ON DETOURS IN GRAPHS[1]

S.F. Kapoor, H.V. Kronk and D.R. Lick

(received December 10, 1967)

1. Introduction. A path of maximum length in a graph G is referred to as a detour path of G and the length of such a path is called the detour number of G. It is not surprising that the study of detour paths is closely associated with the problem of investigating hamiltonian paths in graphs. Evidently few results have been obtained in this area, although Ore [3] has shown that any two detour paths intersect. It is the purpose of this article to further investigate these concepts. In particular, we obtain bounds for several graph-theoretic parameters in terms of the detour number and also present formulae for the detour numbers of several important classes of graphs.

2. Basic definitions and preliminary results. Let G be a connected graph and u and v any two points of G. The distance between u and v is defined to be the length of a shortest path between u and v with endpoints u and v and is denoted $d(u, v)$. Let $\nabla(u, v)$ denote a path between u and v having maximum length. Such a path is called a detour path between u and v and its length is denoted by $\partial(u, v)$. The distance functions d and ∂ are metrics on the point set V of G. For any point u in G, we define $\partial(u) = \max_{v \in V} \partial(u, v)$. By a detour path in G is meant a path in G of maximum length. The length of a detour path in G denoted $\partial(G)$, is called the detour number of G, i.e. $\partial(G) = \max\{\partial(u): u \in V\}$. For example, if G is any graph on p points having a hamiltonian path, then $\partial(G) = p - 1$. It is also easy to see that for any connected graph G having $p \geq 3$ points, $\partial(G) \geq 2$, with equality holding if and only if G is a triangle or a star.

Our first result gives a bound for the detour number of a graph in terms of the minimum degree of its points. The method of proof we use is essentially due to Ore [3].

[1] The authors thank Professor Gary Chartrand for his helpful suggestions.

https://doi.org/10.4153/CMB-1968-022-8 Published online by Cambridge University Press

PROPOSITION 1. _If_ G _is a connected graph with_ p _points having minimum degree_ r, _then_ $\partial(G) \geq \min(p - 1, 2r)$.

Proof. If $\partial(G) = p - 1$, the result clearly follows. Thus we assume $\partial(G) < \min(p - 1, 2r)$. Let P be a path of length $\partial(G) = \partial$ whose points are successively $v_o, v_1, \ldots, v_\partial$. The subgraph G' induced by the set of points of P cannot contain a cycle having all points of G', for otherwise there would necessarily exist a point v not in G' adjacent with some point v_i in G' producing a path of length $\partial + 1$. Similarly, v_o and v_∂ are adjacent only to points of G', but not to each other. By hypotheses, the sum of the degrees of v_o and v_∂ is at least 2r. Since $\partial(G) < 2r$, there must exist points v_{i-1} and v_i in G, where v_i is adjacent to v_o and v_{i-1} is adjacent to v_∂. This however, implies the existence of the cycle $v_o v_i v_{i+1} \ldots v_\partial v_{i-1} v_{i-2} \ldots v_1 v_o$ which contains all points of G', but we have seen that this is impossible. Therefore a contradiction arises and the desired result follows.

Since every n-connected graph has minimum degree at least n, we obtain the following corollary.

COROLLARY 1a. _If_ G _is an n-connected graph with_ p _points_, _then_ $\partial(G) \geq \min(p - 1, 2n)$.

As with the metric d, one can define a radius and centre with respect to ∂. The _detour radius_ of a graph G, denoted $r_\partial(G)$, is defined to be the number $\min_{u \in V} \partial(u)$ and the set $C_\partial(G) = \{v \in V \mid \partial(v) = r_\partial(G)\}$ is called the _detour centre_ of G.

A _block of a graph_ G is a maximal connected subgraph of G containing no cutpoints.

PROPOSITION 2. _If_ G _is a connected graph_, _then_ $C_\partial(G)$ _lies in a block of_ G.

Proof. Let G be a connected graph with detour number $\partial(G) = \partial$, and assume that $C_\partial(G)$ fails to lie in any block of G. Then G has a cutpoint v with the property that at least two components G_1 and G_2 of the graph G - v (obtained from G by deleting v and all incident lines) contain points of $C_\partial(G)$. Let P_1 be a path of length $\partial(v)$ having an endpoint at v. Since v is a cutpoint, at least one of the subgraphs G_1 and G_2, say G_2, contains no points of P_1. Let u be a point of G_2 belonging to $C_\partial(G)$. If P_2 is a path having endpoints

https://doi.org/10.4153/CMB-1968-022-8 Published online by Cambridge University Press

at u and v, then the paths P_1 and P_2 determine a path P_3 having length exceeding $\partial(v)$, i.e., $\partial(u) > \partial(v)$. This, however, contradicts the fact that u belongs to $C_\partial(G)$. Thus $C_\partial(G)$ is contained in a block of G.

Since every block of a tree contains two points, we obtain the following.

COROLLARY 2a. The detour centre of a tree consists of one point or two adjacent points.

It is an elementary observation that for a tree, the detour number equals its diameter, every detour path is a diametrical path, and the detour centre coincides with the centre. Such is not the case in general however. For the graph of Figure 1, $\{c_1, c_2, c_3\}$ constitutes the centre and $\{d_1, d_2\}$ the detour centre.

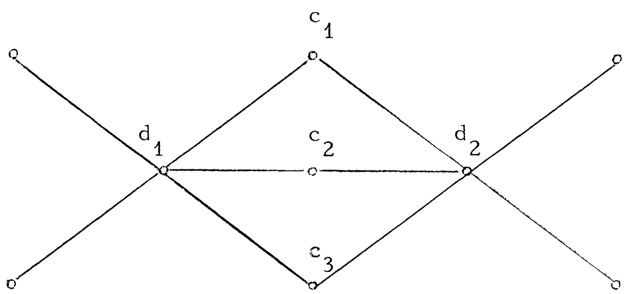

Figure 1

3. Detour paths and Hamiltonian graphs. A graph with a hamiltonian path is called traceable while a hamiltonian graph is one containing a hamiltonian cycle. Clearly, every hamiltonian graph is traceable. It is also immediate that if G is a traceable graph with p points, then $\partial(G) = p - 1$, and conversely.

A graph G is detour-connected if for every two distinct points u and v of G, there exists a detour path with u and v as endpoints. If G is a detour-connected, traceable graph, then every pair of points are joined by a hamiltonian path. Such graphs are called hamiltonian-connected and have been studied by Ore [4].

PROPOSITION 3. For any graph G, G is detour-connected if and only if it is hamiltonian-connected.

Proof. It is obvious that every hamiltonian-connected graph is detour-connected. For the converse, let G be a detour-connected graph. If G has only two points, then certainly G is hamiltonian-connected. If G has more than two points, then let u and v be any

https://doi.org/10.4153/CMB-1968-022-8 Published online by Cambridge University Press

two adjacent points of G. Since G is detour-connected, there exists a detour path P having u and v as endpoints. We now claim that P contains all points of G implying G is traceable and therefore hamiltonian-connected. To see this consider the cycle C determined by the path P and the line uv. If C does not contain all points of G, then, since G is connected, there exists a point w not on C but adjacent with a point of C. This produces a path of length one greater than that of P, which leads to a contradiction. Thus C and therefore P contains all points of G.

The preceding proof also provides the following corollary.

COROLLARY 3a. _If G is a detour-connected graph with_ $p \geq 3$ _points, then G is hamiltonian._

4. _The detour number and other graph-theoretic parameters._ A graph G is _homeomorphic from_ a graph H if it is possible to obtain G from H by inserting new points of degree 2 into lines of H. In [1], a graph was said to have property P_n, $n \geq 1$, if it fails to contain a subgraph homeomorphic from either the complete graph K_{n+1} or the complete bipartite graph $K([\frac{n+2}{2}], \{\frac{n+2}{2}\})$, where $[x]$ and $\{x\}$ denote the greatest integer not exceeding x and the least integer not less than x, respectively. As was shown in [1], the first 4 values of n correspond to totally disconnected graphs, forests, outerplanar graphs, and planar graphs.

For each graph G and positive integer n there is associated a number $\chi^{(n)}(G)$, defined as the minimum number of subsets into which the point set of G may be partitioned so that each subset induces a subgraph with property P_n. For n = 1, 2, 3, 4, these parameters have been referred to as chromatic number, point-arboricity, point-outerthickness, and point-thickness. It is possible to give bounds for all of these parameters in terms of the detour number.

PROPOSITION 4. _For any graph G and positive integer n,_

$$\chi^{(n)}(G) \leq 1 + [\frac{\partial(G)}{n}].$$

Proof. Let n be an arbitrary but fixed positive integer. If the graph G has property P_n, then $\chi^{(n)}(G) = 1$ so that the desired inequality holds. Otherwise we proceed as follows.

Let V_1 be a set with a minimum number of points such that the subgraph $G - V_1$ of G has property P_n. Let G_1 be the subgraph of G induced by V_1.

We now claim that $\partial(G_1) \leq \partial(G) - n$, for let Q be a path in G_1 of length $\partial(G_1)$ having endpoints u and v. If v, say, is added to the point set of $G - V_1$, the resulting induced subgraph necessarily does not have property P_n due to the minimality of V_1. In this subgraph then v belongs either to a subgraph homeomorphic from K_{n+1} or one homeomorphic from $K([\frac{n+2}{2}], \{\frac{n+2}{2}\})$. In either case there exists a path Q' of length n with v as endpoint. Hence Q and Q' determine a path Q'' of length $\partial(G_1) + n$ so that $\partial(G_1) \leq \partial(G) - n$.

Next let V_2 be a set with the minimum number of points such that the subgraph $G_1 - V_2$ of G_1 has property P_n. Also let G_2 be the subgraph of G_1 induced by V_2. As before, we have $\partial(G_2) \leq \partial(G_1) - n \leq \partial(G) - 2n$.

We continue the above procedure until finally arriving at a subgraph G_k for which $\partial(G_k) < n$. We also have $\partial(G_k) \leq \partial(G) - kn$. Clearly, G_k has property P_n. Thus each of the subgraphs $G - V_1, G_1 - V_2, \ldots, G_{k-1} - V_k, G_k$ has property P_n. Therefore $\chi^{(n)}(G) \leq k + 1$. On the other hand, $\partial(G) \geq \partial(G_k) + kn$ so that $1 + [\frac{\partial(G)}{n}] \geq k + 1$. This completes the proof.

5. <u>Complete n-partite graphs</u>. The <u>complete n-partite graph</u> $K(p_1, p_2, \ldots, p_n)$, $p_1 \leq p_2 \leq \ldots \leq p_n$, has its point set V partitioned into n subsets V_i, where $|V_i| = p_i$ and $\sum_{i=1}^{n} p_i = p$ and such that two points u and v are adjacent if and only if $u \in V_j$ and $v \in V_k$, $j \neq k$. The class of complete n-partite graphs contains such familiar graphs as the complete graphs and the complete bipartite graphs.

PROPOSITION 5. <u>If the graph</u> $G = K(p_1, p_2, \ldots, p_n)$, <u>then</u> $\partial(G) = \min(2p - 2p_n, p - 1)$.

<u>Proof</u>. We consider two cases.

<u>Case 1</u>. $p - p_n \geq p_n - 1$. This implies that $p - p_n \geq (p-1)/2$. Clearly, $\min \deg G = \sum_{i=1}^{n-1} p_i = p - p_n \geq (p-1)/2$. Thus by Proposition 1, $\partial(G) = p - 1$.

Case 2. $p - p_n < p_n - 1$. In this case $\min(2p - 2p_n, p - 1) = 2p - 2p_n$. Since the number of points in V_n exceeds the number of points in $V - V_n$ by at least two, there exists a path P beginning and ending in V_n such that every point in $V - V_n$ is on P. The length of such a path P is $2p - 2p_n$. Thus $\partial(G) \geq 2p - 2p_n$. If $\partial(G) > 2p - 2p_n$, then there exists a path P' having at least $2p - 2p_n + 2$ points. However, at least $p - p_n + 2$ of these points must belong to V_n, implying that P' contains two consecutive points from V_n, which is contrary to the definition of G.

COROLLARY 5a. The graph $G = K(p_1, p_2, \ldots, p_n)$ is traceable if and only if $p \geq 2p_n - 1$.

6. Unsolved problems. As mentioned in the introduction, Ore has shown that every two detour paths in a graph intersect. If G is a traceable graph with p points; i.e., if $\partial(G) = p - 1$, then all p points of G belong to each detour path. It is natural to inquire whether all detour paths intersect if $\partial(G) < p - 1$. In particular, if $\partial(G) = p - 2$ and all the detour paths have a point in common, this implies that no graph is hypo-traceable. (A graph G with p points is hypo-traceable if it is not traceable, but every induced subgraph with $p - 1$ points is traceable.) Hypo-hamiltonian graphs are known to exist and have been investigated by Herz, Duby and Vigué [2].

The graph in Figure 2 shows that a diametrical path (namely, $v_1v_2v_3v_4v_5$) need not contain points of the centre. Whether the analogous situation holds for detour paths and the detour centres is unknown.

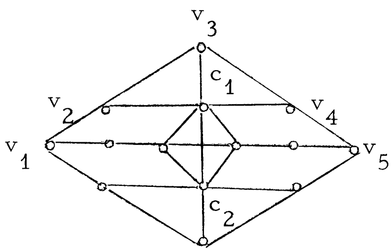

Figure 2

Added in proof. H. Walther has constructed a graph in which all the detour paths don't have a point in common. His construction will appear in the Journal of Combinatorial Theory.

REFERENCES

1. G. Chartrand, D. Geller and S. Hedetniemi, Graphs with forbidden subgraphs. To appear.

2. J. C. Herz, J. J. Duby and F. Vigué, Recherche systématique des graphes hypohamiltoniens. Théorie des graphes, journées internationales d'études. (Dunod, Paris, 1967) 153-159.

3. O. Ore, Theory of graphs (Chapters 2, 3). Amer. Math. Soc. Colloq. Publ., 38 (Providence, 1962).

4. O. Ore, Hamiltonian connected graphs. J. Math. Pures Appl. 42 (1963) 21-27.

Western Michigan University
SUNY at Binghamton

https://doi.org/10.4153/CMB-1968-022-8 Published online by Cambridge University Press