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EASTON'S RESULTS VIA ITERATED BOOLEAN-VALUED EXTENSIONS

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The purpose of this article is to show how the main result of Easton [**1**] can be obtained as a special case of a general theory, which was developed in [**6**], of Boolean-valued models of ZF when the Boolean algebra is a proper class in the ground model. Indeed [**1**] was the motivating example for [**6**]. Thus the present article together with [**6**] contain a presentation of Easton's forcing argument in the context of Boolean-valued models. This presentation is not, however, an automatic translation of Easton's argument from the language of forcing to that of Boolean-valued models. In fact, we hope to illuminate the "black magic" referred to in Rosser [**8**, p. 169].

In [**1**], the model is presented using a ramified language, and the techniques of forcing are extended to accommodate the specific proper class of forcing conditions required. The essential properties of this class are pooled together in Lemma 25 of [**1**], called "Easton's Lemma", which is then used to prove that the Axioms of Replacement and Power Set are satisfied and to derive the desired facts about cardinal arithmetic in the extended model.

Our approach can be described as a generalization in two directions of the techniques of Solovay and Tennenbaum [**12**] involving iterated Boolean-valued extensions. In one direction, iteration is performed through On-many stages. In another direction, the algebra used at limit stages is not required, as in [**12**], to be the Dedekind-Mac Neille completion of the direct limit of the algebras used at preceding stages; this second generalization is inspired by ideas of Takeuti [**13**] which amount to iterating the $V^{(\Gamma)}$ construction rather than the $V^{(\mathscr{B})}$ construction.

From our point of view, we find that the $\mathscr{B}$-validity of the Axiom of Replacement follows from the very construction of the model as a $\mathscr{B}$-valued extension for a $\mathscr{B}$ which arises from a strongly homogeneous direct system of complete Boolean algebras of length On. (Definitions and details to follow.) Our proof of the $\mathscr{B}$-validity of the Power Set Axiom does use Easton's Lemma but differs in two respects from the one in [**1**]: (1) it does not depend on first proving the $\mathscr{B}$-validity of the Replacement Axiom, and (2) it provides a more explicit representation for the power set of a set in the model; this simplifies the subsequent arguments about cardinal arithmetic in the model.

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1. Since we intend to show that the Boolean algebra corresponding to Easton's proper class of forcing conditions satisfies the hypotheses of the principal theorems from [**6**], we begin by recalling these theorems.

1.1 *Definition*. Let $\mathscr{B}_\alpha = \langle B_\alpha, \ldots \rangle$ $(\alpha < \kappa \leqq On)$ be complete Boolean algebras and $i_{\alpha\beta} : B_\alpha \to B_\beta$ $(\alpha \leqq \beta < \kappa)$ be complete monomorphisms; then

$$\{\langle B_\alpha | \alpha < \kappa\rangle, \langle i_{\alpha\beta} | \alpha \leqq \beta < \kappa\rangle\}$$

is called a *direct system of complete Boolean algebras* (dscBa) *of length* κ if

(i) for all $\alpha < \kappa$, $i_{\alpha\alpha}$ is the identity on B_α, and

(ii) for all $\alpha \leqq \beta \leqq \gamma < \kappa$, $i_{\alpha\gamma} = i_{\beta\gamma} \circ i_{\alpha\beta}$.

Each dscBa gives rise, in a natural way, to a Boolean algebra $\mathscr{B}$, whose underlying set, B, is $\bigcup_{\alpha<\kappa} B_\alpha$. This procedure is standard and involves identifying an element $b \in B_\alpha$ and all of its "images" $i_{\alpha\beta}(b) \in B_\beta$. When $\kappa = On$, $\mathscr{B}$ is a proper class and is *small complete*, meaning that for every sub*set*

$$A \subseteq \bigcup_{\alpha \in On} B_\alpha,$$

sup A and inf A exist and are elements of $\bigcup_{\alpha\in On} B_\alpha$. When $\kappa \in On$, $\mathscr{B}$ is in general just cf(κ)-complete, meaning that for every subset $A \subseteq B$, if $|A| <$ cf(κ), then sup A and inf A exist and are elements of B. We will denote the Dedekind-MacNeille completion of a Boolean algebra $\mathscr{D} = \langle D, \ldots \rangle$ by $\tilde{\mathscr{D}} = \langle \tilde{D}, \ldots \rangle$.

1.2 *Definition*. *A* dscBa $\{\langle B_\alpha | \alpha < \kappa\rangle, \ldots\}$ is called *strongly homogeneous* (shdscBa) if we are given $G_\alpha, H_\alpha, j_{\alpha\beta}$ $(\alpha \leqq \beta < \kappa)$ satisfying the following six conditions:

(i) G_α is a subgroup of Aut(B_α), the group of all automorphisms of B_α;

(ii) For $\alpha \leqq \beta < \kappa$, G_α is a subgroup of G_β and $j_{\alpha\beta} : G_\alpha \to G_\beta$ are monomorphisms satisfying $j_{\beta\gamma} \circ j_{\alpha\beta} = j_{\alpha\gamma}$ $(\alpha < \beta < \gamma < \kappa)$ and for all $g \in G_\alpha$ and $b \in B_\alpha$, $j_{\alpha\beta}(g)(b) = g(b)$. Using these monomorphisms, we always assume $G_\alpha \subseteq G_\beta$ for $\alpha < \beta < \kappa$. $g(b)$ is uniquely defined and no confusion is likely;

(iii) H_α is a subgroup of $G_{\alpha+1}$;

(iv) for all $h \in H_\alpha$ and $g \in G_\alpha$, $gh = hg$;

(v) for all $h \in H_\alpha$ and $b \in B_\alpha$, $h(b) = b$;

(vi) for all $b \in B_{\alpha+1}$, if $h(b) = b$ for all $h \in H_\alpha$ then $b \in B_\alpha$.

This definition was first introduced by Takeuti [**13**] for other purposes.

Given a shdscBa of length κ (the notation from 1.2 will always be used in this context), we let $G = \bigcup_{\alpha<\kappa} G_\alpha$; it is obvious that G is a group. Let $\tilde{H}_\alpha$ be the subgroup of G generated by $\bigcup_{\alpha\leqq\beta<\kappa} H_\beta$. From 1.2 (v) we conclude that any element $b \in B_\alpha$ is invariant under every member of $\tilde{H}_\alpha$.

1.3 *Definition*. When $\mathscr{B}$ is a proper class of sets arising as in Definition 1.1

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from a dscBa of length On we define

$$
\begin{aligned}
{}^{R}V_0^{(\mathscr{B})} &= \phi \\
{}^{R}V_\lambda^{(\mathscr{B})} &= \bigcup_{\alpha<\lambda} {}^{R}V_\alpha^{(\mathscr{B})} \text{ if } \lambda \text{ is a limit ordinal} \\
{}^{R}V_{\alpha+1}^{(\mathscr{B})} &= \{u \in B_{\alpha+1}{}^{\mathrm{dom}(u)} | \mathrm{dom}(u) \subseteq {}^{R}V_\alpha^{(\mathscr{B})}\} \\
{}^{R}V^{(\mathscr{B})} &= \bigcup_{\alpha \in On} {}^{R}V_\alpha^{(\mathscr{B})}.
\end{aligned}
$$

Each ${}^{R}V_\alpha^{(\mathscr{B})}$ is a set and ${}^{R}V^{(\mathscr{B})}$ is a class of sets definable in ZF.

$L({}^{R}V^{(\mathscr{B})})$ is the language of ZF augmented by a formal constant symbol for each element of ${}^{R}V^{(\mathscr{B})}$. If χ is an atomic formula of $L({}^{R}V^{(\mathscr{B})})$, $||\chi||$ is defined as usual, see [**11**]. But when $\mathscr{B}$ is a proper class, we are in fact faced with sups and infs over subclasses of $\mathscr{B}$ that are proper classes in the usual definitions

$$
\begin{aligned}
||\exists x \phi(x)|| &= \sum_{u \in {}^{R}V^{(\mathscr{B})}} ||\phi(u)||, \\
||\forall x \phi(x)|| &= \prod_{u \in {}^{R}V^{(\mathscr{B})}} ||\phi(u)||.
\end{aligned}
$$

We shall nonetheless retain these definitions of $||\chi||$ for arbitrary formulas of $L({}^{R}V^{(\mathscr{B})})$; it is proved in [**6**] that under the hypotheses of Theorem 1.5 (see below), $||\chi|| \in B$ even though we have only assumed that $\mathscr{B}$ is small complete.

1.4 *Definition.* The following property of a shdscBa of length κ is called the *Automorphism Reflection Principle* (ARP).

ARP: For all $\alpha < \kappa$ and all β such that $\alpha < \beta < \kappa$, if $b \in B_\beta$ is invariant under every member of $\tilde{H}_\alpha \cap G_\beta$, then $b \in B_\alpha$.

1.5 THEOREM. *If $\mathscr{B}$ is a proper class arising from a shdscBa of length On satisfying* ARP, *then the axioms of ZFC minus the Power Set Axiom are $\mathscr{B}$-valid in ${}^{R}V^{(\mathscr{B})}$.*

Proof. This is Theorem 2.9 of [**6**].

1.6 *Definition.* In [**12**], an operator, #, is introduced in the following context: let $\mathscr{B}'$ be a complete subalgebra of the complete Boolean algebra $\mathscr{B}$. For $b \in B$, define

$$\#(b) = \inf\{b' \in B' | b' \geqq b\}.$$

1.7 LEMMA [**12**].
(1) $\#(x) \geqq x$;
(2) $x = 0 \Leftrightarrow \#(x) = 0$;
(3) *if* $x \in B'$, $\#(x) = x$;
(4) *if* $y \in B'$ *and* $y \geqq x$ *then* $y \geqq \#(x)$;
(5) *if* $y \in B'$ *and* $x \in B$ *then* $y \cdot \#(x) = \#(y \cdot x)$;
(6) *if* $y \in B'$ *and* $x \in B$ *then* $y \cdot x = 0 \leftrightarrow y \cdot \#(x) = 0$.

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Proof. (1) – (4) are immediate and (6) follows from (5) and (2) so we prove (5). $y \cdot x \leqq y \cdot \#(x)$ from (1) and $y \cdot \#(x) \in B'$ so from (4) we get $\#(y \cdot x) \leqq y \cdot \#(x)$. If this inequality were strict, we would have

$$x = y \cdot x + -y \cdot x \leqq \#(y \cdot x) + \#(-y \cdot x) < y \cdot \#(x) + -y \cdot \#(x) = \#(x)$$

But $\#(y \cdot x) + \#(-y \cdot x) \in B'$, and this contradicts the minimality of $\#(x)$.

If $\mathscr{B}_0 \subseteq \mathscr{B}_1 \subseteq \mathscr{B}_2$ are complete Boolean algebras with $\mathscr{B}_i$ a complete subalgebra of $\mathscr{B}_{i+1}$ $(i = 0, 1)$ and we let

$$\#^1(b) = \inf \{b' \in B_1 | b' \geqq b\}, \#^0(b) = \inf\{b' \in B_0 | b' \geqq b\}$$

it is easy to see that $\#^0(b) = \#^0(\#^1(b))$. Thus if $\{\langle \mathscr{B}_\alpha | \alpha \in On\rangle, \langle i_{\alpha\beta} | \alpha \leqq \beta \in On\rangle\}$ is a dscBa and $\mathscr{B} = \langle \bigcup_{\alpha \in On} B_\alpha, \ldots \rangle$ is the limit algebra, then $\#^\alpha(b) = \inf\{b' \in B_\alpha | b' \geqq b\}$ is well-defined for $b \in B = \bigcup_{\alpha \in On} B_\alpha$.

1.8 *Definition.* An algebra $\mathscr{B} = \langle \bigcup_{\alpha \in On} B_\alpha, \ldots \rangle$ arising from a dscBa of length On is said to satisfy the *Power Set Law* (PSL) if:

$$\text{PSL}: \forall \alpha \, \exists \beta \, \forall \{b_\gamma | \gamma < \aleph_\alpha\} \forall p > 0 \quad \exists 0 < q \leqq p$$
$$\forall \gamma < \aleph_\alpha \quad \#^\beta(q \cdot b_\gamma) \cdot q \leqq b_\gamma$$

where α, β range over On, b_γ, p, q range over B, and $\{b_\gamma | \gamma < \aleph_\alpha\}$ ranges over subsets of B of cardinality $\aleph_\alpha$.

1.9 THEOREM. *If B arises from a dscBa of length On satisfying* PSL, *then the Power Set Axiom is $\mathscr{B}$-valid in ${}^R V^{(\mathscr{B})}$.*

Proof. This is Theorem 3.3 of [**6**].

1.10 COROLLARY (to the proof of Theorem 1.9). *Let $\mathscr{B}$ satisfy the hypothesis of Theorem* 1.9 *and let $x \in V$ with $|x| = \aleph_\alpha$. Then $||P(\hat{x}) = B_\beta{}^{\mathrm{dom}(\hat{x})} \times \{1\}|| = 1$ in ${}^R V^{(\mathscr{B})}$, where β is the ordinal corresponding to α via* PSL.

An informal statement of this corollary is that no new subsets of $\hat{x}$ are introduced beyond the βth stage in the iteration.

2. If T is a topological space, we denote the complete Boolean algebra of regular open subsets of T by $\mathscr{B}_T$. The reader is referred to Halmos [**2**] on Sikorski [**9**] for the basic relationships between T and $\mathscr{B}_T$. The definitions and results to follow in this section can be found in [**13**].

2.1 LEMMA. *Let X and Y be topological spaces and $f : X \to Y$ be open, continuous and onto (o.c.o). If $D \subseteq Y$ then*

(i) $f^{-1}``\bar{D} = \overline{(f^{-1}``D)}$, *and*
(ii) $f^{-1}``D^0 = (f^{-1}``D)^0$

Proof. This is a straightforward exercise.

2.2 THEOREM. *Let X and Y be topological spaces and $f : X \to Y$ be o.c.o. Then f induces a complete monomorphism $i : \mathscr{B}_Y \to \mathscr{B}_X$ defined by*

$$i(b) = f^{-1}``b \quad \text{for } b \in \mathscr{B}_Y.$$

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Proof. Lemma 2.1 assures us that $i(b) \in \mathscr{B}_X$. The fact that f^{-1} preserves all set-theoretic operations makes i a complete monomorphism.

2.3 *Definition.* Let $T_\alpha (\alpha < \kappa \leqq On)$ be topological spaces and $p_{\alpha\beta} : T_\beta \to T_\alpha (\alpha \leqq \beta < \kappa)$ be o.c.o. We say that

$$\{\langle T_\alpha | \alpha < \kappa\rangle, \langle p_{\alpha\beta} | \alpha \leqq \beta < \kappa\rangle\}$$

is an *o.c.o. inverse system of topological spaces of length* κ if

(i) for all $\alpha < \kappa$, $p_{\alpha\alpha}$ is the identity on T_α, and

(ii) for all $\alpha \leqq \beta \leqq \gamma < \kappa$, $p_{\alpha\gamma} = p_{\alpha\beta} \circ p_{\beta\gamma}$.

If $\{\langle T_\alpha | \alpha < \kappa\rangle, \langle p_{\alpha\beta} | \alpha \leqq \beta < \kappa\rangle\}$ is an o.c.o. inverse system of topological spaces of length κ, and we take $\mathscr{B}_\alpha = \mathscr{B}_{T_\alpha}$ and $i_{\alpha\beta} : B_\alpha \to B_\beta$ to be the complete monomorphism induced by $p_{\alpha\beta}$, it follows from Theorem 2.2 that $\{\langle \mathscr{B}_\alpha | \alpha < \kappa\rangle, \langle i_{\alpha\beta} | \alpha \leqq \beta < \kappa\rangle\}$ is a dscBa (see Definition 1.1) of length κ.

This fact is to be used in §3, where we shall introduce Easton's algebra by first defining the appropriate o.c.o. inverse system of length *On*.

3. We begin this section with a statement of the result of Easton [**1**] concerning variants of the *GCH* which are consistent with *ZFC*.

3.1 Theorem. [**1**]. *Let* $F : On \to On$ *be a function satisfying*

(i) $\alpha \leqq \beta \to F'\alpha \leqq F'\beta$, *and*

(ii) $\mathrm{cf}(\aleph_{F'\alpha}) > \aleph_\alpha$ *for all* α *such that* $\aleph_\alpha$ *is regular.*

If $\mathscr{M}$ *is a model of* $ZF + GCH$, *then there exists an extension* $\mathscr{N} \supseteq \mathscr{M}$ *having the same cardinals as* $\mathscr{M}$ *and satisfying* $ZFC + 2^{\aleph_\alpha} = \aleph_{F'\alpha}$ *for every* α *such that* $\aleph_\alpha$ *is regular.*

Actually, in [**1**], this theorem is stated and proved for models of GB + the class form of the Axiom of Choice. Also, an additional condition on F, that F be absolute with respect to the extension, is required; any "reasonably defined" F will satisfy this condition, which is somewhat cumbersome to make precise; we refer to Levy and Solovay [**4**] for a discussion of this point.

3.2 *Definition.* Let $F : On \to On$ be a function satisfying (i) and (ii) of Theorem 3.1. For notational reasons, we also assume

(iii) if $\aleph_\lambda$ is singular then

$$F'\lambda = \sum_{\alpha<\lambda} F'\alpha;$$

we make this convention here so that the present definition and following lemma can be stated in notation that is uniform for both the regular and singular cases.

For each $\alpha \in On$, $j < \aleph_{F'\alpha}$, $i = 0, 1$ let ${}^\alpha B_j{}^i$ denote

$$\{f \in 2^{\aleph_{F'\alpha}} | f(j) = i\};$$

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for each $J \subseteq \aleph_{F'\alpha}$ and $g \in 2^J$ let

$$\bigcap_{j\in J} {}^{\alpha}B_j^{g(j)} \text{ denote } \{ f \in 2^{\aleph_{F'\alpha}} | \textit{for all } j \in J, f(j) = g(j)\}.$$

For each $\alpha \in On$, let T_α be the topological space whose underlying set is $2^{\aleph_{F'\alpha}}$ and which has as a basis for the topology the collection of all subsets of the form $\bigcap_{j\in J} {}^{\alpha}B_j^{g(j)}$ for $g \in 2^J$ and $J \subseteq \aleph_{F'\alpha}$ satisfying $|J \cap \aleph_{F'\gamma}| < \aleph_\gamma$ for every $\gamma \leqq \alpha$ such that $\aleph_\gamma$ is regular. A subset $J \subseteq \aleph_{F'\alpha}$ satisfying $|J \cap \aleph_{F'\gamma}| < \aleph_\gamma$ for every $\gamma \leqq \alpha$ such that $\aleph_\gamma$ is regular will be called an *F-thin* subset of $\aleph_{F'\alpha}$ and the topology above will be called the *F-thin topology* on $2^{\aleph_{F'\alpha}}$. The reason for this terminology is the following: in Kunen [**3**], a subset $J \subseteq \aleph_\alpha$ is called *thin* if $|J \cap \aleph_\gamma| < \aleph_\gamma$ for every $\gamma \leqq \alpha$ such that $\aleph_\gamma$ is regular.

3.3 LEMMA. *Let $\alpha < \beta$ and T_α, T_β be the spaces just defined; then the maps $p_{\alpha\beta} : T_\beta \to T_\alpha$ determined by $p_{\alpha\beta}(f) = f \restriction \aleph_{F'\alpha}$ are open, continuous and onto (o.c.o.).*

Proof. This is clear from the definition of the topologies involved.

3.4 *Definition.* If $\alpha < \beta < \gamma$, the preceding maps satisfy $p_{\alpha\gamma} = p_{\alpha\beta} \circ p_{\beta\gamma}$, so that if we let $\mathscr{E}_\alpha = \mathscr{B}_{T_\alpha}$, the complete Boolean algebra of regular open subsets of T_α, it will follow from the closing sentences of §2 that

$$\{\langle E_\alpha | \alpha \in On\rangle, \langle i_{\alpha\beta} | \alpha \leqq \beta \in On\rangle\}$$

is a direct system of complete Boolean algebras of length On. Easton's algebra, $\mathscr{E}$ is defined to be the small complete Boolean algebra $\langle \bigcup_{\alpha\in On} E_\alpha, \dots \rangle$ arising from this dscBa in the manner of Definition 1.1.

3.5 *Remark.* Let Γ_α be as in Definition 15 of [**1**] and let P_α be the partially ordered set $\{p | p \text{ is a set of conditions and } p \subseteq \Gamma_\alpha\}$, where the term "set of conditions" is as in Definition 4 of [**1**] and the ordering is by inclusion. This partial order induces a topology on P_α and $\mathscr{B}_{T_\alpha}$ is isomorphic to the algebra of regular open subsets of P_α with this topology.

3.6 *Definition.* The basic open subsets of T_α are also closed, hence regular open, and so are elements of E_α. We denote this collection of basic open subsets of T_α by S_α, and note that S_α is dense in E_α; thus any $b \in E_\alpha$, $b > 0$, can be expressed as a supremum of elements of S_α. In accordance with the convention following Definition 1.1, an element $\bigcap_{j\in J} {}^{\alpha}B_j^{g(j)} \in S_\alpha$ will not be distinguished from its images in E_β $(\beta > \alpha)$, so we drop the upper left superscript on these basic elements, and simply write $\bigcap_{j\in J} B_j^{g(j)}$. We also write $S = \bigcup_{\alpha\in On} S_\alpha$.

3.7 *Definition.* Consider E_α and let $j < \aleph_{F'\alpha}$. The map π_j which interchanges B_j^0 and B_j^1 and leaves all other $B_{j'}^i (j' \neq j, i = 0, 1)$ fixed extends to an automorphism of E_α in a natural way. Define G_α to be the subgroup of $\mathrm{Aut}(E_\alpha)$ of all finite products of the π_j $(j < \aleph_{F'\alpha})$. Define H_α to be the subgroup of $G_{\alpha+1}$ generated by those π_j for which $\aleph_{F'\alpha} \leqq j < \aleph_{F'\alpha+1}$. H_α is the trivial subgroup if $F'\alpha = F'_{\alpha+1}$.

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3.8 Theorem. *E_α, G_α, H_α ($\alpha \in On$) as defined above form a shdscBa of length On which, moreover, satisfies* ARP.

Proof. It remains to verify conditions (i) - (vi) of Definition 1.2 as well as ARP. Conditions (i) - (v) are immediate from the definitions; we recall that (vi) is the special case of ARP in which $\beta = \alpha + 1$, so we complete the proof by establishing ARP.

Let $\alpha < \beta$ and let $b \in E_\beta$ be invariant under all the automorphisms in $\tilde{H}_\alpha \cap G_\beta$. Write b as a supremum of basic elements of E_β, $b = \sum_{i \in I} b_i$ where $b_i = \bigcap_{j \in J_i} B_j^{g_i(j)}$ with $J_i \subseteq \aleph_{F'\beta}$. Let

$$b_i^\alpha = \bigcap_{j \in J_i \cap \aleph_{F'\alpha}} B_j^{g_i(j)};$$

clearly $b_i^\alpha \geqq b_i$. Fix i temporarily. If $b_i^\alpha > b_i$ then there exists $j_0 \in J_i \cap (\aleph_{F'\beta} - \aleph_{F'\alpha})$. Let

$$c_i = \bigcap_{j \in J_i - \{j_0\}} B_j^{g_i(j)}$$

so that $b_i = c_i \cdot B_{j_0}^{g_i(j_0)}$. Since $\pi_{j_0} \in \tilde{H}_\alpha \cap G_\beta$, we have

$$b = \pi_{j_0} b = \sum_{i \in I} \pi_{j_0} b_i \geqslant \pi_{j_0} b_i = c_i \cdot B_{j_0}^{1 - g_i(j_0)}.$$

Thus

$$b \geqq b_i + \pi_{j_0} b_i = c_i \cdot B_{j_0}^{g_i(j_0)} + c_i \cdot B_{j_0}^{1 - g_i(j_0)} = c_i.$$

Repeating this argument for each $j \in J_i \cap (\aleph_{F'\beta} - \aleph_{F'\alpha})$, we eventually conclude $b \geqq b_i^\alpha$. Now this procedure applies to any $i \in I$, so we have

$$b = \sum_{i \in I} b_i \leqslant \sum_{i \in I} b_i^\alpha \leqslant b.$$

Thus $b \in E_\alpha$ because $b = \sum_{i \in I} b_i^\alpha$ is a supremum of elements of E_α.

3.9 *Remark.* To illustrate the fact that for limit ordinals, λ, we need not have $E_\lambda = (\bigcup_{\alpha < \lambda} E_\alpha)^\sim$, consider the function $F : On \to On$ given by $F'\alpha = \alpha + 2$ if $\aleph_\alpha$ is regular, $F'\lambda = \sum_{\alpha < \lambda} F'\alpha$ if $\aleph_\lambda$ is singular. Let $J_0 = \{\aleph_n + 1 | n < \omega\} \subseteq \aleph_{F'\omega}$; then J_0 is F-thin so $b = \bigcap_{j \in J_0} B_j^{g(j)} \in E_\omega$. If $E_\omega = (\bigcup_{n < \omega} E_n)^\sim$, then by density there exist $n < \omega$, $b' \in E_n$ with $b' \leqq b$; but this is clearly impossible since b restricts a set of coordinates which is cofinal in $\aleph_\omega$ $(= \aleph_{F'\omega})$.

3.10 Lemma [**1**]. *Let $\alpha \in On$, $p \in S$ and $\{b_\gamma | \gamma < \aleph_\alpha\} \subseteq E = \bigcup_{\alpha \in On} E_\alpha$ be given, where E_α and S are as defined in* 3.4 *and* 3.6 *respectively. Then there exists $0 < q \leqq p$ and a set $\Pi \subseteq S$ such that*

(i) $|\Pi| \leqq \aleph_\alpha$,

(ii) $\forall\, 0 < q'' \leqq q\ \forall \gamma < \aleph_\alpha\ \exists B \in \Pi$ *such that* $0 < B \cdot q'' \wedge (B \cdot q \leqq b_\gamma \vee B \cdot q \leqq -b_\gamma)$, *and*

(iii) $|\Pi| \subseteq S_\alpha$.

Proof. This is Lemma 25 of [**1**] modulo change of notation from the context

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of forcing to that of Boolean-valued models. A proof can be found in [**1**], or in the Appendix to Section Two of McAloon [**5**], or as part of Theorem 8.7 of Rosser [**8**]. We shall refer to 3.10 as "Easton's Lemma".

3.11 THEOREM. *Easton's algebra,* $\mathscr{E}$, *satisfies* PSL.

Proof. For the convenience of the reader, we restate PSL (Definition 1.6).

$$\text{PSL: } \forall\alpha\,\exists\beta\,\forall\{b_\gamma|\gamma < \aleph_\alpha\}\,\forall p > 0\,\exists 0 < q \leqq p\,\forall\gamma < \aleph_\alpha \quad \#^\beta(q\cdot b_\gamma)\cdot q \leqq b_\gamma$$

We will show that E satisfies PSL when β is taken to be α itself. Thus we would like to prove

$$\forall p > 0\,\forall\{b_\gamma|\gamma < \aleph_\alpha\}\,\exists 0 < q \leqq p\,\forall\gamma < \aleph_\alpha\,\#^\alpha(q\cdot b_\gamma)\cdot q \leqq b_\gamma$$

which we abbreviate as $\forall p > 0\ \psi(p,\alpha)$.

Since S is dense in E, it suffices to prove $\psi(p,\alpha)$ for $p \in S$. Conclusion (i) of Easton's Lemma will not be required for this; it is used only in Easton's proof that the extension has precisely the same cardinals as the ground model.

Suppose we have a violation of PSL for α. Thus we have $p \in S$, $\{b_\gamma|\gamma < \aleph_\alpha\}$ such that

$$\forall 0 < q' \leqq p\,\exists\gamma < \aleph_\alpha\,\exists 0 < q'' \leqq q' \quad q'' \leqq \#^\alpha(q\cdot b_\gamma)\cdot -b_\gamma.$$

This will then be true in particular when we take as q' the specific $0 < q \leqq p$ which satisfies, in addition, the conditions of Easton's Lemma. Thus $\exists 0 < q \leqq p\,\exists\gamma < \aleph_\alpha\,\exists 0 < q'' \leqq q\,\exists\,\Pi$ such that

(1) $q'' \leqq \#^\alpha(q\cdot b_\gamma)\cdot -b_\gamma$,

(2) $\exists B \in \Pi \quad 0 < B\cdot q'' \wedge (B\cdot q \leqq b_\gamma \vee B\cdot q \leqq -b_\gamma)$,

(3) $\Pi \subseteq S_\alpha$.

We will now produce a contradiction. Note that $B\cdot q$ is contained entirely in b_γ or entirely in $-b_\gamma$. Now $q'' \leqq q\cdot -b_\gamma$ so $0 < B\cdot q'' \leqq B\cdot q\cdot -b_\gamma$, and hence $0 < B\cdot q \leqq -b_\gamma$. On the other hand, $q'' \leqq \#^\alpha(q\cdot b_\gamma)$, so $0 < B\cdot q'' \leqq B\cdot\#^\alpha(q\cdot b_\gamma)$. Since $B \in \Pi \subseteq S_\alpha$, $B \in E_\alpha$; by the properties of $\#^\alpha$ proved in Lemma 1.7, $0 < B\cdot\#^\alpha(q\cdot b_\gamma) = \#^\alpha(B\cdot q\cdot b_\gamma)$, which implies $0 < B\cdot q\cdot b_\gamma$, and hence $0 < B\cdot q \leqq b_\gamma$.

3.12 THEOREM. ${}^RV^{(E)}$ *is a model of* $ZFC + 2^{\aleph_\alpha} = \aleph_{F'\alpha}$ *for every* α *such that* $\aleph_\alpha$ *is regular.*

Proof. That ${}^RV^{(E)}$ is a model of ZFC follows by combining Theorems 1.5, 1.9, 3.8 and 3.11. The remainder of the proof makes use of the fact that the extension has precisely the same cardinals as the ground model (for this, consult [**1**]); thus the cardinality of $2^{\aleph_\alpha}$ in the extension will be equal, by Corollary 1.10, to the cardinality of ${E_\alpha}^{\aleph_\alpha}$ in the ground model. Recall that we are assuming GCH in the ground model, and that $\mathrm{cf}(\aleph_{F'\alpha}) > \aleph_\alpha$, so the desired result will follow if $|E_\alpha|$ in the ground model is $\aleph_{F'\alpha}$ for those α such that $\aleph_\alpha$ is regular.

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To see that $|E_\alpha| = \aleph_{F'\alpha}$ in this case, first observe that E_α satisfies the $\aleph_\alpha{}^+$ chain condition (consult Lemma 10.3 of Shoenfield [**10**], noting that the F-thin topology on $2^{\aleph_{F'\alpha}}$ is coarser than the one in use there), thus every element of E_α is a supremum of $\leqq \aleph_\alpha$ many elements of S_α. Also

$$S_\alpha \subseteq T = \left\{ \bigcap_{j\in J} B_j^{g(j)} \mid J \subseteq \aleph_{F'\alpha} \wedge |J| < \aleph_\alpha \wedge g \in 2^J \right\};$$

the set of all such J's has cardinality $\aleph_{F'\alpha}$ to the weak power $\aleph_\alpha$ and each such J gives rise to 2^γ (where $|J| = \gamma$) different elements of T, hence $|S_\alpha| \leqq |T| \leqq \aleph_{F'\alpha} \cdot \aleph_\alpha = \aleph_{F'\alpha}$.

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