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A NOTE ON FIXED POINT SETS AND WEDGES

BY

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ABSTRACT. A space Z is said to have the complete invariance property (CIP) provided that every nonempty closed subset of Z is the fixed point set of some continuous self-mapping of Z. In this paper it is shown that there exists a one-dimensional contractible planar continuum having CIP whose wedge with itself at a specified point is contractible, planar, and does not have CIP.

1. **Introduction.** The problem of determining when a nonempty closed subset of a topological space Z can be the fixed point set of a continuous self-mapping or homeomorphism of Z has been investigated in [1] through [9]. In [9, p. 553] Ward defines a space Z to have the *complete invariance property* (CIP) provided that each nonempty closed subset of Z is the fixed point set of a continuous self-mapping of Z. Examples in [2, section 3] show that the wedge of two continua, each having CIP, does not necessarily have CIP, even when the continua are locally connected or one-dimensional and unicoherent. However, none of these continua is acyclic. In [2, 3.5] it is asked if CIP is preserved by wedging acyclic continua, one-dimensional acyclic continua, or contractible continua.

The purpose of this note is to show that there is a one-dimensional contractible planar continuum X having CIP whose wedge with itself at a specified point does not have CIP. Thus, the questions in [2, 3.5] mentioned above are answered. In addition, the wedge is contractible and planar.

2. **The example.** Let $X=[\bigcup_{n=0}^{\infty} A_n]\cup S$ be the continuum in the plane R^2 pictured in Fig. 1, where $a_0=(-1,0)$, $a_n=(-1,2^{-n})$ for each $n=1,2,\ldots,A_n$ is the convex arc in R^2 from $(-3,0)$ to a_n for each $n=0,1,2,\ldots,$ and S is the convex arc in R^2 from a_0 to $(0,0)$. Note that for each $n=1,2,\ldots,$

$$A_n=\{(x,y)\in R^2\colon -3\le x\le -1 \quad \text{and} \quad y=2^{-n-1}(x+3)\}.$$

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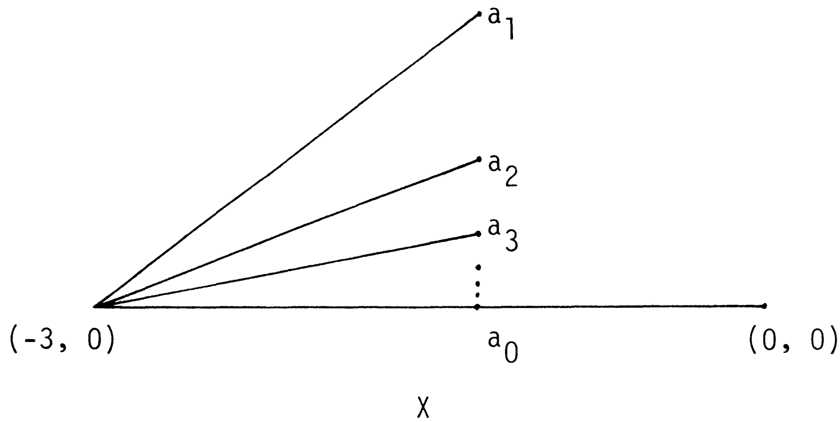

Figure 1

It is easy to see that X is a one-dimensional contractible continuum. We leave it to the reader to verify that X has CIP.

The wedge of X with itself at $(0, 0)$ is the quotient space obtained from the union of two disjoint copies of X by identifying the points corresponding to $(0, 0)$. This wedge is homeomorphic to the continuum W drawn in Fig. 2,

$$W = X \cup \{(-x, y) \in R^2 : (x, y) \in X\}.$$

It is easy to see that W is contractible. We now show that W does not have CIP. Let

$$K = \{(x, y) \in W : x = -2 \quad \text{or} \quad x = +2\}.$$

Suppose that there is a continuous function $f : W \to W$ such that f has fixed point set equal to K. Then, using the uniform continuity of f, it follows that there exists ε, $0 < \varepsilon < 1$, such that for

$$q_n = (-2 + \varepsilon, 2^{-n-1}[1 + \varepsilon]) \in A_n,\ n = 1, 2, \ldots,$$

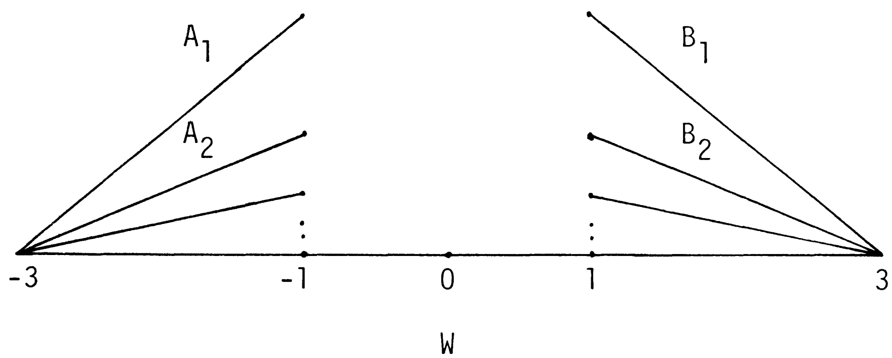

Figure 2

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$f(q_n)\in A_n$ for each $n=1,2,\ldots$. Hence, since K is the fixed point set of f, the first coordinate of $f(q_n)$ is strictly less than $-2+\varepsilon$ for each $n=1,2,\ldots$ (otherwise, for some $n=1,2,\ldots,f$ would have a fixed point $(x,y)\in A_n$ such that $x\geq -2+\varepsilon$). Thus, since $\{q_n\}_{n=1}^{\infty}$ converges to $q=(-2+\varepsilon,0)$, we see that $f(q)$ is a point of the form $(x_0,0)$ where $x_0<-2+\varepsilon$. Similarly, there exists $\varepsilon', 0<\varepsilon'<1$, such that for $q'=(2-\varepsilon',0)$ we have $f(q')=(x_0',0)$ where $x_0'>2-\varepsilon'$. Let α be the arc in W with end points q and q'. By comparing points of α with their images under f while going along α from q to q', it follows easily that f must have a fixed point in α. However, since $\alpha\cap K=\varnothing$, this is a contradiction.

Remarks. In relation to the choice of K at the beginning of the proof above, we mention that $\{(x,y)\in W: x=-1 \text{ or } x=+1\}$ is the fixed point set of a continuous self-mapping of W.

Our continuum X is one-dimensional and acyclic, but X is not locally connected. Every one-dimensional, acyclic, locally connected continuum has CIP by [5, 3.1]; hence, any wedge of two such continua must *also* have CIP. However, we do not know if CIP is always preserved by wedging two locally connected continua which are contractible or planar. The reader is referred to [1] and [2] for other questions about locally connected continua and CIP.

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