Glasgow Math. J. **47** (2005) 339–345. © 2005 Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089505002570. Printed in the United Kingdom

GENERALIZED FOURIER EXPANSIONS OF DIFFERENTIABLE FUNCTIONS ON THE SPHERE

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(Received 22 September, 2004; accepted 26 January, 2005)

Abstract. We show that the Fourier expansion in spherical h-harmonics (from Dunkl's theory) of a function f on the sphere converges uniformly to f if this function is sufficiently differentiable.

2000 *Mathematics Subject Classification.* Primary 42C10. Secondary 33C50.

1. Introduction. The theory of Dunkl's operators has found in recent years numerous applications in mathematics and mathematical physics (see the references in [**4**] and [**6**]). One of its starting points was the study of generalized spherical harmonics associated to a finitely generated reflection group and a multiplicative function $h \geq 0$. Among the many results for classical spherical harmonics carried over to these spherical h-harmonics is the following ([**7**, Theorem 3.1], [**6**, Theorem 5.5]): the Fourier expansion (in spherical h-harmonics) of any continuous function f on $\mathbf{S}^{N-1}$ is uniformly summable in Cesàro means of order δ to f on $\mathbf{S}^{N-1}$ as long as $\delta > \deg h + (N-2)/2$. Similar results about the Fourier expansion of $f \in L^p(\mathbf{S}^{N-1})$ have also been obtained.

Quite surprisingly, it seems that such questions for f differentiable on $\mathbf{S}^{N-1}$ have been neglected until now. The aim of this work is to make a step in this direction. More precisely, we show in Section 4 that the Fourier expansion of $f \in C^{2q}(\mathbf{S}^{N-1})$ converges to f uniformly on $\mathbf{S}^{N-1}$ as long as $q > \deg h/2 + N/4$.

For that we follow the approach of [**5**] in the classical case, which induces us to define in Section 3 an h-analogue of the Laplace-Beltrami operator on the sphere. In Section 2 we recall the basic facts in the theory of h-harmonics.

2. Preliminaries. For a vector v in $\mathbf{R}^N \setminus \{0\}$ ($N \geq 2$) we define the reflection $\sigma_v \in O(N)$ by

$$x\sigma_v := x - 2\langle x, v\rangle v/\|v\|^2$$

for all $x \in \mathbf{R}^N$, where $\langle x, v\rangle$ is the Euclidean scalar product of x and v, and $\|v\| := \langle v, v\rangle^{1/2}$. Thus $v\sigma_v = -v$ and $x\sigma_v = x$ if and only if x is perpendicular to v.

Suppose now G is a finite subgroup of $O(N)$ generated by reflections. Let $\{\sigma_1, \dots, \sigma_m\}$ be all reflections in G. We choose vectors $v_1, \dots, v_m$ in $\mathbf{R}^N$ such that $\sigma_j = \sigma_{v_j}$ for $j = 1, \dots, m$ and $\|v_i\| = \|v_j\|$ whenever σ_i is conjugate to σ_j in G. Next we take $\alpha_1, \ldots, \alpha_m \in \mathbf{R}_{\geq 0}$ with $\alpha_i = \alpha_j$ whenever σ_i is conjugate to σ_j in G and let

$$h(x) = h_\alpha(x) := \prod_{j=1}^{m} |\langle x, v_j\rangle|^{\alpha_j};$$

this is a G-invariant function, homogeneous of degree $|\alpha| := \alpha_1 + \cdots + \alpha_m$.

We write $\mathbf{S}^{N-1}$ the unit sphere in $\mathbf{R}^N$ and $d\sigma_{N-1}$ the measure on $\mathbf{S}^{N-1}$ induced by the Lebesgue measure on $\mathbf{R}^N$, so that $\omega_{N-1} := \int_{\mathbf{S}^{N-1}} d\sigma_{N-1}(\eta) = 2\pi^{N/2}/\Gamma(N/2)$. We define a G-invariant measure on $\mathbf{S}^{N-1}$ by

$$d\sigma_h(\eta) := H_\alpha h_\alpha^2(\eta)\, d\sigma_{N-1}(\eta),$$

with the constant H_α so chosen that $d\sigma_h$ is normalized. We write

$$\langle f, g\rangle_2 := \int_{\mathbf{S}^{N-1}} f(\eta)\overline{g(\eta)}\, d\sigma_h(\eta),$$

the usual scalar product in $L^2(\mathbf{S}^{N-1}, d\sigma_h)$ and $\|f\|_2$ the associated norm.

For $i = 1, \ldots, N$ we write $\mathcal{D}_i$ for Dunkl's operator defined by

$$\mathcal{D}_i f(x) := \partial_i f(x) + \sum_{j=1}^{m} \alpha_j \frac{f(x) - f(x\sigma_j)}{\langle x, v_j\rangle} \langle v_j, e_i\rangle,$$

where $\partial_i := \partial/\partial x_i$ and $(e_i)_l = \delta_{il}$. Dunkl's operators form a family of commuting first order difference-differential operators which play here a role similar to $\partial_1, \ldots, \partial_N$. In particular the *h-Laplacian* is

$$\Delta_h := \sum_{i=1}^{N} \mathcal{D}_i^2.$$

Let $\mathcal{P}_l$ denote the space of homogeneous polynomials of degree $l \in \mathbf{N}_0$ on $\mathbf{R}^N$. Then $\mathcal{D}_i\mathcal{P}_l \subset \mathcal{P}_{l-1}$ and $\Delta_h\mathcal{P}_l \subset \mathcal{P}_{l-2}$. Moreover, if $P \in \mathcal{P}_l$, $\langle P, Q\rangle_2 = 0$ for all $Q \in \cup_{k=0}^{l-1}\mathcal{P}_k$ if and only if $\Delta_h P = 0$. The elements of $\mathcal{H}_l := \{P \in \mathcal{P}_l : \Delta_h P = 0\}$ are called *h-harmonic polynomials of degree l*. We have

$$d_l = d_l^{(N)} := \dim \mathcal{H}_l = \binom{l+N-1}{l} - \binom{l+N-3}{l-2}.$$

When $|\alpha| = 0$ (that is, $h \equiv 1$), we get classical spaces and operators; in particular $\mathcal{D}_i = \partial_i$ and Δ_h is the usual Laplacian Δ.

Concerning all of the above we refer the reader to [**4**].

3. The *h*-Laplace-Beltrami operator. If f is a function on $\mathbf{S}^{N-1}$, we write $f\uparrow$ for the homogeneous function of degree 0 defined on $\mathbf{R}^N \setminus \{0\}$ by $(f\uparrow)(x) := f(x/\|x\|)$. Conversely, if g is a function defined on $\mathbf{R}^N \setminus \{0\}$ we write $g\downarrow$ for its restriction to $\mathbf{S}^{N-1}$. We say that a function f on $\mathbf{S}^{N-1}$ is in $C^q(\mathbf{S}^{N-1})$ $(q \in \mathbf{N}_0)$ if $f\uparrow \in C^q(\mathbf{R}^N \setminus \{0\})$. When $f \in C^q(\mathbf{S}^{N-1})$ with $q \geq 2$ we can define ${}_{\mathbf{S}}\Delta_h f \in C^{q-2}(\mathbf{S}^{N-1})$ by

$${}_{\mathbf{S}}\Delta_h f := (\Delta_h(f\uparrow))\downarrow\,.$$

We call ${}_{\mathbf{S}}\Delta_h$ the *h-Laplace-Beltrami operator on* $\mathbf{S}^{N-1}$; it commutes with the action of G. We write $\mathcal{S}H_l(\mathbf{S}^{N-1}) := \{P\downarrow : P \in \mathcal{H}_l\}$ $(l \in \mathbf{N}_0)$; its elements are called *spherical h-harmonics of degree l* and its dimension is $d_l^{(N)}$.

https://doi.org/10.1017/S0017089505002570 Published online by Cambridge University Press

LEMMA 1. *If* $\lambda > 0$ *and* $f \in C^2(\mathbf{R}^N \setminus \{0\})$ *is homogeneous of degree* ϕ, *then*

$$\Delta_h(\|\cdot\|^{-\lambda} f) = -\lambda(2\phi + 2|\alpha| + N - \lambda - 2)\|\cdot\|^{-\lambda-2} f + \|\cdot\|^{-\lambda}\Delta_h f.$$

Proof. This is proved in [**4**, Lemma 5.1.9 p. 178] with the unnecessary restriction that f be a polynomial.

PROPOSITION 1. *Let* $l \in \mathbf{N}_0$. *For every* $Y \in \mathcal{S}H_l(\mathbf{S}^{N-1})$,

$${}_{\mathbf{S}}\Delta_h Y = -l(l + 2|\alpha| + N - 2)Y.$$

Proof. By hypothesis, there exists $P \in \mathcal{H}_l$ with $Y = P{\downarrow}$. Since P is homogeneous of degree l, $Y{\uparrow}(x) = Y(x/\|x\|) = P(x/\|x\|) = \|x\|^{-l}P(x)$. Therefore

$$\begin{aligned}\Delta_h(Y{\uparrow}) &= \Delta_h(\|\cdot\|^{-l}P) \\ &= -l(2l + 2|\alpha| + N - l - 2)\|\cdot\|^{-l-2}P + \|\cdot\|^{-l}\Delta_h P \\ &= -l(l + 2|\alpha| + N - 2)\|\cdot\|^{-l-2}P,\end{aligned}$$

using Lemma 1 for the second equality and $\Delta_h P = 0$ for the third. Hence

$$\begin{aligned}{}_{\mathbf{S}}\Delta_h Y &= [\Delta_h(Y{\uparrow})]{\downarrow} \\ &= [-l(l + 2|\alpha| + N - 2)\|\cdot\|^{-l-2}P]{\downarrow} \\ &= -l(l + 2|\alpha| + N - 2)\cdot 1 \cdot Y.\end{aligned}$$

PROPOSITION 2. *The h-Laplace-Beltrami operator is self-adjoint; in other words, for all* $f, g \in C^2(\mathbf{S}^{N-1})$,

$$\langle {}_{\mathbf{S}}\Delta_h f, g\rangle_2 = \langle f, {}_{\mathbf{S}}\Delta_h g\rangle_2.$$

Proof. According to [**2**, p. 35] we have $\Delta_h = L_h - D_h$, where

$$D_h\psi(x) := \sum_{j=1}^{m} \alpha_j \frac{\psi(x) - \psi(x\sigma_j)}{\langle x, v_j\rangle^2}\|v_j\|^2.$$

and

$$L_h\psi := (\Delta(\psi h) - \psi\Delta h)/h$$

Let us define ${}_{\mathbf{S}}L_h$ on $C^2(\mathbf{S}^{N-1})$ by ${}_{\mathbf{S}}L_h f := (L_h(f{\uparrow})){\downarrow}$. We will show that ${}_{\mathbf{S}}L_h$ is self-adjoint. We take $f, g \in C^2(\mathbf{S}^{N-1})$ and apply Green's formula to $F := f{\uparrow}$, $G := \overline{g}{\uparrow}$ and $\Omega := B(0, r) \setminus B(0, 1/2)$ (where $r > 1/2$):

$$\begin{aligned}&\int_\Omega [(L_h F)G - F(L_h G)](x)\, H_\alpha\, h^2(x)\, dx \\ &= \int_\Omega [\{(\Delta(Fh) - F\Delta h)/h\}G - F\{(\Delta(Gh) - G\Delta h)/h\}](x)\, H_\alpha\, h^2(x)\, dx \\ &= \int_\Omega [(\Delta(Fh)\cdot(Gh) - (Fh)\cdot\Delta(Gh)](x) H_\alpha\, dx \\ &= \int_{\partial\Omega} [(\partial_\nu(Fh)\cdot(Gh) - (Fh)\cdot\partial_\nu(Gh)](y)\, H_\alpha\, d\sigma_{N-1}(y) =: I\end{aligned}$$

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Here Fh and Gh are homogeneous (of degree $|\alpha|$). Now, if ψ is homogeneous and ν is the outer normal vector to $\partial B(0,\rho)$, then

$$\partial_\nu \psi(y) = \langle \operatorname{grad} \psi(y), \nu(y)\rangle = \langle \operatorname{grad} \psi(y), y/\|y\|\rangle = \|y\|^{-1} \deg \psi \cdot \psi(y)$$

by Euler's formula. Therefore the integral I is equal to

$$\begin{aligned}
&\int_{\partial B(0,r)} [r^{-1}|\alpha|(Fh)(Gh) - (Fh)r^{-1}|\alpha|(Gh)](y)\, H_\alpha\, d\sigma_{N-1}(y) \\
&\qquad - \int_{\partial B(0,1/2)} [\,2|\alpha|(Fh)(Gh) - (Fh)2|\alpha|(Gh)](y)\, H_\alpha\, d\sigma_{N-1}(y) \\
&= 0 - 0 = 0.
\end{aligned}$$

We have thus proved that

$$\int_{1/2}^{r} \int_{\mathbf{S}^{N-1}} [(L_h F)G - F(L_h G)](\rho y)\, H_\alpha\, h^2(\rho y)\, d\sigma_{N-1}(y)\, \rho^{N-1} d\rho = 0$$

for all $r > 1/2$. Let us differentiate this equality with respect to r and then evaluate at $r = 1$; we get

$$\int_{\mathbf{S}^{N-1}} [(L_h F)G - F(L_h G)](y)\, H_\alpha\, h^2(y)\, d\sigma_{N-1}(y) = 0,$$

that is,

$$\int_{\mathbf{S}^{N-1}} [{}_{\mathbf{S}}L_h f \cdot \overline{g} - f \cdot \overline{{}_{\mathbf{S}}L_h g}\,](y)\, d\sigma_h(y) = 0.$$

Next, if we define ${}_{\mathbf{S}}D_h$ on $C^2(\mathbf{S}^{N-1})$ by ${}_{\mathbf{S}}D_h f := (D_h(f\uparrow))\downarrow$, then it is self-adjoint by [**2**, Proposition 1.2]. To conclude, we note that ${}_{\mathbf{S}}\Delta_h = {}_{\mathbf{S}}L_h - {}_{\mathbf{S}}D_h$.

4. Fourier expansions. Given $\eta \in \mathbf{S}^{N-1}$, the mapping $\Lambda : \mathcal{S}H_l(\mathbf{S}^{N-1}) \to \mathbf{C}$ defined by $\Lambda(Y) := Y(\eta)$ is a linear form on the finite dimensional hermitian space $\mathcal{S}H_l(\mathbf{S}^{N-1})$ with the scalar product $\langle\ ,\ \rangle_2$. Hence there exists $P_l(\cdot,\eta) \in \mathcal{S}H_l(\mathbf{S}^{N-1})$ such that $Y(\eta) = \Lambda(Y) = \langle Y, P_l(\cdot,\eta)\rangle_2$ for all $Y \in \mathcal{S}H_l(\mathbf{S}^{N-1})$; P_l is called the *reproducing kernel* of $\mathcal{S}H_l(\mathbf{S}^{N-1})$.

If $f \in L^2(\mathbf{S}^{N-1}, d\sigma_h)$ and $l \in \mathbf{N}_0$, we write $\Pi_l(f)$ for the orthogonal projection of f on $\mathcal{S}H_l(\mathbf{S}^{N-1})$; we call the series

$$\sum_{l=0}^{+\infty} \Pi_l(f)$$

the *Fourier expansion* of f (in spherical h-harmonics). For any orthonormal basis $(E_1^l, \ldots, E_{d_l}^l)$ of $\mathcal{S}H_l(\mathbf{S}^{N-1})$,

$$\Pi_l(f) = \sum_{j=1}^{d_l} \langle f, E_j^l\rangle_2\, E_j^l.$$

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Moreover

$$P_l(\cdot,\eta)=\sum_{j=1}^{d_l}\langle P_l(\cdot,\eta),E_j^l\rangle_2\,E_j^l=\sum_{j=1}^{d_l}\overline{\langle E_j^l,P_l(\cdot,\eta)\rangle_2}\,E_j^l=\sum_{j=1}^{d_l}\overline{E_j^l(\eta)}\,E_j^l$$

and

$$\Pi_l(f)(\eta)=\langle \Pi_l(f),P_l(\cdot,\eta)\rangle_2=\langle f,P_l(\cdot,\eta)\rangle_2.$$

PROPOSITION 3. *The Fourier expansion of any $f\in L^2(\mathbf{S}^{N-1},d\sigma_h)$ converges to f in $L^2(\mathbf{S}^{N-1},d\sigma_h)$.*

Proof. It suffices to show that $\oplus_{l=0}^{+\infty}\mathcal{S}H_l(\mathbf{S}^{N-1})$ is dense in $L^2(\mathbf{S}^{N-1},d\sigma_h)$. But, according to [**2**, Theorem 1.7], for every $g\in\mathcal{P}_n$ we can write

$$g(x)=\sum_{j=0}^{\lfloor n/2\rfloor}\|x\|^{2j}g_{n-2j}(x)$$

with $g_{n-2j}\in\mathcal{H}_{n-2j}$; hence $\oplus_{j=0}^{n}\mathcal{S}H_j(\mathbf{S}^{N-1})\supset\{P\!\downarrow: P\in\mathcal{P}_l,\ 0\le l\le n\}$. Since $\{P\!\downarrow: P\in\mathcal{P}_l,\ l\in\mathbf{N}_0\}$ is dense in $L^2(\mathbf{S}^{N-1},d\sigma_h)$, the proof is complete.

PROPOSITION 4. *For every $f\in C^2(\mathbf{S}^{N-1})$ and $l\in\mathbf{N}_0$,*

$$\Pi_l({}_{\mathbf{S}}\Delta_h f)=-l(l+2|\alpha|+N-2)\,\Pi_l(f).$$

Proof. If $\eta\in\mathbf{S}^{N-1}$ we get, by Propositions 1 and 2,

$$\begin{aligned}\Pi_l({}_{\mathbf{S}}\Delta_h f)(\eta)&=\langle {}_{\mathbf{S}}\Delta_h f,P_l(\cdot,\eta)\rangle_2\\&=\langle f,{}_{\mathbf{S}}\Delta_h P_l(\cdot,\eta)\rangle_2\\&=\langle f,-l(l+2|\alpha|+N-2)P_l(\cdot,\eta)\rangle_2\\&=-l(l+2|\alpha|+N-2)\,\langle f,P_l(\cdot,\eta)\rangle_2\\&=-l(l+2|\alpha|+N-2)\,\Pi_l(f)(\eta).\end{aligned}$$

LEMMA 2. *For all $l\in\mathbf{N}_0$ and $\zeta,\eta\in\mathbf{S}^{N-1}$, $|P_l(\zeta,\eta)|\le d_l^{(2|\alpha|+N)}$.*

Proof. According to [**7**, Theorem 3.2],

$$P_l(\zeta,\eta)=\frac{l+|\alpha|+(N-2)/2}{|\alpha|+(N-2)/2}\,V\big[C_l^{(|\alpha|+(N-2)/2)}(\langle\cdot,\eta\rangle)\big](\zeta),$$

where $C_l^{(\lambda)}$ denotes the Gegenbauer polynomial defined by

$$\frac{1-r^2}{(1-2tr+r^2)^{\lambda+1}}=\sum_{k=0}^{+\infty}\frac{k+\lambda}{\lambda}\,C_k^{(\lambda)}(t)\,r^k$$

and V is the intertwining operator defined uniquely as being linear with $V\mathcal{P}_n\subset\mathcal{P}_n$, $V1=1$ and $\mathcal{D}_i\circ V=V\circ\partial_i$ (see [**3**]). But V is positive [**6**, Theorem 1.2], which implies that

$$\begin{aligned}\big|V\big[C_l^{(|\alpha|+(N-2)/2)}(\langle\cdot,\eta\rangle)\big](\zeta)\big|&\le\sup_{\|y\|\le1}C_l^{(|\alpha|+(N-2)/2)}(\langle y,\eta\rangle)\\&\le C_l^{(|\alpha|+(N-2)/2)}(1),\end{aligned}$$

since $|C_l^{(\lambda)}(t)|\le C_l^{(\lambda)}(1)$ for all $|t|\le1$.

Now, if $|\alpha| = 0$ we are in the classical case, where

$$\frac{l + (N-2)/2}{(N-2)/2}\, C_l^{((N-2)/2)}(1) = P_l(\eta, \eta)$$

[**8**, p. 187] and $P_l(\eta, \eta) = d_l^{(N)}$ [**1**, Proposition 5.27]. This completes the proof.

LEMMA 3. *Let $f \in L^2(\mathbf{S}^{N-1}, d\sigma_h)$ and $l \in \mathbf{N}_0$. For all $\eta \in \mathbf{S}^{N-1}$ we have*

$$|\Pi_l(f)(\eta)| \leq \sqrt{d_l^{(2|\alpha|+N)}} \cdot \|f\|_2.$$

Proof. Let $(E_1^l, \ldots, E_{d_l}^l)$ be an orthonormal basis of $\mathcal{S}H_l(\mathbf{S}^{N-1})$. Then

$$\begin{aligned} |\Pi_l(f)(\eta)| &= \left| \sum_{j=1}^{d_l} \langle f, E_j^l\rangle_2\, E_j^l(\eta) \right| \\ &\leq \sum_{j=1}^{d_l} |\langle f, E_j^l\rangle_2| \cdot |E_j^l(\eta)| \\ &\leq \left(\sum_{j=1}^{d_l} |\langle f, E_j^l\rangle_2|^2 \right)^{1/2} \cdot \left(\sum_{j=1}^{d_l} |E_j^l(\eta)|^2 \right)^{1/2} \end{aligned}$$

by the Cauchy-Schwarz inequality. Moreover,

$$\sum_{j=1}^{d_l} |\langle f, E_j^l\rangle_2|^2 \leq \|f\|_2^2$$

by Bessel's inequality and

$$\sum_{j=1}^{d_l} |E_j^l(\eta)|^2 = \sum_{j=1}^{d_l} \overline{E_j^l(\eta)}\, E_j^l(\eta) = P_l(\eta, \eta) \leq d_l^{(2|\alpha|+N)}$$

by Lemma 2.

LEMMA 4. *Let $q \in \mathbf{N}_0$. There exists a constant $C_q > 0$ depending only on q, h and N such that, for all $l \in \mathbf{N}_0$, $f \in C^{2q}(\mathbf{S}^{N-1})$ and $\eta \in \mathbf{S}^{N-1}$,*

$$|\Pi_l(f)(\eta)| \leq C_q \left\| {}_{\mathrm{S}}\Delta_h^q f \right\|_2 \cdot l^{|\alpha|+N/2-2q-1}.$$

Proof. On the one hand, the preceding lemma implies that

$$\left|\Pi_l\left({}_{\mathrm{S}}\Delta_h^q f\right)(\eta)\right| \leq \sqrt{d_l^{(2|\alpha|+N)}} \cdot \left\| {}_{\mathrm{S}}\Delta_h^q f \right\|_2.$$

On the other hand, Proposition 4 implies that

$$\left|\Pi_l\left({}_{\mathrm{S}}\Delta_h^q f\right)(\eta)\right| = l^q(l + 2|\alpha| + N - 2)^q \cdot |\Pi_l(f)(\eta)|.$$

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Therefore

$$|\Pi_l(f)(\eta)| \leq \frac{\sqrt{d_l^{(2|\alpha|+N)}}}{l^q(l+2|\alpha|+N-2)^q} \left\| \mathrm{s}\Delta_h^q f \right\|_2.$$

To complete the proof we use the bound $d_l^n \leq 2l^{n-2} + O(l^{n-3})$ when $l \to +\infty$.

PROPOSITION 5. *Let $q \in \mathbf{N}$ with $q > |\alpha|/2 + N/4$. The Fourier expansion of any $f \in C^{2q}(\mathbf{S}^{N-1})$ converges to f uniformly on $\mathbf{S}^{N-1}$.*

Proof. If $q > |\alpha|/2 + N/4$, then $|\alpha| + N/2 - 2q - 1 < -1$; hence, by the preceding lemma, the Fourier expansion $\sum_{l=0}^{+\infty} \Pi_l(f)$ converges absolutely and uniformly on $\mathbf{S}^{N-1}$ to a continuous function we denote by g. But the series converges to g also in $L^2(\mathbf{S}^{N-1}, d\sigma_h)$, since

$$\|\phi - \psi\|_2 \leq \|\phi - \psi\|_\infty \cdot \sqrt{H_\alpha\, \omega_{N-1}}\, \|h\|_\infty$$

for $\phi, \psi \in C(\mathbf{S}^{N-1})$. From Proposition 3 it follows that $f = g$ almost everywhere and then everywhere by continuity.

ACKNOWLEDGEMENT. Work supported by the Swiss National Science Foundation.

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