).\end{aligned}\tag{4.17}$$

As verified in Proposition A.2 in the Appendix, the inequality $r(\tilde{Q}_{m,\kappa},\tilde{B}_{m,\kappa})\leq 0$ is valid for any $(m,\kappa)\in[1,\infty)\times(0,1)$. Furthermore, the function $r(\tilde{Q}_{m,\kappa},\tilde{B}_{m,\kappa})$ vanishes if and only if $m=1$. In particular, both $\tilde{Q}_{1,\kappa}$ and $\tilde{B}_{1,\kappa}$ have $\sigma(1,\kappa)=\kappa/(2\kappa+3)$ as their roots. For $m>1$, the polynomials $\tilde{Q}_{m,\kappa}$ and $\tilde{B}_{m,\kappa}$ do not share any common root. Hence, $F\geq 0$ on $[1,\infty)\times(0,1)$ and the function F vanishes if and only if $m=1$. Therefore, for $m\geq 2$, the inequality

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$Q|_{\mathcal{L}_\kappa\cap\{P=0\}\cap\{B=0\}}\geq 0$ is valid and the equality occurs only at p_0^+ and a point $\Phi\cap\{B=0\}$. The proof is complete. $\square$

With the help of the proceeding proposition, we are ready to prove the following lemma.

LEMMA 4.5. *For $m\geq 2$, integral curves γ_{s_1} that are in the interior of $\mathcal{S}$ can only escape through some point in $\mathcal{E}\cap\{H\geq 0\}\cap\{X_1-X_2=0\}$.*

Proof. The boundary $\partial\mathcal{S}$ is a union of the following five sets:

$$\mathcal{S}\cap\{X_1-X_2=0\},\quad \mathcal{S}\cap\{X_2=0\},\quad \mathcal{S}\cap\{A=0\},\quad \mathcal{S}\cap\{B=0\},\quad \mathcal{S}\cap\{P=0\}.$$

By Proposition 4.1, if a γ_{s_1} escapes $\mathcal{S}$ through some point with vanished X_2-coordinate, the point must lie in $\mathcal{S}\cap\{A=0\}\cap\{P=0\}$. Hence, we aim to show that V points inward when restricted to the last three parts of $\partial\mathcal{S}$.

A straightforward computation shows that

$$\begin{aligned}
&\langle\nabla A,V\rangle|_{A=0}\\
&=A\left(2H\left(G+\frac{1}{n}(1-H^2)\right)-2X_1-(4m+2)X_2\right)\\
&\quad+\frac{A}{X_1+(2m/3)X_2}\left(R_1+\frac{2m}{3}R_2-\left(1+\frac{2m}{3}\right)\frac{1}{n}(1-H^2)\right)+\frac{Y}{X_1+(2m/3)X_2}P\\
&=\frac{Y}{X_1+(2m/3)X_2}P\quad\text{since } A=0\\
&\geq 0.
\end{aligned}\tag{4.18}$$

It is confirmed that $V|_{A=0}$ points to the interior of $\mathcal{S}$.

Since

$$\begin{aligned}
&\langle\nabla B,V\rangle|_{B=0}\\
&=-2\frac{H}{n}(H^2-1)\left(G+\frac{1}{n}(1-H^2)\right)\\
&\quad-\frac{2n^2(2m+3)(m-1)}{m(16m^2+14m+3)}YZ\left(2H\left(G+\frac{1}{n}(1-H^2)\right)-2X_2\right)\\
&\quad\text{by (2.11)}\\
&=2BH\left(G+\frac{1}{n}(1-H^2)\right)+\frac{4n^2(2m+3)(m-1)}{m(16m^2+14m+3)}YZX_2\\
&=\frac{4n^2(2m+3)(m-1)}{m(16m^2+14m+3)}YZX_2\quad\text{since } B=0\\
&\geq 0,
\end{aligned}\tag{4.19}$$

it is clear that $V|_{B=0}$ points to the interior of $\mathcal{S}$.

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Since

$$
\begin{aligned}
&\langle \nabla P, V\rangle|_{P=0}\\
&= P\left(H\left(3G+\frac{3}{n}(1-H^2)-1\right)+\frac{4m}{3}X_2\right)+(X_1-X_2)Q\\
&= (X_1-X_2)Q \quad \text{since } P=0\\
&\geq 0 \quad \text{by Proposition 4.4,}
\end{aligned}
\tag{4.20}
$$

we learn that $V|_{P=0}$ also points to the interior of $\mathcal{S}$.

Finally, we need to exclude the possibility of non-transversal passing of a γ_{s_1} through some point with $X_1 \neq X_2$. By (4.19) and Proposition 4.1, such a point does not exist on $\mathcal{S}\cap\{B=0\}$. Suppose there were such a point on $\mathcal{S}\cap\{P=0\}$; then $P=Q=0$ at that point by (4.20). Then by Proposition 4.4, we know that such a point is either p_0^+ or a point on Φ, which is impossible. Suppose the non-transversal passing point exists on $\mathcal{S}\cap\{A=0\}$; then by (4.18) and Proposition 4.1, we must have $A=P=0$ at that point, which is also impossible. $\square$

To show that some γ_{s_1} is initially in the set $\mathcal{S}$ we need the following technical proposition.

PROPOSITION 4.6. *Define $\check{A} := YX_2-((m+2)/m)Z(X_1+(2m/3)X_2)$. For $m\geq 2$, the function Q is positive on the set*

$$\check{\mathcal{S}} = \mathcal{E}\cap\{X_1-X_2>0\}\cap\{X_2>0\}\cap\{A>0\}\cap\{\check{A}<0\}\cap\{P<0\}.$$

Proof. We consider $\check{\mathcal{S}}$ as a union of slices

$$\check{\mathcal{S}} = \bigcup_{\kappa\in(0,1)} \check{\mathcal{L}}_\kappa, \quad \check{\mathcal{L}}_\kappa := \check{\mathcal{S}}\cap\{X_2-\kappa X_1=0\}.$$

For each $\check{\mathcal{L}}_\kappa$ with $\kappa\in(0,1)$, replace X_2 with κX_1. Then from (2.9) and $P<0$ we have

$$
\begin{aligned}
0 &> \left(\frac{1-\kappa}{3n}(32\kappa^2m^2-12\kappa^2m+48m\kappa-18\kappa+27)-\frac{12m}{n(n-1)}(1-\kappa)^3\right)X_1^2\\
&\quad -\left(2\kappa+\frac{6(1-\kappa)}{n-1}\right)Y^2+(4m+8)\left(1-4m\frac{1-\kappa}{n-1}\right)YZ+\left(\frac{1-\kappa}{n-1}12m-(4m\kappa+6)\right)Z^2\\
&= \frac{(1-\kappa)(16\kappa^2m^2-10\kappa^2m+24\kappa m-6\kappa+9)}{6m+3}X_1^2\\
&\quad -\left(2\kappa+\frac{6(1-\kappa)}{n-1}\right)Y^2+(4m+8)\left(1-4m\frac{1-\kappa}{n-1}\right)YZ+\left(\frac{1-\kappa}{n-1}12m-(4m\kappa+6)\right)Z^2.
\end{aligned}
\tag{4.21}
$$

The coefficient for X_1^2 in (4.21) is obviously positive for any $(m,\kappa)\in[2,\infty)\times(0,1)$. On the other hand, use (2.9) to replace the constant term in Q with homogeneous polynomials in X_1, Y and Z. The polynomial Q restricted on $\check{\mathcal{L}}_\kappa$ becomes

$$
\begin{aligned}
Q|_{\check{\mathcal{L}}_\kappa} &= \frac{-64\kappa^3m^3+(40\kappa-96)\kappa^2m^2+(36\kappa^2+60\kappa-108)\kappa m+54(\kappa-1)}{18m+9}X_1^3\\
&\quad + X_1\left(\left(2\kappa+\frac{6}{2m+1}\right)Y^2-\frac{8m+16}{2m+1}YZ-\left(12m\kappa+\frac{12m}{2m+1}\right)Z^2\right).
\end{aligned}
\tag{4.22}
$$

It is obvious that the coefficient for X_1^3 in (4.22) is negative for any $(m,\kappa)\in[2,\infty)\times(0,1)$. Note that if (4.21) reaches equality, then one can write X_1^2 as a homogeneous polynomial in Y and Z.

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Moreover, substituting the X_1^2 by the homogeneous polynomial in (4.22) gives the formula for $X_1Y^2\tilde{Q}(Z/Y)$ as in (4.14). Therefore, from (4.21) we obtain

$$Q|_{\check{\mathcal{L}}_\kappa} > X_1Y^2\tilde{Q}\left(\frac{Z}{Y}\right).$$

Note that $\tilde{Q}(Z/Y)$ above is defined on $\check{\mathcal{S}}$ instead of $\mathcal{S}\cap\{P=0\}$. On the other hand, the inequalities $A>0$ and $\check{A}<0$ become

$$\frac{3m\kappa}{3(m+2)+2m(m+2)\kappa} < \frac{Z}{Y} < \frac{m\kappa}{3+2m\kappa}.$$

As shown in (4.15), it is clear that the coefficient q_2 is negative. It suffices to show that $\tilde{Q}$ is positive at $m\kappa/(3+2m\kappa)$ and $3m\kappa/(3(m+2)+2m(m+2)\kappa)$ to prove that the polynomial is positive on the open interval in between. From (4.16) it is clear that $\tilde{Q}(m\kappa/(3+2m\kappa))>0$. For any $(m,\kappa)\in[2,\infty)\times(0,1)$, a straightforward computation shows that

$$\begin{aligned}
&\tilde{Q}\left(\frac{3m\kappa}{3(m+2)+2m(m+2)\kappa}\right)\\
&\quad=\frac{512(m-1)\kappa^2}{3(1-\kappa)(2\kappa m+3)(2\kappa^2m(8m-5)+6\kappa(4m-1)+9)(m+2)^2}\\
&\qquad\times\left(\left(m^3+\frac{11}{8}m^2-\frac{19}{4}m-\frac{3}{8}\right)m^2\kappa^3\right.\\
&\qquad\left.+\frac{3}{2}\left(m^3-\frac{5}{8}m^2+3\,m-\frac{3}{4}\right)m\kappa^2+\left(\frac{9}{8}m^3-\frac{99}{64}m^2+\frac{27}{16}m-\frac{27}{32}\right)\kappa+\left(\frac{27}{64}m^2+\frac{27}{32}\right)\right)\\
&\quad>0.
\end{aligned}\tag{4.23}$$

The proof is complete. $\square$

We are ready to prove the following lemma.

LEMMA 4.7. *For $m\geq 2$, the integral curve γ_{s_1} is initially in the interior of $\mathcal{S}$ if $s_1\in(3/(m-1),9(5m+3)(4m^2+4m+3)/n^2(2m+3)(m-1))$.*

Proof. With the linearized solution (3.1), we have

$$\begin{aligned}
X_1(\gamma_{s_1})&=\frac{1}{3}-(8m^2+18m+18+(4m+8)ms_1)e^{(2/3)\eta}+O(e^{(2/3+\epsilon)\eta}),\\
X_2(\gamma_{s_1})&=(-9+3s_1(m+2))e^{(2/3)\eta}+O(e^{(2/3+\epsilon)\eta}), &(4.1)\\
A(\gamma_{s_1})&=\frac{m+3}{m}((m-1)s_1-3)e^{(2/3)\eta}+O(e^{(2/3+\epsilon)\eta}),\\
B(\gamma_{s_1})&=\frac{2n^2(2m+3)(m-1)}{m(2m+1)(8m+3)}\left(\frac{9(5m+3)(4m^2+4m+3)}{n^2(2m+3)(m-1)}-s_1\right)e^{(2/3)\eta}+O(e^{(2/3+\epsilon)\eta}),\\
P(\gamma_{s_1})&=O(e^{(2/3+\epsilon)\eta}),\\
Q(\gamma_{s_1})&=O(e^{(2/3+\epsilon)\eta}), &(4.24)
\end{aligned}$$

near p_0^+. Hence, functions X_2, A and B are positive along γ_{s_1} near p_0^+ if $s_1\in(3/(m-1),9(5m+3)(4m^2+4m+3)/n^2(2m+3)(m-1))$. Furthermore, the last two equalities in (4.24) show that ∇P and ∇Q are perpendicular to the linearized solution (3.1) at p_0. Hence, the integral curve γ_{s_1} is tangent to $\{P=0\}$ and $\{Q=0\}$ for any s_1. It takes a little bit more work

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to show that the function P is initially positive along γ_{s_1}. Recall in (4.20), we have

$$P' = P\left(H\left(3G + \frac{3}{n}(1-H^2) - 1\right) + \frac{4m}{3}X_2\right) + (X_1 - X_2)Q. \tag{4.25}$$

Note that

$$\begin{aligned}&\left(H\left(3G + \frac{3}{n}(1-H^2) - 1\right) + \frac{4m}{3}X_2\right)(\gamma_{s_1})\\ &\quad = -(48m^2 + 84m + 20m(m+2)s_1)e^{(2/3)\eta} + O(e^{(2/3+\epsilon)\eta})\end{aligned}$$

near p_0^+. If P were negative initially along γ_{s_1} near p_0^+, the first term in (4.25) would be positive. Moreover, from the linearized solution (3.1), we have

$$\check{A}(\gamma_{s_1}) = -\frac{1}{m}(6m + 6 - (m+2)(m-1)s_1)e^{(2/3)\eta} + O(e^{(2/3+\epsilon)\eta}).$$

Since $9(5m+3)(4m^2+4m+3)/n^2(2m+3)(m-1) < 6(m+1)/(m+2)(m-1)$ for $m \geq 2$, we know that $\check{A}(\gamma_{s_1})$ is initially negative along γ_{s_1}. Based on the assumption that P is initially negative along γ_{s_1}, we know that γ_{s_1} is initially in $\check{\mathcal{S}}$. By Proposition 4.6, we know that the second term in (4.25) is also positive and so is P', which is a contradiction. Therefore, the function P must be positive initially along γ_{s_1} near p_0^+. □

According to § 4 in [Böh98], the existence of the heterocline that joins $p_0^{\pm}$ relies on the number of critical points of $\sqrt{Z/Y}$ that appear before the turning point. The number is originally denoted by $\sharp C_w(\bar{h})$ in [Böh98], where w is the ratio $f_1/f_2 = \sqrt{Z/Y}$ and $\bar{h}$ corresponds to the initial data f in (2.3). We introduce the following modified definitions of $\sharp C_w(\bar{h})$ and W-intersection points.

Definition 4.8. For a γ_{s_1} that is not a heterocline (i.e., a γ_{s_1} that is not defined on $\mathbb{R}$ or $\lim_{\eta\to\infty} \gamma_{s_1} \neq p_0^-$), let $\sharp C(\gamma_{s_1})$ be the number of critical points of the function $\sqrt{Z/Y}$ along γ_{s_1} that appear in $\mathcal{E} \cap \{H > 0\} \cap \{Y - Z > 0\}$.

Definition 4.9. A point where γ_{s_1} or ζ_{s_2} intersects $\mathcal{E} \cap \{H > 0\} \cap \{Y - Z = 0\}$ is called a W-intersection point.

We have the following proposition.

Proposition 4.10. *Any γ_{s_1} with $s_1 > -3$ (or ζ_{s_2} with $s_2 > 0$) has a turning point at $\mathcal{E} \cap \{H = 0\}$ or a W-intersection point at $\mathcal{E} \cap \{Y - Z = 0\}$.*

Proof. An integral curve γ_{s_1} with $s_1 > -3$ (or ζ_{s_2} with $s_2 > 0$) is initially in the interior of the compact set $\mathcal{E} \cap \{H \geq 0\} \cap \{Y - Z \geq 0\}$. Suppose the integral curve does not have any turning point or any W-intersection point. Then it must be defined on $\mathbb{R}$. Furthermore, such an integral curve is in $\mathcal{E} \cap \{H < 1\}$ initially. From (2.11), along the integral curve we eventually have $H^2 < 1 - \epsilon$ for some $\epsilon > 0$. Hence,

$$H' = (H^2 - 1)\left(\frac{1}{n} + \frac{12m}{n}(X_1 - X_2)^2\right) \leq \frac{H^2 - 1}{n} < -\frac{\epsilon}{n}$$

eventually, meaning the function H must at vanish some point. We reach a contradiction. Each integral curve without a turning point must have a W-intersection point. □

We rephrase Lemma 4.4 in [Böh98] with these new definitions.

Theorem 4.11. *If γ_{s_1} is not a heterocline for any $s_1 \in [a, b]$, then $\sharp C(\gamma_{s_1})$ is a constant for all $s_1 \in [a, b]$.*

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Immediately, we have the following proposition

PROPOSITION 4.12. *The quantities $\sharp C(\gamma_0)$ and $\sharp C(\zeta_{1/(2m+6)})$ are both zero.*

Proof. From the defining equations of $\mathcal{B}_{QK}$, we have $X_1 = Y - 2Z$ and $X_2 = -Z$ along γ_0. Then $H^2 < 1$ becomes $(3Y - (n+3)Z)^2 < 1$. From (3.14) we have

$$\begin{aligned}\frac{12m}{n}(Y-Z)^2 + 6Y^2 + 4m(4m+8)YZ - 12mZ^2 &> \frac{n-1}{n}(3Y-(4m+6)Z)^2\\ \Leftrightarrow n(n+9)(n-1)(Y-Z)Z &> 0.\end{aligned} \tag{4.26}$$

Hence, $X_1 - X_2 = Y - Z > 0$ along γ_0. Similarly, we have $X_2 - X_1 = Y - Z > 0$ along $\zeta_{1/(2m+6)}$. Note that $X_1 - X_2$ vanishes at the critical point p_1^- and stays positive along γ_0. $\square$

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Consider a γ_{s_1} with $s_1 \in (3/(m-1), 9(5m+3)(4m^2+4m+3)/n^2(2m+3)(m-1))$. From Lemma 4.7 we know that such a γ_{s_1} is initially in $\mathcal{S}$. From (2.11) we know that such a γ_{s_1} must exit $\mathcal{S}$. From Lemma 4.5 we know that such a γ_{s_1} exits $\mathcal{S}$ through the face $\mathcal{E} \cap \{H > 0\} \cap \{Y - Z \geq 0\} \cap \{X_1 - X_2 = 0\}$. It is clear that $(\sqrt{Z/Y})' = \sqrt{Z/Y}(X_1 - X_2)$. Since Z is initially positive along γ_{s_1} with $s_1 > 3/(m-1)$, we know that $\sharp C(\gamma_{s_1})$ is exactly the number of times that γ_{s_1} intersects $\mathcal{E} \cap \{H > 0\} \cap \{Y - Z \geq 0\} \cap \{X_1 - X_2 = 0\}$. Hence, $\sharp C(\gamma_{s_1}) \geq 1$ for $s_1 \in (3/(m-1), 9(5m+3)(4m^2+4m+3)/n^2(2m+3)(m-1))$.

On the other hand, the integral curve γ_0 joins p_0^+ and p_1^-. From Proposition 4.12, the function $X_1 - X_2$ stays positive along γ_0. Hence, we have $\sharp C(\gamma_0) = 0$. Therefore, by Theorem 4.11 there exists some $s_\star \in (0, 9(5m+3)(4m^2+4m+3)/n^2(2m+3)(m-1))$ such that $\gamma_{s_\star}$ is a heterocline that joins $p_0^\pm$. Theorem 1.1 is proved. $\square$

Remark 4.13. With some small modifications, the polynomial $\mathcal{S}$ can be applied to prove the existence of positive Einstein metrics on F^{m+1}, a cohomogeneity-one space formed by the group triple $(Sp(m)U(1), Sp(m)Sp(1), Sp(m+1))$. Furthermore, some non-existence results can also be obtained from the defining polynomial P. For some cohomogeneity-one spaces, the function P is negative along all γ_{s_1} in $\mathcal{E} \cap \{X_1 - X_2 \geq 0\}$, essentially forcing $X_1 - X_2$ to be positive along these integral curves. For example, there is no Spin(9)-invariant cohomogeneity-one positive Einstein metric on $\mathbb{OP}^2\sharp\overline{\mathbb{OP}}^2$. A systematic study of the existence problem on all cohomogeneity-one spaces with two isotropy summands will be presented in later work.

Remark 4.14. One can recover Böhm's metric on $\mathbb{HP}^2\sharp\overline{\mathbb{HP}}^2$ by enlarging the set $\mathcal{S}$ for $m = 1$. Specifically, we can increase the coefficient for Y in the polynomial A properly so that, first, integral curves γ_{s_1} with large enough s_1 are in the enlarged $\mathcal{S}$ initially; and second, integral curves that are in the enlarged $\mathcal{S}$ must exit through the face $\mathcal{S} \cap \{X_1 - X_2 = 0\}$. Note that the derivative of the new polynomial A still depends on the non-negativity of the same polynomial P. Hence, $\sharp C(\gamma_{s_1}) \geq 1$ with large enough s_1 while $\sharp C(\gamma_0) = 0$, and Theorem 4.11 can be applied to prove the existence.

We end this section with the following remark to discuss the motivation in defining $\mathcal{S}$.

Remark 4.15. Inspired by Corollary 5.8 in [Böh98] and by the fact that p_2^+ is a stable node, we realized that taking the limit $s_1 \to \infty$ for γ_{s_1} may not provide enough information to prove the existence theorem for all m. From (3.2) we know that s_1 is related to the initial condition f. Numerical data in Table 2 in [Böh98] indicate that the winding angle of γ_{s_1} around Φ may not be monotonic as s_1 increases. Hence, it is reasonable to find a bounded interval of s_1 for which the winding angle of γ_{s_1} is large enough.

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From (4.19) and (4.24), it appears that the upper bound for s_1 can be easily controlled by changing the coefficient of YZ in B. Looking for an appropriate lower bound for s_1, on the other hand, is relatively more difficult. Originally we choose to obtain the lower bound from $f_2\dot{f}_2 \geq f_1\dot{f}_1$. The inequality is equivalent to $YX_2 \geq ZX_1$. Although the inequality is geometrically motivated, showing it can be maintained before the winding angle gets large enough seems to be too difficult. We eventually define the polynomial A, whose first and second derivatives are relatively easier to control by polynomials P and Q.

5. Limiting winding angle

From [Win17] we know that γ_∞ joins p_0^+ and p_2^+. By the symmetry of (2.7), we know that there exists $\bar{\gamma}_\infty$ that joins p_2^- and p_0^-. As Φ joins $p_2^\pm$, we have the set $\{\gamma_\infty, \Phi, \bar{\gamma}_\infty\}$ of heteroclines that joins $p_0^\pm$. From this perspective, the critical point p_2^+ is anticipated to play an important role in the qualitative analysis. Intuitively speaking, if p_2^+ were a stable focus in the Ricci-flat subsystem, the integral curve γ_{s_1} would wind around Φ more frequently as s_1 increases. From § 3, however, we learn that p_2^+ is a stable node in the Ricci-flat subsystem. Hence, the winding behavior of γ_{s_1} around Φ is less obvious. The new coordinate change helps us estimate the limiting winding angle of γ_{s_1} as $s_1 \to \infty$ and establish Theorem 1.2. On the other hand, another set $\{\zeta_\infty, \Phi, \bar{\zeta}_\infty\}$ of heteroclines joins $p_1^\pm$. It is natural to ask if some heterocline other than ζ_0 joins $p_1^\pm$. The new coordinate change also helps us to answer this question and prove Theorem 1.3.

We introduce some known estimates in the Ricci-flat system in the following. The Ricci-flat subsystem on $\mathcal{E} \cap \{H = 1\}$ is simply a subsystem of (2.16) in [Chi21], with $Y_1 \to \sqrt{2}$, $Y_2 \to \sqrt{2}Y$ and $Y_3 \to 2\sqrt{2}Z$. From Lemma 4.4 in [Chi21], we learn that the compact set

$$\hat{\mathcal{B}}_{\mathrm{RF}} := \mathcal{E} \cap \{H = 1\} \cap \{Y - Z \geq 0\} \cap \{X_2 - X_1 + 2Y - 2Z \geq 0\} \cap \left\{X_1 \leq \tfrac{1}{2}\right\} \cap \{X_2 \geq 0\}$$

is invariant. Critical points p_0^+ and p_1^+ are on the boundary of $\hat{\mathcal{B}}_{\mathrm{RF}}$ while p_2^+ is in the interior. Straightforward computations show that γ_∞ and ζ_∞ are initially in $\hat{\mathcal{B}}_{\mathrm{RF}}$. From Lemma 5.7 in [Chi21], it is known that these two integral curves converge to p_2^+. We construct the following invariant set introduced in [Chi19], which gives us more information on γ_∞ near p_2^+.

Proposition 5.1. *The set*

$$\mathcal{B}_{\mathrm{RF}} := \mathcal{E} \cap \{H = 1\} \cap \{Y - (2m+3)Z \geq 0\} \cap \{X_2 - X_1 + Y - (2m+3)Z \geq 0\}$$

is compact and invariant.

Proof. The compactness is derived from $Y - (2m+3)Z \geq 0$ and (2.9).

Since

$$\begin{aligned}
&\langle \nabla(Y - (2m+3)Z), V\rangle_{\mathcal{B}_{RF} \cap \{Y-(2m+3)Z=0\}}\\
&\quad = (Y - (2m+3)Z)\left(H\left(G + \frac{1}{n}(1 - H^2)\right) - X_1 - (4m+6)Z\right)\\
&\qquad + (4m+6)Z(X_2 - X_1 + Y - (2m+3)Z)\\
&\quad = (4m+6)Z(X_2 - X_1 + Y - (2m+3)Z) \quad \text{since } Y - (2m+3)Z = 0\\
&\quad \geq 0,
\end{aligned} \tag{5.1}$$

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the vector field V points inward on the boundary $\mathcal{B}_{\mathrm{RF}}\cap\{Y-(2m+3)Z=0\}$. As for $\mathcal{B}_{\mathrm{RF}}\cap\{X_2-X_1+Y-(2m+3)Z=0\}$, from (2.9) we have

$$\begin{aligned}&\frac{12m}{n}(Y-(2m+3)Z)^2+6Y^2+4m(4m+8)YZ-12mZ^2=\frac{n-1}{n}\\&\quad\Leftrightarrow 18(2m+1)Y^2+8m(8m^2+16m+3)YZ+24m(2m^2+4m+3)Z^2=4m+2.\end{aligned}\tag{5.2}$$

Then we have

$$\begin{aligned}&\langle\nabla(X_2-X_1+Y-(2m+3)Z),V\rangle_{\mathcal{B}_{\mathrm{RF}}\cap\{X_2-X_1+Y-(2m+3)Z)=0\}}\\&\quad=(X_2-X_1+Y-(2m+3)Z))\left(H\left(G+\frac{1}{n}(1-H^2)-1\right)+\frac{4m}{n}Y+\frac{8m^2+24m+18}{n}Z\right)\\&\qquad+\frac{1}{n}(Y-(2m+3)Z)((4m+2)H-(12m+6)Y-(8m^2+16m+12)Z)\\&\quad=\frac{1}{n}(Y-(2m+3)Z)(4m+2-(12m+6)Y-(8m^2+16m+12)Z)\end{aligned}$$

since $H=1$ and $X_2-X_1+Y-(2m+3)Z=0$. (5.3)

With $Y,Z\geq 0$, showing $4m+2-(12m+6)Y-(8m^2+16m+12)Z\geq 0$ is equivalent to showing $(4m+2)^2\geq((12m+6)Y+(8m^2+16m+12)Z)^2$. Note that $(4m+2)^2$ is simply the left-hand side of (5.2) multiplied by $4m+2$. Hence, one can obtain the non-negativity by verifying

$$\begin{aligned}&(4m+2)(18(2m+1)Y^2+8m(8m^2+16m+3)YZ+24m(2m^2+4m+3)Z^2)\\&\quad\geq((12m+6)Y+(8m^2+16m+12)Z)^2\\&\quad\Leftrightarrow 16n(m-1)((4m^2+8m+3)YZ+(2m^2+4m+3)Z^2)\geq 0.\end{aligned}\tag{5.4}$$

Note that the equality is obtained for $m=1$. Hence, (5.3) is non-negative and identically vanishes if $m=1$. Hence, $\mathcal{B}_{\mathrm{RF}}$ is invariant. □

Remark 5.2. With (2.9) one can easily show that

$$\begin{aligned}\mathcal{B}_{\mathrm{Spin}(7)}&=\mathcal{E}\cap\{Y-2Z-X_1=0\}\cap\{3Z-X_2=0\}\\&=\mathcal{E}\cap\{H=1\}\cap\{X_2-X_1+Y-5Z=0\}.\end{aligned}\tag{5.5}$$

Therefore, the fact that (5.3) identically vanishes for $m=1$ recovers the invariant set $\mathcal{B}_{\mathrm{Spin}(7)}$ as in (3.4).

Remark 5.3. By (5.1) and (5.3), one can also show that the set

$$\tilde{\mathcal{B}}_{\mathrm{RF}}:=\mathcal{E}\cap\{H=1\}\cap\{Y-(2m+3)Z\leq 0\}\cap\{X_2-X_1+Y-(2m+3)Z\leq 0\}.$$

is also invariant. Furthermore, we have the following proposition.

Proposition 5.4. *For $m\geq 2$, the integral curve γ_∞ is in $\mathcal{B}_{\mathrm{RF}}$ initially. For $m=1$, the integral curve γ_∞ stays on the boundary of $\mathcal{B}_{\mathrm{RF}}$. For $m\geq 1$, the integral curve ζ_∞ is in $\tilde{\mathcal{B}}_{\mathrm{RF}}$ initially.*

Proof. The statement is clear by (3.3) and (3.13). Note that for $m=1$, the function X_2-X_1+Y-5Z is identically zero on γ_∞. □

To obtain more information on how γ_{s_1} and ζ_{s_2} wind around Φ as $s_1,s_2\to\infty$, we consider the 'cylindrical' coordinates

$$r\sin(\theta)=X_1-X_2,\quad r\cos(\theta)=\sqrt{\frac{2m+2}{2m+3}}(Y-(2m+3)Z).$$

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The system (2.7) is transformed into

$$\begin{bmatrix} H\\ r\\ \theta\\ Y\end{bmatrix}' = \begin{bmatrix} (H^2-1)\left(\frac{1}{n}+\frac{12m}{n}r^2\sin^2(\theta)\right)\\ rH\left(\frac{1}{n}+\frac{12m}{n}r^2\sin^2(\theta)\right)-Hr\sin^2(\theta)-\frac{H}{n}r\cos^2(\theta)+\left(\frac{m+2}{m+1}+\frac{3}{n}\right)r^2\sin(\theta)\cos^2(\theta)\\ 2\sqrt{\frac{2m+2}{2m+3}}Y+\frac{1}{m+1}r\cos(\theta)-\frac{n-1}{n}H\sin(\theta)\cos(\theta)-\left(\frac{m+2}{m+1}+\frac{3}{n}\right)r\cos(\theta)\sin^2(\theta)\\ Y\left(\frac{12m}{n}Hr^2\sin^2(\theta)-\frac{4m}{n}r\sin(\theta)\right)\end{bmatrix} \tag{5.6}$$

with conservation law (2.9) rewritten as

$$C_{\Lambda\geq 0}\colon \frac{12m}{n}r^2\sin^2(\theta)+6Y^2+4m(4m+8)Y\tilde{Z}-12m\tilde{Z}^2=1-\frac{1}{n},$$
$$\tilde{Z}=\left(\frac{Y}{2m+3}-\frac{r\cos(\theta)}{\sqrt{(2m+2)(2m+3)}}\right).$$

The set $\mathcal{E}$ is then defined in the (H,r,θ,Y)-coordinate accordingly. The variable r tells us the distance from a point to Φ and θ records the winding angle around Φ.

With the new conservation law, setting $r=0$ implies $Y=(2m+3)z_0$. Restricting (5.6) to the invariant set $\mathcal{E}\cap\{r=0\}$ gives the subsystem

$$\begin{bmatrix} H\\ \theta\end{bmatrix}' = \begin{bmatrix} (H^2-1)\frac{1}{n}\\ 2\sqrt{(2m+2)(2m+3)}z_0-\frac{n-1}{n}H\sin(\theta)\cos(\theta)\end{bmatrix}, \quad r=0, \quad Y=(2m+3)z_0. \tag{5.7}$$

This subsystem is essentially the integral curve Φ. Straightforward computations show that (5.6) has the following four sequences of critical points in $\mathcal{E}\cap\{r=0\}$:

$$\begin{aligned} &\{A_i^{\pm}:=(\pm 1,0,a_i^{\pm},(2m+3)z_0)\}_{i\in\mathbb{Z}}, \quad a_i^{\pm}=\pm\arctan\left(-\frac{\delta_1}{2\sqrt{(2m+3)(2m+2)}z_0}\right)+i\pi,\\ &\{B_i^{\pm}:=(\pm 1,0,b_i^{\pm},(2m+3)z_0)\}_{i\in\mathbb{Z}}, \quad b_i^{\pm}=\pm\arctan\left(-\frac{\delta_2}{2\sqrt{(2m+3)(2m+2)}z_0}\right)+i\pi. \end{aligned} \tag{5.8}$$

Furthermore, for each $i\in\mathbb{Z}$ we have

$$\begin{aligned} a_i^+\in\left(i\pi,\frac{\pi}{4}+i\pi\right), \quad b_i^+\in\left(\frac{\pi}{4}+i\pi,\frac{\pi}{2}+i\pi\right),\\ b_i^-\in\left(-\frac{\pi}{2}+i\pi,-\frac{\pi}{4}+i\pi\right), \quad a_i^-\in\left(-\frac{\pi}{4}+i\pi,i\pi\right). \end{aligned} \tag{5.9}$$

Remark 5.5. Computations show that A_i^+ and B_i^+ are transformed respectively from the two stable eigenvectors u_1 and u_2 of the linearization at p_2^+. Recall from § 3 that both u_1 and u_2 are tangent to $\mathcal{E}\cap\{H=1\}$, and the corresponding eigenvalues δ_1 and δ_2 are real numbers and we have $\delta_2<\delta_1<0$. Each linearized solution to the Ricci-flat subsystem around p_2^+ must have $e^{\delta_2\eta}\ll e^{\delta_1\eta}$ as $\eta\to\infty$. Hence, integral curves γ_∞ and ζ_∞ converge to p_2^+ along u_1. Hence, it is not surprising that A_i^+ are sinks and B_i^+ are saddles in the subsystem of (5.6) restricted

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to $\mathcal{E}\cap\{H=1\}$. Furthermore, for the subsystem (5.7), critical points A_i^+ are saddles, and B_i^+ are sources.

Thanks to the invariant set $\mathcal{B}_{\mathrm{RF}}$ in Proposition 5.1, whose boundary contains p_2^+, we are now ready to show that both γ_∞ and ζ_∞ do not wind fully around p_2^+.

Proposition 5.6. *Consider the (H,r,θ,Y)-coordinate. For $m\geq 2$, the integral curve γ_∞ joins the critical points p_0^+ and A_0^+. For $m=1$, the variable θ remains a constant along γ_∞ and the integral curve joins p_0^+ and B_0^+. For $m\geq 1$, the integral curve ζ_∞ joins the critical points p_1^+ and A_1^+.*

Proof. In the new coordinate, the critical point p_0^+ is

$$\left(1,\frac{1}{3}\sqrt{\frac{4m+5}{2m+3}},\arctan\left(\sqrt{\frac{2m+3}{2m+2}}\right),\frac{1}{3}\right).$$

As γ_∞ joins p_0^+ and p_2^+ in the (X_1,X_2,Y,Z)-coordinate, it is clear that the integral curve joins p_0^+ and one of A_i^+ or B_i^+ in the new coordinate. From Propositions 5.1 and 5.4 we know that $r\cos(\theta)\geq 0$ and $\sqrt{(2m+3)/(2m+2)}r\cos(\theta)-r\sin(\theta)\geq 0$ along γ_∞. In particular, the second inequality reaches equality so that $\theta=\arctan(\sqrt{(2m+3)/(2m+2)})$ along γ_∞ for $m=1$. Note that $0<a_0^+<\arctan(\sqrt{(2m+3)/(2m+2)})\leq b_0^+$ and the last inequality reaches equality only for $m=1$. The θ-coordinate for p_0^+ is positive and $\{\theta\geq 0\}$ is clearly invariant. Hence, for $m\geq 2$, the integral curve γ_∞ converges to A_0^+; for $m=1$, the integral curve γ_∞ converges to B_0^+ as $\theta=b_0^+$ along γ_∞.

In the (H,r,θ,Y)-coordinate, the critical point p_1^+ is

$$\left(1,\sqrt{\frac{2m+2}{2m+3}}\frac{2m+2}{n},\pi,\frac{1}{n}\right).$$

It is established in [Chi21] that ζ_∞ is an integral curve that joins p_1^+ and p_2^+ in the (X_1,X_2,Y,Z)-coordinate. Therefore, in the (H,r,θ,Y)-coordinate, the integral curve ζ_∞ joins p_1^+ and one of A_i^+ or B_i^+. As ζ_∞ is in $\tilde{\mathcal{B}}_{\mathrm{RF}}$ initially, by Remark 5.3 we know that $\cos(\theta)=\sqrt{(2m+2)/(2m+3)}(Y-(2m+3)Z)\leq 0$ and $\sqrt{(2m+3)/(2m+2)}r\cos(\theta)-r\sin(\theta)\leq 0$ along ζ_∞. Hence, we know that $\pi/2\leq\theta\leq\arctan(\sqrt{(2m+3)/(2m+2)})+\pi\leq b_1^+$ along ζ_∞. Therefore, ζ_∞ converges to A_1^+ for $m\geq 2$. We claim that the integral curve ζ_∞ also converges to A_1^+ for $m=1$. Recall Remarks 4.3 and 5.2. The integral curve χ also converges to p_2^+ and along χ we have $X_2-X_1\geq 0$ and $X_2-X_1+Y-5Z=0$. Hence, χ converges to B_1^+ in the (H,r,θ,Y)-coordinate. As the linearization at B_1^+ has only one stable eigenvector, we know that ζ_∞ converges to A_1^+. □

We claim the following lemma.

Lemma 5.7. *Let Π be the integral curve of the subsystem (5.7) that emanates from A_0^+. Let $(0,0,\theta_*,(2m+3)z_0)$ be the midpoint of Π at which it passes through $H=0$. For $m\geq 2$, we have $\theta_*=\lim_{s_1\to\infty}\theta(\gamma_{s_1}\cap\{H=0\})$.*

Proof. We think of (5.6) on $\mathcal{E}$ as a dynamical system in the three-dimensional $Hr\theta$-space with Y as a function in (H,r,θ). As mentioned in Remark 5.5, each A_i^+ is a saddle whose linearization has two stable eigenvectors and one unstable eigenvector. Furthermore, the two stable eigenvectors are parallel to $\mathcal{E}\cap\{H=1\}$, meaning that each A_i^+ is a sink in the Ricci-flat subsystem and Π is the only integral curve that emanates from A_i^+ in $\mathcal{E}\cap\{H<1\}$. By the Hartman–Grobman theorem, there is a local homeomorphism $(\phi,\mathcal{U})$ defined around A_0^+ through which the system

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(5.6) is topologically equivalent to the linear dynamical system

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{5.10}$$

In particular, we have

$$\phi(A_0^+) = (0,0,0), \quad \phi(\mathcal{U} \cap \{H = 1\}) \subset \{(x,y,0) \mid x,y \in \mathbb{R}\}, \quad \phi(\mathcal{U} \cap \Pi) \subset \{(0,0,z) \mid z \in \mathbb{R}\}.$$

Let D_ϵ be an open disk on the xy-plane with radius ϵ_1. Let U_1 be the cylinder $D_{\epsilon_1} \times (-\epsilon_2, \epsilon_2]$. Note that integral curves in $U_1 \cap \{z > 0\}$ can only escape through the face $D_{\epsilon_1} \times \{\epsilon_2\}$. Choose small enough ϵ_1 and ϵ_2 so that $\mathcal{U}_1 := \phi^{-1}(U_1)$ is contained in $\mathcal{U}$. Then $p := \phi^{-1}(0,0,\epsilon_2)$ is a point on $\mathcal{U} \cap \Pi$.

Let $\mathcal{U}_0$ be an open neighborhood around the point $(0,0,\theta_*)$ in the $Hr\theta$-space. By the continuous dependence, there exists an open set $\mathcal{U}_2 \ni p$ in $\mathcal{U}$ such that any point in $\mathcal{U}_2$ lies on an integral curve that enters $\mathcal{U}_0$. It is clear that $p \in \mathcal{U}_1 \cap \mathcal{U}_2$. Modify U_1 by shrinking ϵ_1 so that $D_{\epsilon_1} \times \{\epsilon_2\}$ is contained in $\phi(\mathcal{U}_2)$ while leaving ϵ_2 unchanged. Then correspondingly with the modified $\mathcal{U}_1$, an integral curve in $\mathcal{U}_1 \cap \{H < 1\}$ must enter $\mathcal{U}_2$. Since γ_∞ converges to A_0^+, there exists a point $q \in \gamma_\infty \cap \mathcal{U}_1$. By the continuous dependence, there exists a large enough N such that $s_1 > N$ implies γ_{s_1} must enter $\mathcal{U}_1 \cap \{H < 1\}$ and hence $\mathcal{U}_0$. □

Note that proving Lemma 5.7 for $m = 1$ is more subtle. As γ_∞ converges to B_0^+ and there is an obvious integral curve that joins B_0^+ and A_0^+, a more delicate analysis is needed to show that γ_{s_1} with a large enough s_1 must enter U_0. On the other hand, since ζ_∞ converges to A_1^+ for $m \geq 1$, we have the following corollary to Lemma 5.7.

Corollary 5.8. *For $m \geq 1$, the number $\theta_* + \pi$ is the limiting winding angle of ζ_{s_2} around Φ while $H > 0$ as $s_2 \to \infty$.*

Lemma 5.9. *For $m \geq 2$, there exist at least two Einstein metrics on $\mathbb{HP}^{m+1}\sharp\overline{\mathbb{HP}}^{m+1}$ if $\theta_* < \pi$. For $m \geq 1$, there exist at least two Einstein metrics on $\mathbb{S}^{4m+4}$ if $\theta_* > \pi$.*

Proof. Consider $\mathbb{HP}^{m+1}\sharp\overline{\mathbb{HP}}^{m+1}$ for $m \geq 2$. Let $\tilde{s}_1 = \max\{3/(m-1), s_\star\}$, where $\gamma_{s_\star}$ is the heterocline in the proof of Theorem 1.1 that joins $p_0^\pm$. The proof of Theorem 1.1 shows that we have $\sharp C(\gamma_{s_1}) \geq 1$ for $s_1 \in (\tilde{s}_1, 9(5m+3)(4m^2+4m+3)/n^2(2m+3)(m-1))$. If $\theta_* < \pi$, then $\lim_{s_1 \to \infty} \sharp C(\gamma_{s_1}) = 0$ by Lemma 5.7. By Theorem 4.11 there exists a heterocline $\gamma_{s_{\star\star}}$ for some $s_{\star\star} \in (\tilde{s}_1, \infty)$ and $s_{\star\star} \neq s_\star$.

Consider $\mathbb{S}^{4m+4}$ for $m \geq 1$. If $\theta_* > \pi$, then $\Pi + (0,0,\pi,(2m+3)z_0)$ is an integral curve that emanates from A_1^+ and passes $(0,0,\theta_* + \pi, (2m+3)z_0)$ and $\theta_* + \pi > 2\pi$. By Proposition 4.10, any ζ_{s_2} with $s_2 > 0$ has either a turning point or a W-intersection point. We learn from the linearized solution (3.6) that along ζ_{s_2} with $s_2 > 0$ the functions $Y - Z$ and $X_2 - X_1$ are positive initially. Since $(Y/Z)' = 2(Y/Z)(X_2 - X_1)$ by (3.9), the function $Y - Z = Z(Y/Z - 1)$ can only have a zero after $X_2 - X_1$ changes sign. In particular, the function $X_2 - X_1$ must vanish first before any W-intersection point can occur.

Replace (s_1, γ_{s_1}) by (s_2, ζ_{s_2}) in Definition 4.8 and Theorem 4.11. For $\theta_* > \pi$ we have $\lim_{s_2 \to \infty} \sharp C(\zeta_{s_2}) \geq 1$ by Corollary 5.8. On the other hand, from Proposition 4.12 we know that $X_2 - X_1 > 0$ along $\zeta_{1/(2m+6)}$ and hence $\sharp C(\zeta_{1/(2m+6)}) = 0$. By Theorem 4.11 there exists some $s_\bullet \in (1/(2m+6), \infty)$ such that $\zeta_{s_\bullet}$ is a heterocline.

Assume $\theta_* > \pi$ so that such an $s_\bullet$ exists. We claim that the Einstein metric $\hat{g}_{\mathbb{S}^{4m+4}}$ represented by $\zeta_{s_\bullet}$ is not the standard sphere metric. If $\hat{g}_{\mathbb{S}^{4m+4}}$ were the standard sphere metric, it would

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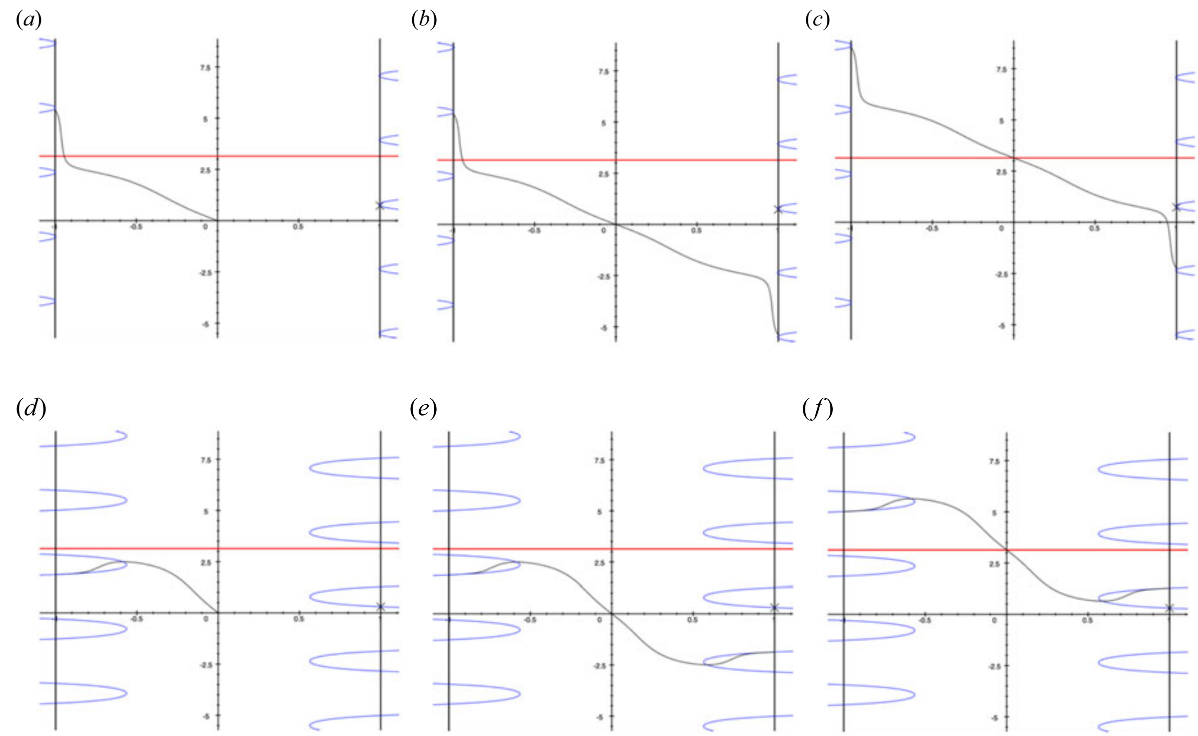

Figure 1. Each graph shows the horizontal line $\theta=\pi$ and the level curve $2\sqrt{(2m+2)(2m+3)}z_0-((n-1)/n)H\sin(\theta)\cos(\theta)=0$. The cross at $\{H=1\}$ is A_0^+. The plots show that for $m=1$, the graph of Π must be above $\Psi+(0,\pi)$ and hence $\theta_*>\pi$. On the other hand, for $m=5$, the graph of Π is below $\Psi+(0,\pi)$ and hence $\theta_*<\pi$. (a) $m=1,\ \Psi(\eta),\eta\geq 0$. (b) $m=1,\ \Psi(\eta),\eta\in\mathbb{R}$. (c) $m=1,\ \Psi(\eta)+(0,\pi),\eta\in\mathbb{R}$. (d) $m=5,\ \Psi(\eta),\eta\geq 0$. (e) $m=5,\ \Psi(\eta),\eta\in\mathbb{R}$. (f) $m=5,\ \Psi(\eta)+(0,\pi),\eta\in\mathbb{R}$.

have constant sectional curvature. In particular, we must have $(\ddot{f}_2/f_2)(f_1/\ddot{f}_1)=1$. From (3.7) we must have $s_2=0$. Hence, $\hat{g}_{\mathbb{S}^{4m+4}}$ is not the standard sphere metric. The proof is complete. $\square$

From Lemma 5.9 we learn that the number θ_* plays an important role in proving the existence of Einstein metrics on $\mathbb{HP}^{m+1}\sharp\overline{\mathbb{HP}}^{m+1}$ and $\mathbb{S}^{4m+4}$. We can apply the Runge–Kutta algorithm to estimate Π. Then one can only set the initial step near A_0^+, making the approximation less accurate as m increases. To bypass this issue, we make use of the symmetry of (5.7) and estimate θ_* using the fourth-order Runge–Kutta algorithm with a well-defined initial step.

Now consider the $H\theta$-plane. It is obvious that (5.7) admits $\mathbb{Z}_2$-symmetry in the sign of (H,θ). The system also admits translation symmetry $(H,\theta)\to(H,\theta+i\pi)$ for any $i\in\mathbb{Z}$. Let $O_i^\pm$ be either $A_i^\pm$ or $B_i^\pm$ and correspondingly, let $o_i^\pm$ be either $a_i^\pm$ or $b_i^\pm$. Let Ψ be the integral curve with the initial condition $\Psi(0)=(0,0)$. In general, the integral curve Ψ must converge to some O_i^- with $i\geq 1$. By symmetry, we know that Ψ is defined on $\mathbb{R}$ and joins O_{-i}^+ and O_i^-. Then $\Psi+(0,\pi)$ is an integral curve that joins O_{-i+1}^+ and O_{i+1}^- and passes through $(0,\pi)$, forming a barrier for estimating θ_*. In particular, if Ψ converges to O_1^-, then we have $o_1^-<\pi$. Then $\Psi+(0,\pi)$ passes $(0,\pi)$ and joins either A_0^+ and A_2^- or B_0^+ and B_2^-. In both cases, the integral curve Π passes through $(0,\theta_*)$ for some $\theta_*\leq\pi$. On the other hand, if Ψ converges to O_i^- with $i\geq 2$, then $\Psi+(0,\pi)$ passes $(0,\pi)$ and joins O_{-i+1}^+ and O_{i+1}^-. But $o_{-i+1}^+<a_0^-$, hence Π passes through $(0,\theta_*)$ for some $\theta_*>\pi$. We present two sets of graphs on $H\theta$-plane for Ψ and $\Psi+(0,\pi)$ in Figure 1 to illustrate our argument. Note that if Ψ converges to A_1^-, then $\Pi=\Psi+(0,\pi)$ and

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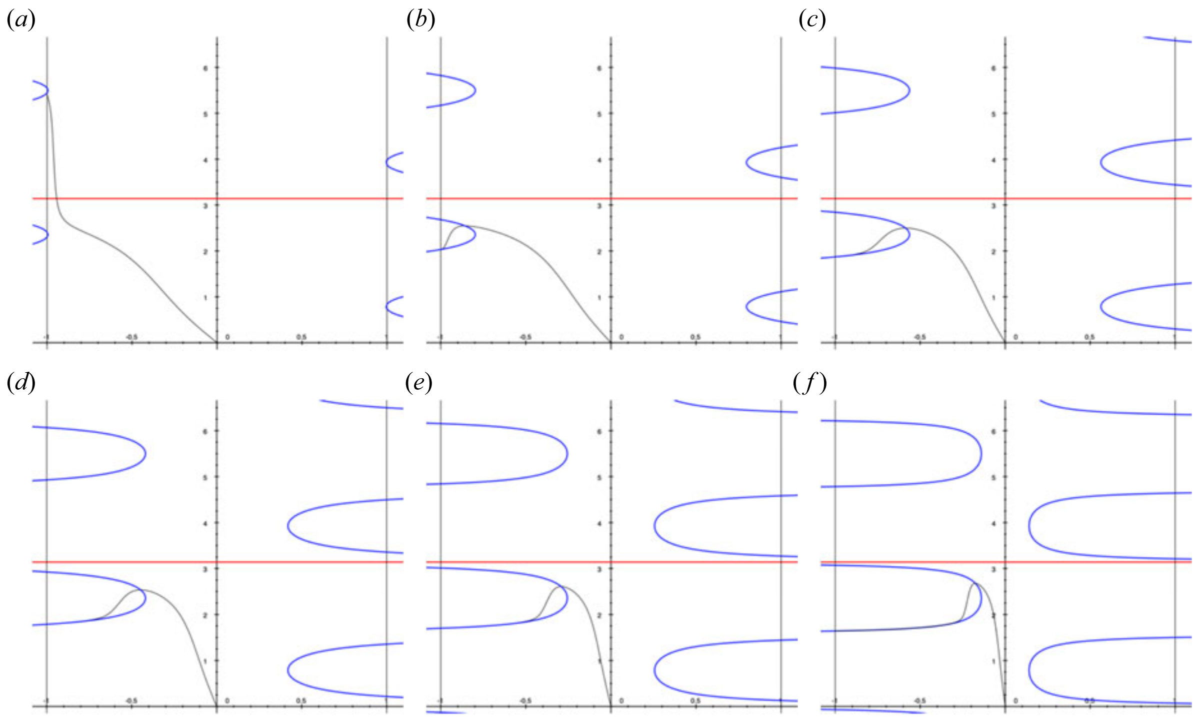

FIGURE 2. (a) $m = 1$. (b) $m = 2$. (c) $m = 5$. (d) $m = 10$. (e) $m = 29$. (f) $m = 100$.

we have $\theta_* = \pi$. In such a situation, a more delicate analysis is needed to obtain $\lim_{s_1\to\infty} \sharp C(\gamma_{s_1})$ and $\lim_{s_2\to\infty} \sharp C(\zeta_{s_2})$. In fact, Lemma 5.9 implies that as long as Ψ does not converge to A_1^- for a fixed m, we have either a second Einstein metric on $\mathbb{HP}^{m+1}\sharp\overline{\mathbb{HP}}^{m+1}$ or a new Einstein metric on $\mathbb{S}^{4m+4}$.

Fortunately, the fourth-order Runge–Kutta algorithm shows that for $m \in [2, 100]$, the integral curve Ψ converges to B_1^-. Hence, Π must pass $(0, \theta_*)$ for some $\theta_* < \pi$. Therefore, by Lemma 5.9, the second Einstein metric exists on $\mathbb{HP}^{m+1}\sharp\overline{\mathbb{HP}}^{m+1}$ for $m \in [2, 100]$. The function H in (5.7) can be solved explicitly. By Remark 2.4, it is clear that $H = -\tanh(2\eta/n)$. Hence, the above discussion can be summarized into a more compact statement as in Theorem 1.2. Since Ψ converges to one of the O_i^-, from (5.9) the inequality $\Omega < 3\pi/4$ in Theorem 1.2 essentially means that Ψ converges to B_1^-. As shown by the algorithm, for $m = 1$ the integral curve Ψ converges to B_2^-, meaning that Π must pass $(0, \theta_*)$ for some $\theta_* > \pi$. We show some plots of Ψ for different m on the $H\theta$-plane in Figure 2, generated by the fourth-order Runge–Kutta algorithm with step size 0.01 in Grapher.

In the following lemma, we prove that $\Omega > \pi$ for $m = 1$. Therefore, the inequality $\theta_* > \pi$ is indeed valid for $m = 1$.

LEMMA 5.10. *Let Ψ be an integral curve to the dynamical system*

$$\begin{bmatrix} H \\ \theta \end{bmatrix}' = \tilde{V}(H, \theta) := \begin{bmatrix} (H^2 - 1)\dfrac{1}{7} \\ \dfrac{4\sqrt{5}}{21} - \dfrac{3}{7} H \sin(2\theta) \end{bmatrix} \tag{5.11}$$

with $\Psi(0) = (0, 0)$. Then $\lim_{\eta\to\infty} \theta(\Psi) > \pi$.

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Proof. Let $H_1 = 2/(e^{\sqrt{5}\pi/6}+1)-1 \approx -0.527$ and $H_2 = 2/(\pi-2-2a_1^-) \approx -0.543$. Define the function

$$\Theta(H) = \begin{cases} \dfrac{1}{H}+1+a_1^-, & -1 \leq H \leq H_2, \\ \dfrac{\pi}{2}, & H_2 < H < H_1, \\ \dfrac{3\sqrt{5}}{5}\ln\left(\dfrac{1-H}{1+H}\right), & H_1 \leq H \leq 0. \end{cases}$$

Recall that in (5.8) we have $a_1^- = \pi - \arctan(2\sqrt{5}/5)$. Then we have

$$\sin(a_1^-) = \frac{2}{3}, \quad \cos(a_1^-) = -\frac{\sqrt{5}}{3}, \quad \sin(2a_1^-) = -\frac{4\sqrt{5}}{9}, \quad \cos(2a_1^-) = \frac{1}{9}.$$

The function Θ is non-increasing. The numbers H_1 and H_2 are chosen so that $\{\theta - \Theta(H) = 0\}$ is a continuous curve that joins the origin and A_1^-.

We show that $\tilde{V}$ restricted to $\{\theta - \Theta(H) = 0\}$ points upward. For $(H,\theta) \in (H_1, 0)\times(0,\pi/2)$, we have

$$\begin{aligned} \left\langle \nabla\left(\theta - \frac{3\sqrt{5}}{5}\ln\left(\frac{1-H}{1+H}\right)\right), \tilde{V}\right\rangle &= \frac{4\sqrt{5}}{21} - \frac{3}{7}H\sin(2\theta) + \frac{6\sqrt{5}}{5}\frac{1}{1-H^2}\frac{H^2-1}{7} \\ &= \frac{2\sqrt{5}}{105} - \frac{3}{7}H\sin(2\theta) \\ &> 0. \end{aligned} \tag{5.12}$$

Therefore, the function $\theta - \Theta(H)$ remains positive along Ψ as H decreases from 0 to H_1. The computation above also shows that the function $\theta - \Theta(H)$ is positive along $\Psi(\eta)$ once the integral curve leaves the origin. If $\theta = \pi/2$, then $\langle \nabla\theta, \tilde{V}\rangle = 4\sqrt{5}/21 > 0$. Hence, $\theta > \pi/2$ along Ψ as H decreases from H_1 to H_2.

Finally, as H decreases from H_2 to -1, the function $\Theta(H)$ increases from $\pi/2$ to a_1^-. We first claim that for $\theta \in [\pi/2, a_1^-]$, the inequality

$$D(\theta) := \sin(2\theta) - \left(\theta - \frac{\pi}{2}\right)\left(\frac{7}{4}(\theta - a_1^-) + \frac{\sin(2a_1^-)}{a_1^- - \pi/2}\right) \geq 0 \tag{5.13}$$

is valid. It is obvious that $D(a_1^-) = D(\pi/2) = 0$. Hence, there exists some $\theta_1 \in (\pi/2, a_1^-)$ such that $D'(\theta_1) = 0$ by the mean value theorem. On the other hand, we have

$$D'(\theta) = 2\cos(2\theta) - \left(\frac{7}{4}(\theta - a_1^-) + \frac{\sin(2a_1^-)}{a_1^- - \pi/2}\right) - \frac{7}{4}\left(\theta - \frac{\pi}{2}\right), \quad D''(\theta) = -4\sin(2\theta) - \frac{7}{2}.$$

Hence, $D'' > 0$ on $(\pi/2 + \frac{1}{2}\arcsin(\frac{7}{8}), \pi - \frac{1}{2}\arcsin(\frac{7}{8}))$. Since $\pi - \frac{1}{2}\arcsin(\frac{7}{8}) \approx 2.609 > 2.412 \approx a_1^-$, it is clear that D' decreases in $(\pi/2, \pi/2 + \frac{1}{2}\arcsin(\frac{7}{8}))$ and increases in $(\pi/2 + \frac{1}{2}\arcsin(\frac{7}{8}), a_1^-)$. Since

$$\begin{aligned} D'\left(\frac{\pi}{2}\right) &= -2 - \left(\frac{7}{4}\left(\frac{\pi}{2} - a_1^-\right) + \frac{\sin(2a_1^-)}{a_1^- - \pi/2}\right) \approx 0.654 > 0, \\ D'(a_1^-) &= 2\cos(2a_1^-) - \frac{\sin(2a_1^-)}{a_1^- - \pi/2} - \frac{7}{4}\left(a_1^- - \frac{\pi}{2}\right) \approx -0.068 < 0, \end{aligned} \tag{5.14}$$

we know that D' only vanishes once in $[\pi/2, a_1^-]$. Therefore, the function D is indeed positive for $\theta \in (\pi/2, a_1^-)$.

Then from (5.13) we have

$$
\begin{aligned}
&\left\langle \nabla\left(\theta - \frac{1}{H} - 1 - a_1^-\right), \tilde{V} \right\rangle\Big|_{\theta - 1/H - 1 - a_1^- = 0} \\
&\quad = \frac{4\sqrt{5}}{21} - \frac{3}{7} H \sin(2\theta) + \frac{1}{7H^2}(H^2 - 1) \\
&\quad \geq \frac{4\sqrt{5}}{21} - \frac{3}{7} H \left(\theta - \frac{\pi}{2}\right)\left(\frac{7}{4}(\theta - a_1^-) + \frac{\sin(2a_1^-)}{a_1^- - \pi/2}\right) + \frac{1}{7H^2}(H^2 - 1) \\
&\quad = \frac{4\sqrt{5}}{21} - \frac{3}{7} H \left(\frac{1+H}{H} + a_1^- - \frac{\pi}{2}\right)\left(\frac{7}{4}\left(\frac{1+H}{H}\right) + \frac{\sin(2a_1^-)}{a_1^- - \pi/2}\right) + \frac{1}{7H^2}(H^2 - 1) \\
&\qquad \text{since } \theta - \frac{1}{H} - 1 - a_1^- = 0 \\
&\quad = \frac{4\sqrt{5}}{21}(1+H) - \left(\frac{3}{4}\left(a_1^- - \frac{\pi}{2}\right) + \frac{3}{7}\frac{\sin(2a_1^-)}{a_1^- - \pi/2}\right)(1+H) - \frac{3}{4}\frac{(1+H)^2}{H} + \frac{1}{7H^2}(H^2 - 1).
\end{aligned} \tag{5.15}
$$

Since $4\sqrt{5}/21 - (\frac{3}{4}(a_1^- - \pi/2) + \frac{3}{7}(\sin(2a_1^-)/(a_1^- - \pi/2))) \approx 0.3015 > 0.3$, the computation above continues as

$$
\begin{aligned}
\left\langle \nabla\left(\theta - \frac{1}{H} - 1 - a_1^-\right), \tilde{V} \right\rangle\Big|_{\theta - 1/H - 1 - a_1^- = 0} &\geq \frac{3}{10}(1+H) - \frac{3}{4}\frac{(1+H)^2}{H} + \frac{1}{7H^2}(H^2 - 1) \\
&= \frac{1+H}{H^2}\left(-\frac{9}{20}H^2 - \frac{17}{28}H - \frac{1}{7}\right).
\end{aligned} \tag{5.16}
$$

A straightforward computation shows that the factor $-\frac{9}{20}H^2 - \frac{17}{28}H - \frac{1}{7}$ is positive on $[-1, -\frac{1}{2}]$. As $H_2 < -\frac{1}{2}$, it is proved that Ψ does not pass the barrier $\theta - \Theta(H) = 0$ where $H \in (-1, H_2)$. Therefore, we must have $\lim_{\eta\to\infty} \theta(\Psi) \geq a_1^-$.

We claim that Ψ does not converge to A_1^-. The linearization of (5.11) at A_1^- is $\begin{bmatrix} -\frac{2}{7} & 0 \\ \frac{4\sqrt{5}}{21} & \frac{2}{21} \end{bmatrix}$, whose only stable eigenvalue and eigenvector are respectively $-\frac{2}{7}$ and $\begin{bmatrix} 2 \\ -\sqrt{5} \end{bmatrix}$. Hence, the linearized solution in $\{H > -1\}$ takes the form $A_1^- + \begin{bmatrix} 2 \\ -\sqrt{5} \end{bmatrix} e^{-(2/7)\eta}$. Suppose Ψ is the integral curve that tends to A_1^-. We must have

$$
(\theta - \Theta(H))(\Psi(\eta)) \sim (\theta - \Theta(H))\left(A_1^- + \begin{bmatrix} 2 \\ -\sqrt{5} \end{bmatrix} e^{-(2/7)\eta}\right) = -e^{-(2/7)\eta}\left(\sqrt{5} - \frac{2}{1 - 2e^{-(2/7)\eta}}\right) < 0
$$

as $\eta \to \infty$, which is a contradiction. Therefore, the integral curve Ψ converges to some O_i^- with $i \geq 2$. As $A_2^- > B_2^- > \pi$, we conclude that $\lim_{\eta\to\infty} \theta(\Psi) > \pi$. □

Theorem 1.3 is proved with Lemma 5.10 established.

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Appendix A.

A.1 Coefficients for $\tilde{B}_{m,\kappa}$ and $\tilde{Q}_{m,\kappa}$

We list coefficients for $\tilde{B}_{m,\kappa}$ and $\tilde{Q}_{m,\kappa}$:

– $\tilde{B}_{m,\kappa}(x) = b_2x^2 + b_1x + b_0$

$$\begin{aligned} b_2 &= -\frac{96\kappa^3m^3 + 264\kappa^2m^2 + 216\kappa m + 54}{(1-\kappa)(2\kappa^2m(8m-5) + 6\kappa(4m-1) + 9)} < 0, \\ b_1 &= \frac{4(m+2)(4\kappa m+3)(2\kappa^2m + 4\kappa m + 3)}{(1-\kappa)(2\kappa^2m(8m-5) + 6\kappa(4m-1) + 9)} - \frac{2(4m+3)^2(2m+3)(m-1)}{m(2m+1)(8m+3)}, \\ b_0 &= -\frac{3(4 + (4m-2)\kappa)(4\kappa m+3)\kappa}{(1-\kappa)(2\kappa^2m(8m-5) + 6\kappa(4m-1) + 9)} < 0. \end{aligned} \tag{A.1}$$

– $\tilde{Q}_{m,\kappa}(x) = q_2x^2 + q_1x + q_0$

$$\begin{aligned} q_2 &= -\frac{4(2\kappa m+3)(32\kappa^3m^3 + \kappa^2m^2(96-68\kappa) + \kappa m(90-84\kappa) + 27(1-\kappa))}{3(1-\kappa)(2\kappa^2m(8m-5) + 6\kappa(4m-1) + 9)} < 0, \\ q_1 &= \frac{16\kappa(m+2)(16\kappa^3m^3 - 18\kappa^3m^2 + 32\kappa^2m^2 - 24\kappa^2m + 27\kappa m - 9\kappa + 9)}{3(1-\kappa)(2\kappa^2m(8m-5) + 6\kappa(4m-1) + 9)} > 0, \\ q_0 &= -\frac{4\kappa^2(32\kappa^2m^3 - 20\kappa^2m^2 - 6\kappa^2m + 48\kappa m^2 - 6\kappa m - 9\kappa + 18m + 9)}{3(1-\kappa)(2\kappa^2m(8m-5) + 6\kappa(4m-1) + 9)} < 0. \end{aligned} \tag{A.2}$$

Proposition A.1. *For any $(m,\kappa) \in [1,\infty) \times (0,1)$, the function $\tilde{B}_{m,\kappa}$ has a real root $\sigma(m,\kappa)$ in the interval $(0, m\kappa/(3+2m\kappa))$. For $m=1$, polynomials $\tilde{B}_{1,\kappa}$ and $\tilde{Q}_{1,\kappa}$ share a common real root $\sigma(1,\kappa) = \kappa/(3+2\kappa)$.*

Proof. From (A.1), computations show that

$$\begin{aligned} &\tilde{B}_{m,\kappa}(0) = b_0 < 0, \\ &\tilde{B}_{m,\kappa}\left(\frac{m\kappa}{3+2m\kappa}\right) = \frac{768(m-1)\kappa B_*}{(8m+3)(2\kappa m+3)(2m+1)(1-\kappa)(2\kappa^2m(8m-5) + 6\kappa(4m-1) + 9)} \end{aligned} \tag{A.3}$$

where

$$\begin{aligned} B_* = &\left(m^4 + \frac{15}{8}m^3 + \frac{5}{16}m^2 - \frac{45}{32}m - \frac{45}{64}\right)m\kappa^3 + \left(\frac{13}{4}m^4 + \frac{163}{32}m^3 + \frac{255}{64}m^2 + \frac{9}{16}m - \frac{27}{64}\right)\kappa^2 \\ &+ \left(\frac{21}{8}m^3 + \frac{147}{64}m^2 + \frac{243}{128}m + \frac{27}{32}\right)\kappa + \frac{21}{32}m^2 - \frac{27}{128} \\ > 0 \end{aligned} \tag{A.4}$$

for any $(m,\kappa) \in [1,\infty) \times (0,1)$. Hence, such a $\sigma(m,\kappa)$ exists. Furthermore, we have

$$\begin{aligned} \tilde{B}_{1,\kappa}(x) &= \frac{2(3+4\kappa)((2\kappa+1)x - \kappa - 2)((2\kappa+3)x - \kappa)}{(2\kappa^2 + 6\kappa + 3)(\kappa - 1)}, \\ \tilde{Q}_{1,\kappa}(x) &= -\frac{4((12\kappa^3 - 4\kappa^2 - 21\kappa - 9)x + 2\kappa^3 + 11\kappa^2 + 9\kappa)((2\kappa+3)x - \kappa)}{3(2\kappa^2 + 6\kappa + 3)(\kappa - 1)}. \end{aligned} \tag{A.5}$$

The proof is complete. ☐

A.2 Non-positivity of $r(\tilde{Q}_{m,\kappa}, \tilde{B}_{m,\kappa})$

PROPOSITION A.2. *The resultant* $r(\tilde{Q}_{m,\kappa}, \tilde{B}_{m,\kappa})$ *in* (4.17) *is non-positive for any* $(m,\kappa) \in [1,\infty) \times (0,1)$ *and vanishes if and only if* $m = 1$.

Proof. Recall that

$$\begin{aligned} r(\tilde{Q}_{m,\kappa}, \tilde{B}_{m,\kappa}) &= -\frac{64\kappa^2(m-1)(2\kappa m+3)}{(8m+3)^2(2m+1)^2m^2(1-\kappa)(2\kappa^2m(8m-5)+6\kappa(4m-1)+9)^2}\tilde{r}, \\ \tilde{r} &= 262144\kappa^4m^{10} + (516096\kappa^4 + 679936\kappa^3)m^9 \\ &\quad + (373760\kappa^4 + 1233920\kappa^3 + 675840\kappa^2)m^8 \\ &\quad + (-275904\kappa^4 + 1151040\kappa^3 + 1143744\kappa^2 + 308160\kappa)m^7 \\ &\quad + (-926496\kappa^4 + 248832\kappa^3 + 1432512\kappa^2 + 507456\kappa + 54432)m^6 \\ &\quad + (-800496\kappa^4 - 1256256\kappa^3 + 1472688\kappa^2 + 857520\kappa + 92016)m^5 \\ &\quad + (-281880\kappa^4 - 1525392\kappa^3 + 266328\kappa^2 + 1353024\kappa + 199584)m^4 \\ &\quad + (-33048\kappa^4 - 644436\kappa^3 - 672624\kappa^2 + 957420\kappa + 375192)m^3 \\ &\quad + (-90396\kappa^3 - 433026\kappa^2 + 186624\kappa + 333882)m^2 \\ &\quad + (-74358\kappa^2 - 59049\kappa + 133407)m + 19683(1-\kappa). \end{aligned} \tag{4.17}$$

It is clear that the coefficient for $\tilde{r}$ is non-positive on $[1,\infty) \times (0,1)$ and vanishes if and only if $m = 1$.

Consider the polynomial $\tilde{r}$. Since coefficients for m^i are obviously positive for $k \in (0,1)$ if $i \geq 5$, we must have

$$\begin{aligned} \tilde{r} &> 262144\kappa^4m^4 + (516096\kappa^4 + 679936\kappa^3)m^4 + (373760\kappa^4 + 1233920\kappa^3 + 675840\kappa^2)m^4 \\ &\quad + (-275904\kappa^4 + 1151040\kappa^3 + 1143744\kappa^2 + 308160\kappa)m^4 \\ &\quad + (-926496\kappa^4 + 248832\kappa^3 + 1432512\kappa^2 + 507456\kappa + 54432)m^4 \\ &\quad + (-800496\kappa^4 - 1256256\kappa^3 + 1472688\kappa^2 + 857520\kappa + 92016)m^4 \\ &\quad + (-281880\kappa^4 - 1525392\kappa^3 + 266328\kappa^2 + 1353024\kappa + 199584)m^4 \\ &\quad + (-33048\kappa^4 - 644436\kappa^3 - 672624\kappa^2 + 957420\kappa + 375192)m^3 \\ &\quad + (-90396\kappa^3 - 433026\kappa^2 + 186624\kappa + 333882)m^2 \\ &\quad + (-74358\kappa^2 - 59049\kappa + 133407)m + 19683(1-\kappa) \\ &= (-1132776\kappa^4 + 532080\kappa^3 + 4991112\kappa^2 + 3026160\kappa + 346032)m^4 \\ &\quad + (-33048\kappa^4 - 644436\kappa^3 - 672624\kappa^2 + 957420\kappa + 375192)m^3 \\ &\quad + (-90396\kappa^3 - 433026\kappa^2 + 186624\kappa + 333882)m^2 \\ &\quad + (-74358\kappa^2 - 59049\kappa + 133407)m + 19683(1-\kappa) \\ &\geq (-1132776\kappa^4 + 532080\kappa^3 + 4991112\kappa^2 + 3026160\kappa + 346032)m^3 \\ &\quad + (-33048\kappa^4 - 644436\kappa^3 - 672624\kappa^2 + 957420\kappa + 375192)m^3 \end{aligned}$$

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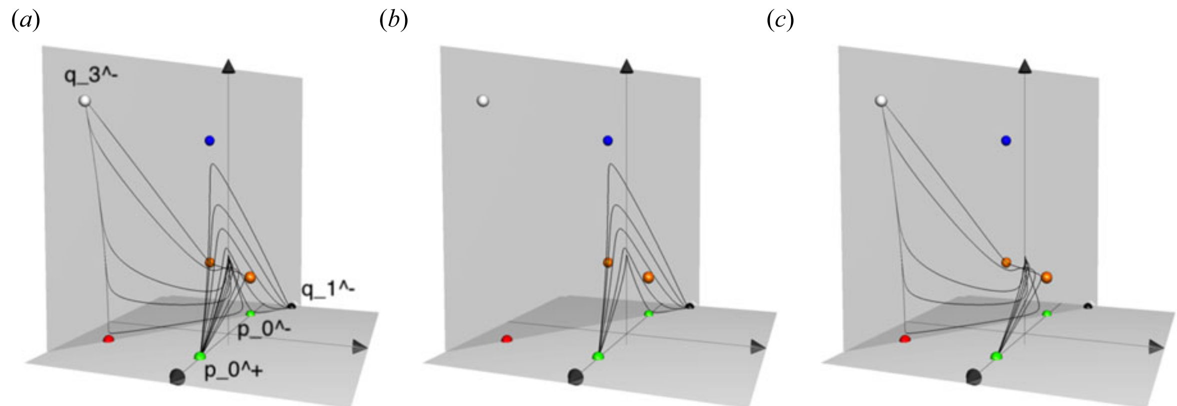

Figure A.1. The set $\mathcal{S}$ is degenerate as shown in Proposition 4.2. Hence, only one integral curve $\gamma_{s_\star}$ is known to join $p_0^\pm$ and it represents the Bohm's metric. The graphs indicate that γ_{s_1} converges to q_1^- for $s_1 \in (0, s_\star)$ and converges to q_3^- for $s_1 \in (s_\star, \infty)$. (a) γ_{s_1} for $m = 1$. (b) $s_1 \in (0, s_\star)$. (c) $s_2 \in (s_\star, \infty)$.

$$
\begin{aligned}
&+ (-90396\kappa^3 - 433026\kappa^2 + 186624\kappa + 333882)m^2 \\
&+ (-74358\kappa^2 - 59049\kappa + 133407)m + 19683(1-\kappa) \\
={}& (-1165824\kappa^4 - 112356\kappa^3 + 4318488\kappa^2 + 3983580\kappa + 721224)m^3 \\
&+ (-90396\kappa^3 - 433026\kappa^2 + 186624\kappa + 333882)m^2 \\
&+ (-74358\kappa^2 - 59049\kappa + 133407)m + 19683(1-\kappa) \\
\geq{}& (-1165824\kappa^4 - 112356\kappa^3 + 4318488\kappa^2 + 3983580\kappa + 721224)m^2 \\
&+ (-90396\kappa^3 - 433026\kappa^2 + 186624\kappa + 333882)m^2 \\
&+ (-74358\kappa^2 - 59049\kappa + 133407)m + 19683(1-\kappa) \\
={}& (-1165824\kappa^4 - 202752\kappa^3 + 3885462\kappa^2 + 4170204\kappa + 1055106)m^2 \\
&+ (-74358\kappa^2 - 59049\kappa + 133407)m + 19683(1-\kappa) \\
>{}& 0.
\end{aligned}
\tag{A.6}
$$

Since $\tilde{r}$ is positive on $[1, \infty) \times (0, 1)$, the proof is complete. □

A.3 Visual summaries

We summarize Theorem 1.1–1.3 in Figures A.1–A.4 generated by Grapher, where integral curves presented are generated by the fourth-order Runge–Kutta algorithm with step size 0.01 and the initial step is set in a neighborhood around p_0^+ or p_1^+. All figures are in the X_1X_2Z-space and the variable Y is eliminated by (2.8).

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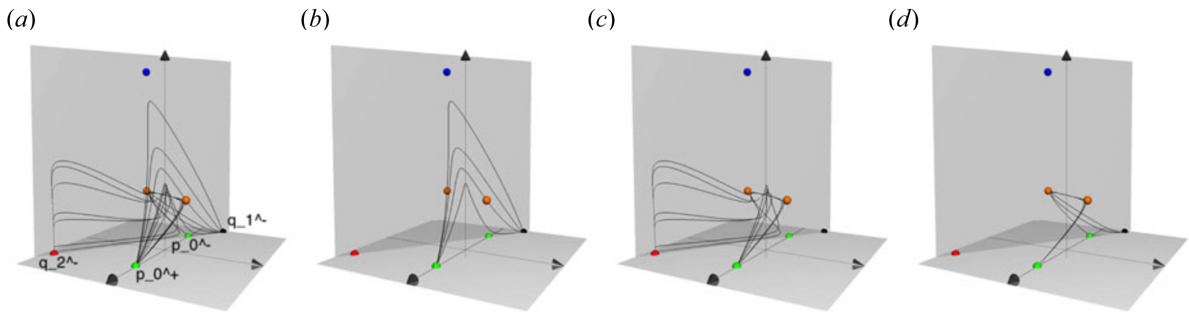

Figure A.2. Theorems 1.1 and 1.2 claim that for $m \geq 2$ there are at least two integral curves $\gamma_{s_\star}$ and $\gamma_{s_{\star\star}}$ that join $p_0^{\pm}$. The graphs indicate that γ_{s_1} converges to q_1^- for $s_1 \in (0, s_\star) \cup (s_{\star\star}, \infty)$ and converges to q_2^- for $s_1 \in (s_\star, s_{\star\star})$. (a) γ_{s_1} for $m = 2$. (b) $s_1 \in (0, s_\star)$. (c) $s_2 \in (s_\star, s_{\star\star})$. (d) $s_2 \in (s_{\star\star}, \infty)$.

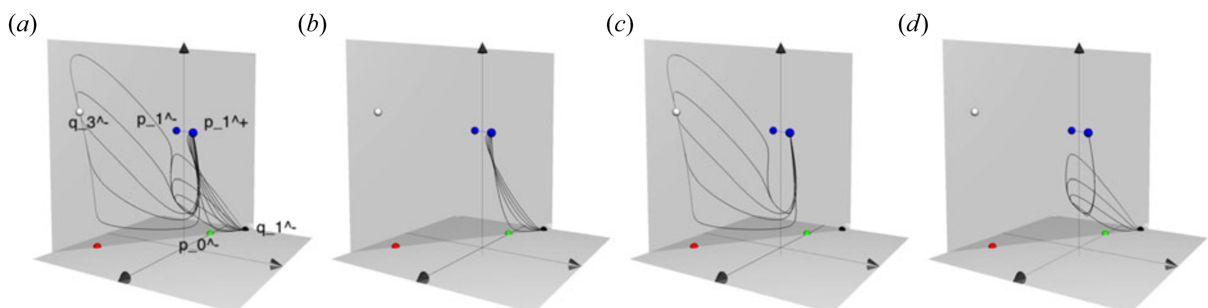

Figure A.3. These plots are a realization of Theorem 1.3. For $m = 1$, the graph of ζ_0 is the straight line that joins $p_1^{\pm}$. As s_2 increases from 0, the integral curve ζ_{s_2} converges to q_1^-, until $\zeta_{1/(2m+6)}$ converges to p_0^-. For $s_2 > 1/(2m+6)$, the integral curve ζ_{s_2} converges to q_3^-, until $s_2 = s_\bullet$ once again joins $p_1^{\pm}$. For the $s_2 > s_\bullet$, the integral curve ζ_{s_2} converges again to q_1^-. (a) ζ_{s_2} for $m = 1$. (b) $s_2 \in (0, 1/(2m+6))$. (c) $s_2 \in (1/(2m+6), s_\bullet)$. (d) $s_2 \in (s_\bullet, \infty)$.

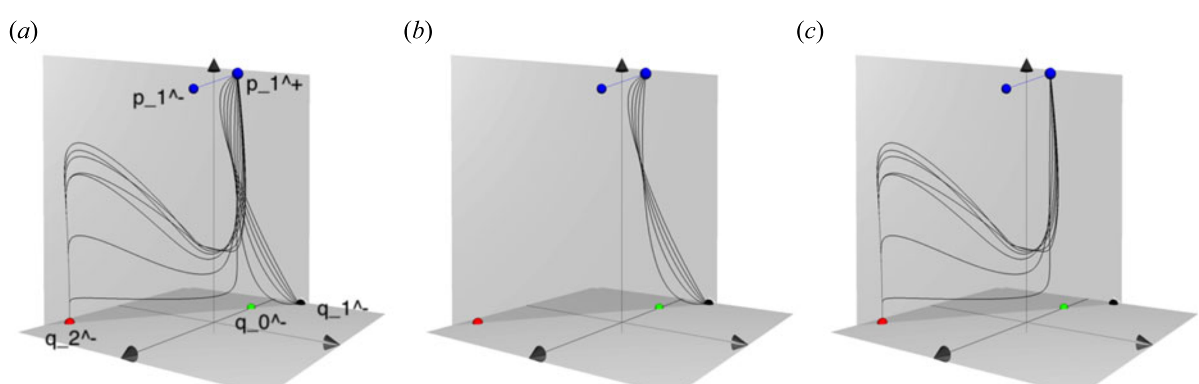

Figure A.4. For $m \geq 2$, the behavior of ζ_{s_2} is relatively simpler. For $s_2 \in (0, 1/(2m+6))$, the integral curve ζ_{s_2} converges to q_1^-. For $s_2 > 1/(2m+6)$, the integral curve ζ_{s_2} converges to q_2^-. (a) ζ_{s_2} for $m = 1$. (b) $s_2 \in (0, 1/(2m+6))$. (c) $s_2 \in (1/(2m+6), \infty)$.

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Conflicts of Interest
None.

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