

TWO SEMIGROUPS OF CONTINUOUS RELATIONS

A. R. BEDNAREK and EUGENE M. NORRIS

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Synopsis

In this paper we define two semigroups of continuous relations on topological spaces and determine a large class of spaces for which Banach–Stone type theorems hold, i.e. spaces for which isomorphism of the semigroups implies homeomorphism of the spaces. This class includes all 0-dimensional Hausdorff spaces and all those completely regular Hausdorff spaces which contain an arc; indeed all of K. D. Magill's S^* -spaces are included. Some of the algebraic structure of the semigroup of all continuous relations is elucidated and a method for producing examples of topological semigroups of relations is discussed.

1. Wide continuous relations

X and Y will always denote topological spaces which need not be Hausdorff. The interior and closure of a set A are denoted A° , A^* respectively. We let $\mathcal{A}(X)$ (respectively, $\mathcal{K}(X)$) denote the collection of all nonvoid subsets (respectively, all nonvoid compact subsets) of X , and consider these collections to be topological spaces in the finite topology of Michael (1951).

A relation from X to Y is a subset of $X \times Y$. If R is a relation from X to Y then R^{-1} is a relation from Y to X , where $R^{-1} = \{(y, x) : (x, y) \in R\}$. If $R \subseteq X \times Y$, $A \subseteq X$ and $B \subseteq Y$, put $AR = \{y : \exists a \in A, (a, y) \in R\}$ and put $RB = BR^{-1}$. A singleton subset will usually be denoted by its element, so that xR will appear in place of $\{x\}R$, and so forth. If $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, put

$$R \circ S = \{(x, z) : \text{for some } y \in Y, (x, y) \in R \text{ and } (y, z) \in S\}.$$

One easily establishes that $(R \circ S) \circ T = R \circ (S \circ T)$ for all relations R, S and T , and that the formulas

$$(1.1) \quad A(R \circ S) = (AR)S$$

$$(1.2) \quad (R \circ S)B = R(SB)$$

$$(1.3) \quad A(R \cup S) = AR \cup AS$$

$$(1.4) \quad (R \cup S)B = RB \cup SB$$

are universally valid. Equations (1.2) and (1.4) are consequences of (1.1) and (1.3) respectively, in light of the identity

$$(1.5) \quad (R \circ S)^{-1} = S^{-1} \circ R^{-1}$$

We note that if R and S are functions then aR and Rb are more commonly denoted by $R(a)$ and $R^{-1}(b)$ respectively, while $a(R \circ S)$ becomes $S(R(a))$. We shall write functions to the left or to the right of their arguments, as is convenient.

If $R \subseteq X \times Y$ we say R is *wide* provided that $X = RY$. To each wide relation $R \subseteq X \times Y$ there corresponds a unique function $f_R: X \rightarrow \mathcal{A}(Y)$, given by $f_R(x) = xR$. The correspondence $\Psi: R \leftrightarrow f_R$ is a bijection between the set of wide relations from X to Y and the set of all functions from X into $\mathcal{A}(Y)$. To each such relation there corresponds uniquely another function $F_R: \mathcal{A}(X) \rightarrow \mathcal{A}(Y)$, given by $F_R(A) = AR$. It is easy to see that $F_R(A) = \cup \{f_R(x): x \in A\}$.

Day and Franklin (1967) say a relation $R \subseteq X \times Y$ is *continuous* provided the following conditions hold:

- (i) RB is closed in X whenever B is closed in Y
- (ii) RB is open in X whenever B is open in Y
- (iii) xR is compact for each $x \in X$.

Each of these three conditions defines a property of relations: (i) is *upper semicontinuity*, (ii) is *lower semicontinuity* (Day and Franklin (1967)) and (iii) is *point-compactness*. In addition, a relation R is called *point-closed* if xR is a closed subset of Y for each $x \in X$. Upper and lower semicontinuity are known to have several equivalent formulations with which we assume the reader to be familiar.

In this paper we shall be concerned with several collections of relations and functions. We denote by $C(X, Y)$ the set of all continuous relations from X into Y ; $CW(X, X)$ is the set of all continuous wide relations from X into Y and $CF(X, Y)$ is the set of all continuous functions from X into Y . Function spaces will always carry the compact-open topology. We shall be interested particularly in the function spaces $CF(X, \mathcal{K}(Y))$ and $CF(\mathcal{K}(X), \mathcal{K}(Y))$. For brevity's sake we write $C(X)$ in place of $C(X, X)$, etc. Day and Franklin (1967) proved that the bijection $\Psi(R) = f_R$ mentioned above maps $CW(X)$ onto $CF(X, \mathcal{K}(X))$. We shall prove that $CW(X)$ is a semigroup under the operation of relation composition. More is actually true, although we shall not

include the additional proof: with the additional operation of pairwise union, over which relation composition distributes, $CW(X)$ is a semiring. We shall need the following well-known lemma, the brief proof of which seems to be new to the literature.

LEMMA 1.1. *If R is a continuous relation from X onto Y , then AR is compact whenever A is compact.*

PROOF. It is no loss to assume that A is a compact subset of RY . Since the function $f_R: RY \rightarrow \mathcal{K}(Y)$ is continuous (a result of Michael (1955), Theorem 9.2), $f_R(A) = \{xR: x \in A\}$ is a compact subset of $\mathcal{K}(Y)$, i.e. $f_R(A) \in \mathcal{K}(\mathcal{K}(Y))$, and hence (Michael (1951), Theorem 2.5.2) $\cup f_R(A) = AR$ is compact.

PROPOSITION 1.2. *$C(X)$ and $CW(X)$ are semigroups under the operation of relation composition.*

PROOF. We omit the routine verification that the composition of upper (respectively, lower) semicontinuous relations is again upper (lower) semicontinuous. If R and S are in $C(X)$ and $x \in X$, then xR is compact, and, by Lemma 1.1, $x(R \circ S) = (xR)S$ is compact. If R and S are wide, it is immediate from (1.1) that $R \circ S$ is wide.

The function space $CF(\mathcal{K}(X))$ is of course a semigroup (in the discrete topology) under function composition. Of particular interest is its subsemigroup $CF_0(\mathcal{K}(X))$ of all functions $f \in CF(\mathcal{K}(X))$ that preserve unions of finite families of subsets of X . It is immediately seen that $CF_0(\mathcal{K}(X))$ is a semigroup: indeed, it is the semigroup of endomorphisms of the topological semilattice $(\mathcal{K}(X), \cup)$. We define $CF_0(\mathcal{K}(X), \mathcal{K}(Y))$ analogously to $CF_0(\mathcal{K}(X))$.

PROPOSITION 1.3. *$CW(X)$ is isomorphic to $CF_0(\mathcal{K}(X))$, if X is Hausdorff.*

PROOF. If $R \in CW(X)$ then, as Michael (1951) proved, $f_R: X \rightarrow \mathcal{K}(X)$ is continuous, where $f_R(x) = xR$. A result of Michael (1951), Theorem 5.10.1) then implies that the induced function $f_R^*: \mathcal{A}(X) \rightarrow \mathcal{A}(\mathcal{K}(X))$, given by $f_R^*(A) = \{f_R(x): x \in A\}$, is also continuous. It is then a consequence of Lemma 1.1 that f_R^* maps $\mathcal{K}(X)$ into $\mathcal{K}(\mathcal{K}(X))$. Michael's "union" function $\sigma: \mathcal{A}(\mathcal{A}(X)) \rightarrow \mathcal{A}(X)$, given by $\sigma(\mathcal{B}) = \cup \mathcal{B}$ for $\mathcal{B} \in \mathcal{A}(\mathcal{A}(X))$, is continuous, provided all spaces carry the finite topology (ibid., Theorem 5.7.2), and therefore the restriction of σ to $\mathcal{K}(\mathcal{K}(X))$ is continuous; according to Theorem 2.5.2 (ibid.), σ carries $\mathcal{K}(\mathcal{K}(X))$ into $\mathcal{K}(X)$. Hence the function $F_R: \mathcal{K}(X) \rightarrow \mathcal{K}(X)$ given by $F_R(A) = AR$ is continuous, since $F_R(A) = \sigma(f_R^*(A))$ for each $A \in \mathcal{K}(X)$. Observe that for any relation R and any

collection \mathcal{B} of sets, $(\cup \mathcal{B})R = \cup \{AR : A \in \mathcal{B}\}$, so that $F_R(\cup \mathcal{B}) = \cup \{F_R(A) : A \in \mathcal{B}\}$; hence, for each $R \in CW(X)$, $F_R \in CF_0(\mathcal{H}(X))$. We now define a function $\Phi: CW(X) \rightarrow CF_0(\mathcal{H}(X))$ by $\Phi(R) = F_R$. Routine checking shows that Φ is 1-1 and a semigroup homomorphism. We will be done if we show that Φ maps $CW(X)$ onto $CF_0(\mathcal{H}(X))$.

If $F \in CF_0(\mathcal{H}(X))$, then, letting $R = \cup_{x \in X} \{x\} \times F(\{x\})$, we will see that $F = F_R$. The restriction of F to the singletons is continuous as a function from X to $\mathcal{H}(X)$ and one sees $F(\{x\}) = F_R(\{x\})$ for all $x \in X$, so $R \in CW(X)$.

Next we will prove that F and F_R agree on the set \mathcal{J} of finite subsets of X , which is dense in $\mathcal{H}(X)$ (ibid, 2.4.1). To this end, let $A \in \mathcal{J}$ and let $\pi: X \times X \rightarrow X$ be the projection mapping (x, y) to y . We compute: $F_R(A) = \pi[(A \times X) \cap R] = \cup_{x \in A} \pi[\{x\} \times F(\{x\})] = \cup_{x \in A} F(\{x\}) = F(A)$. Hence the continuous functions F and F_R agree on all of $\mathcal{H}(X)$, since $\mathcal{H}(X)$ is Hausdorff whenever X is.

The following easily established observation is interesting.

COROLLARY 1.4. *The isomorphism $R \rightarrow F_R$ is order-preserving: if $R \subseteq S$ then $F_R \leq F_S$, where $f \leq g$ means $f(A) \subseteq g(A)$ for every $A \in \mathcal{H}(X)$.*

DEFINITION. Let A be any subset of X . The relation *constantly A* is $K_A = X \times A$.

PROPOSITION 1.5. *A nonvoid rectangular relation $A \times B$ is in $C(X, Y)$ if and only if A is clopen and B is compact; hence $K_B \in CW(X, Y)$ whenever $B \in \mathcal{H}(Y)$.*

PROOF. If $R = A \times B \in C(X, Y)$ then $A = RY$ is clopen since Y is a clopen subset of itself. If $x \in X$, then xR is either void or equal to B , depending on whether $x \in A$ or not; hence B is compact. Conversely, if $C \subseteq Y$, RC is empty or equals A , depending on whether or not $B \cap C$ is void. The upper and lower semicontinuity of R is clear from this, and the compactness of B and \emptyset guarantee the point-compactness of R . The second assertion is clear.

PROPOSITION 1.6. *There is precisely one 0-minimal ideal in $C(X)$, namely $M(X) = \{A \times B : A \text{ is clopen and } B \text{ is compact}\}$. There is precisely one 0-minimal ideal in $CW(X)$, namely $MW(X) = \{K_A : A \in \mathcal{H}(X)\}$.*

PROOF. $M(X)$ is defined in such a way as to include \emptyset , the empty relation on X , which is clearly the zero of $C(X)$. This fact, together with the equations

$$(1.6) \quad R \circ (A \times B) = RA \times B$$

$$(1.7) \quad (A \times B) \circ R = A \times BR,$$

which hold for any relation R and any rectangular relation $A \times B$, proves that $M(X)$ is an ideal of $C(X)$. If I is any ideal of $C(X)$, let $R \in I$ and $(z_1, z_2) \in R$. Then for any nonvoid $A \times B \in M(X)$, $A \times \{z_1\}$ and $X \times B$ are surely in $C(X)$; since $A \times B = A \times \{z_1\} \circ R \circ X \times B \in I$, it follows that $M(X) \subseteq I$. The assertion for $CW(X)$ is proved similarly.

2. A Banach–Stone theorem for $CW(X)$

If h is a bijection from a space X onto a space Y , and Φ is defined by the equation

$$\Phi(R) = h^{-1} \circ R \circ h$$

for each relation R from X to Y , then it is routine to verify that Φ is a finite-union-preserving (in fact, a union-preserving), one-to-one homomorphism of relations; if h is a homeomorphism, then Φ is a finite-union-preserving isomorphism of $CW(X)$ onto $CW(Y)$. We aim to show that the converse of this assertion is true for a quite extensive class of spaces. We begin with the following result.

PROPOSITION 2.1. *If $\Phi: CW(X) \rightarrow CW(Y)$ is any finite-union-preserving isomorphism then there is a bijection $h: X \rightarrow Y$ for which*

$$\Phi(R) = h^{-1} \circ R \circ h$$

for all $R \in CW(X)$.

PROOF. Since Φ is an isomorphism, it maps the 0-minimal ideal of $CW(X)$ onto that of $CW(Y)$; so we may define $H: \mathcal{H}(X) \rightarrow \mathcal{H}(Y)$ by the condition $H(A) = B$ if and only if $\Phi(K_A) = K_B$. It is easily verified that H is a bijection. We include the details of this verification.

(i) H is a function: If $(A, B), (A, C) \in H$, then $Y \times B = K_B = \Phi(K_A) = K_C = Y \times C$, so $B = C$.

(ii) H is one-to-one: If $H(A) = H(B) = C$, then $K_C = \Phi(K_A) = \Phi(K_B)$; since Φ is one-to-one, $K_A = K_B$, whence it follows $A = B$.

(iii) H is onto: If $B \in \mathcal{H}(Y)$, K_B is in the minimal ideal of $CW(X)$, hence is the image under Φ of some K_A in the 0-minimal ideal of $CW(X)$. Then $B = H(A)$.

We will show that H preserves singleton subsets (and thus induces the required function h) by showing that H preserves finite unions. We shall use the fact—a consequence of the definition of H —that $\Phi(K_A) = K_{H(A)}$. To see that H preserves finite unions suppose $\mathcal{B} = \{A_1, \dots, A_n\}$ is a finite subcollection of $\mathcal{H}(X)$ and put $B = \cup \mathcal{B}$. Then $K_B = X \times B$ and $\cup \{X \times A \mid A \in \mathcal{B}\} = \cup \{K_A \mid A \in \mathcal{B}\}$, whence follows

$$\begin{aligned} Y \times H(B) &= K_{H(B)} = \Phi(K_B) = \cup \{\Phi(K_A) \mid A \in \mathcal{B}\} = \cup \{K_{H(A)} \mid A \in \mathcal{B}\} \\ &= \cup \{Y \times H(A) \mid A \in \mathcal{B}\} = Y \times \cup \{H(A) \mid A \in \mathcal{B}\}. \end{aligned}$$

From the extremes of this last we see

$$H(B) = \cup \{H(A) : A \in \mathcal{B}\}$$

(We observe in passing that H preserves arbitrary unions, provided Φ does also.)

Next let $x \in X$ and suppose $y \in H(\{x\})$. There is some $A \in \mathcal{K}(X)$ for which $H(A) = \{y\}$, and it follows that

$$H(\{x\}) = H(\{x\}) \cup \{y\} = H(\{x\}) \cup H(A) = H(\{x\} \cup A);$$

since H is one-to-one, then $\{x\} = \{x\} \cup A$, implying $A \subseteq \{x\}$. Since $A \in \mathcal{K}(X)$, A is nonvoid and therefore $A = \{x\}$. But this implies that $\{y\} = H(A) = H(\{x\})$, i.e. $H(\{x\})$ is a singleton.

The condition $h(x) \in H(\{x\})$ defines a function $h: X \rightarrow Y$ which is clearly a bijection. To see that h has the asserted property, let $R \in CW(X)$ and $y = h(x) \in Y$. Then we compute as follows:

$$\begin{aligned} y\Phi(R) &= y[K_y \circ \Phi(R)] = y[\Phi(K_x) \circ \Phi(R)] = y[\Phi(K_x \circ R)] = y[\Phi(K_{xR})] \\ &= y[\Phi(K_{(yh^{-1}R)})] = y[\Phi(K_{y(h^{-1} \circ R)})] = y[K_{y(h^{-1} \circ R)h}] \\ &= y[K_{y(h^{-1} \circ R \circ h)}] = y[h^{-1} \circ R \circ h]. \end{aligned}$$

Hence $\phi(R) = h^{-1} \circ R \circ h$.

In order to “topologize” proposition 1.7, we need to identify a class of spaces for which that proposition is true for *homeomorphisms* h . The class of CW -spaces defined below is such a class; it is, as we shall see, quite extensive.

DEFINITION. X is a CW -space if and only if $\{Rx \mid x \in X, R \in CW(X)\}$ is a basis for the closed subsets of X .

We note that since the identity function 1_x is in $CW(X)$ that X is necessarily T_1 if it is CW . It will be convenient to refer to the defining basis as the CW -basis of X .

Magill (1967) has defined the class of S^* -spaces as follows: X is an S^* -space if X is T_1 and whenever F is a closed subset of X and $x \notin F$ there is a point $q \in X$ and a continuous function $f: X \rightarrow X$ such that $f(F) = \{q\}$ and $f(x) \neq q$. He proved that X is an S^* -space just in case X is T_1 and the family $\{f^{-1}(x) \mid x \in X, f \in CF(X)\}$ is a basis for the closed subsets of X . Magill proved that among the S^* -spaces are the 0-dimensional Hausdorff spaces as well as those completely regular spaces containing an arc. In light of the following proposition, the class of CW -spaces is rather broad.

PROPOSITION 2.2. *Every S^* -space is a CW-space.*

PROOF. $CF(X) \subseteq CW(X)$, so

$$\{f^{-1}(x): f \in CF(X), x \in X\} \subseteq \{Rx \mid x \in X, R \in CW(X)\};$$

the upper semicontinuity of each R together with the fact that X is T_1 implies that $\{Rx \mid x \in X, R \in CW(X)\}$ is a basis for the closed sets of X .

We may now state and prove the principal result of this section of the paper.

THEOREM 2.3. *Let X and Y be CW-spaces. The following are equivalent:*

- (i) *there is a homeomorphism h from X to Y*
- (ii) *there is a finite-union-preserving isomorphism Φ from $CW(X)$ to $CW(Y)$.*

The homeomorphism and isomorphism satisfy the equation

$$\Phi(R) = h^{-1} \circ R \circ h$$

for all $R \in CW(X)$.

PROOF. That homeomorphism implies isomorphism is clear; we proceed to prove the converse. Suppose that $\Phi: CW(X) \rightarrow CW(Y)$ is an isomorphism and $h: X \rightarrow Y$ is the corresponding bijection from proposition 2.1, i.e. $\Phi(R) = h^{-1} \circ R \circ h$ for all $R \in CW(X)$. We will be done if we show that $(Rx)h = (\Phi(R))(xh)$, for this will show that h induces a bijection between the CW-bases of X and Y and is therefore a homeomorphism.

If $R \in CW(X)$ and $x \in X$, then, writing h to the right,

$$\begin{aligned} (Rx)h &= (xR^{-1})h = x(R^{-1} \circ h) = (R^{-1} \circ h)^{-1}x = (h^{-1} \circ R)x \\ &= (h^{-1} \circ R \circ h \circ h^{-1})x \\ &= (h^{-1} \circ R \circ h)(h^{-1}x) = \Phi(R)(xh). \end{aligned}$$

We pause to record an interesting consequence of Theorem 2.3. Vitanza (1966) studied a notion of inner endomorphism on a semigroup S , that is, such a homomorphism $f: S \rightarrow S$ that for some a and $b \in S$ and for all $x \in S$, $f(x) = axb$. Vitanza showed that an automorphism of S is inner if S is a monoid and the a and b of the preceding definition are inverses relative to the identity of S . In $CW(X)$, of course, the identity function 1_x is the identity and h is a unit relative to 1_x just in case h is a homeomorphism of X onto X .

COROLLARY 2.4. *Let X be a CW-space and Φ an automorphism of $CW(X)$. Then these are equivalent:*

- (i) *Φ is finite-union-preserving*
- (ii) *Φ is inner.*

PROOF. If Φ preserves finite unions then the theorem gives a homeomorphism h on X for which $\Phi(R) = h^{-1} \circ R \circ h$ for all $R \in CW(X)$, so Φ is inner. Conversely there exist, by Vitanza's theorem (op. cit.) and the preceding remarks, a homeomorphism h from X onto X for which $\Phi(R) = h^{-1} \circ R \circ h$ for all $R \in CW(X)$; if $\{R_1, \dots, R_n\}$ is any finite subset of $CW(X)$, then $\Phi(\bigcup_i R_i) = h^{-1} \circ \bigcup_i R_i \circ h = \bigcup_i h^{-1} \circ R_i \circ h = \bigcup_i \Phi(R_i)$, proving that Φ is finite-union-preserving.

Finally, one asks if each isomorphism $\Phi: CW(X) \rightarrow CW(Y)$ is necessarily union-preserving. If so, Theorem 2.3 would be much stronger than it seems to be.

3. Topologies for relation semigroups

The fact that the function $\Phi(R) = F_R$ in the proof of Theorem 1.3 is an isomorphism allows one to produce examples of topological semigroups of relations. For example, if Y is locally compact and Hausdorff then so is $\mathcal{K}(Y)$, and therefore—as is well known—it is the case that $CF(\mathcal{K}(Y))$ is a topological semigroup in the compact-open topology. We pursue this idea briefly.

In general, each choice of topology on $CF(\mathcal{K}(X), \mathcal{K}(Y))$ leads to a topology on $CW(X, Y)$ via Φ , since $CW(X, Y)$ will inherit the subspace topology on $CF_0(\mathcal{K}(X), \mathcal{K}(Y))$. Day and Franklin (1967) give a simple description of the topology \mathcal{U} induced on $CW(X, Y)$ by the bijection $\Psi(R) = f_R$ when $CF(X, \mathcal{K}(Y))$ carries the compact-open topology: \mathcal{U} has for an open subbase all sets $A(K, U) = \{R \in CW(X, Y) : KR \subseteq U\}$ and $B(K, U) = \{R \in CW(X, Y) : K \subseteq RU\}$, where $K \in \mathcal{K}(X)$ and U is open in Y , and Y is a Hausdorff space (op. cit., Prop. 1). We denote by \mathcal{T} the topology on $CW(X, Y)$ induced via Φ when $CF(\mathcal{K}(X), \mathcal{K}(Y))$ carries the compact-open topology. The problem of comparing \mathcal{U} and \mathcal{T} naturally presents itself.

PROPOSITION 3.1. *Let Y be a Hausdorff space. In the notation above, \mathcal{U} is weaker than \mathcal{T} .*

PROOF. (In this proof, the notation $[A, B]$ is used for the set of functions mapping A into B , and $[U], \langle U \rangle$ denote the usual subbasic open sets in $\mathcal{K}(Y)$, namely, $[U] = \{B \in \mathcal{K}(Y) : B \subseteq U\}$ and $\langle U \rangle = \{B \in \mathcal{K}(Y) : B \cap U \neq \emptyset\}$; see Michael (1951).)

Since Y is Hausdorff, so is $\mathcal{K}(Y)$ and therefore (Dugundji, (1965) XII, 5.1(a)) an open subbase for the compact-open topology on $CF_0(\mathcal{K}(X), \mathcal{K}(Y))$ consists of all sets $[\mathcal{C}, [U]], [\mathcal{C}, \langle U \rangle]$ where $\mathcal{C} \in \mathcal{K}(\mathcal{K}(X))$ and U is open in Y . The inverse images of these sets constitute the open subbase for \mathcal{T} . Routine checking shows that the \mathcal{U} -subbasic open sets $A = A(K, U)$ and $B =$

$B(K, U)$ are just the inverse images under Φ of the compact-open sets $\{\{K\}, [U]\}$ and $[\mathcal{B}, \langle U \rangle]$ respectively, where \mathcal{B} is the compact set $\{\{x\}: x \in K\}$ (is compact since it is the image of K via the canonical embedding $x \rightarrow \{x\}$ of X into $\mathcal{K}(X)$). Hence \mathcal{U} is weaker than \mathcal{F} .

THEOREM 3.2. *If X is a locally compact Hausdorff space, then $\mathcal{U} = \mathcal{F}$.*

The proof of this theorem follows from the next proposition and the ensuing discussion.

PROPOSITION 3.3. *If X is locally compact and Hausdorff then $CW(X)$ is a topological semigroup in the topology \mathcal{U} .*

PROOF. (I) If $R \circ S \in A[K, U]$ for K compact and U open in X , then the compact set $K(R \circ S) = (KR)S \subseteq U$; the local compactness of X guarantees the existence of an open relatively compact set V for which $(KR)S \subseteq V \subseteq V^* \subseteq U$. If $y \in KR$ then $yS \subseteq V$ so the upper semicontinuity of S gives (as is well known) an open relatively compact set U_y about y for which it is the case that

$$U_y S \subseteq V,$$

and the lower semicontinuity of S gives that

$$U_y^* S \subseteq (U_y S)^* \subseteq V^* \subseteq U.$$

Since KR is compact there are $y_1, \dots, y_n \in KR$ so that $KR \subseteq U_{y_1} \cup \dots \cup U_{y_n} = W$ and W^* is compact. Then $S \in A[W^*, U]$ and $R \in A[K, W]$. It is immediate that for any $R_1 \in A[K, W]$ and $S_1 \in A[W^*, U]$, $R_1 \circ S_1 \in A[K, U]$.

(II) If $R \circ S \in B[K, U]$, that is, $K \subseteq R(SU)$, then for each $x \in K$, there is some $y_x \in xR$ with the property that $y_x S \cap U$ is nonvoid. The lower semicontinuity of S gives an open set V_x^1 about y_x with the property

$$v \in V_x^1 \text{ implies } vS \cap U \neq \emptyset.$$

Since X is locally compact Hausdorff, there is an open, relatively compact set V_x satisfying

$$y_x \in V_x \subseteq V_x^* \subseteq V_x^1$$

and therefore having the property

$$(1.8) \quad v \in V_x^* \text{ implies } vS \cap U \neq \emptyset.$$

Since $y_x \in V_x \cap xR$, the lower semicontinuity of R gives an open set U_x about x for which

(1.9) $u \in U_x$ implies $uR \cap V_x \neq \emptyset$.

Since K is compact, there exist $x_1, \dots, x_n \in K$ such that $K \subseteq U_{x_1} \cup \dots \cup U_{x_n} = W$; choose the corresponding $y_i = y_{x_i}$ and the corresponding V_i satisfying (1.8); put $V = V_1 \cup \dots \cup V_n$, so that V^* is compact and (again using 1.8) we see that

$$S \in B[V^*, U].$$

Since $K \subseteq W$ then it follows from (1.9) that

$$R \in B[K, V].$$

It is immediate that if $R_1 \in B[K, V]$ and $S_1 \in B[V^*, U]$ then

$$R_1 \circ S_1 \in B[K, U],$$

proving that composition of relations is continuous. \mathcal{U} from its definition is a Hausdorff topology, so the proposition follows.

The bijection Φ from the space $CW(X)$ with topology \mathcal{U} onto the set $CF_0(\mathcal{H}(X))$ induces a topology \mathcal{U}_0 on $CF_0(\mathcal{H}(X))$ which, in light of Proposition 3.1, is weaker than the compact open topology. One can easily prove the next result.

PROPOSITION 3.3. *If F is a topological semigroup of functions on a space Y and if Y is embedded in F , then the evaluation map $P: F \times Y \rightarrow Y$, given by $P(f, y) = f(y)$ is continuous.*

In particular, when X is locally compact and Hausdorff then $\mathcal{H}(X)$ is embedded in the function space of $CF_0(\mathcal{H}(X))$ with topology \mathcal{U}_0 , and from the preceding proposition the evaluation map is continuous and \mathcal{U}_0 is therefore jointly continuous on compacta. But for locally compact Hausdorff spaces it is well known that the compact-open topology is the weakest jointly continuous topology and is therefore weaker than \mathcal{U}_0 . Going back to $CW(X)$ via Φ , then, we see that \mathcal{T} is weaker than \mathcal{U} . This, together with Proposition 3.1, establishes Theorem 3.2.

Although we can consider $CW(X)$ to be a topological semigroup with topology \mathcal{U} whenever X is locally compact and Hausdorff, the general problem of topologizing relation semigroups seems to be a difficult one. For example, it is known that the closed relations on a compact space form a semigroup; each closed relation is itself a point of the hyperspace $\mathcal{H}(X \times X)$. Examples show, even if X is taken to be the closed unit interval, that composition is not continuous in the hyperspace topology.

4. Isomorphism and homeomorphism for $C(X)$

In this section we direct our attention primarily to spaces which are Hausdorff and 0-dimensional. Such spaces of course are regular.

The following easily established result, needed below, generalizes a well-known result on continuous Hausdorff-valued functions and closed graphs. We omit its straightforward proof.

LEMMA 4.1. *A point-closed upper semicontinuous relation with regular range space is closed.*

The semigroup $K(X)$ of clopen relations on X recently studied by Bednarek and Magill (1973) for compact X is easily seen (in the compact case) to be a subsemigroup of $C(X)$; indeed, on any space, if R is an open relation then RA is open for any set A , and if R is closed and A is compact then RA (and AR) are closed. The latter assertion is a consequence of a well-known result of A. D. Wallace. It now follows that $K(X) \subseteq C(X)$ when X is compact. It would be interesting to have a purely algebraic description of $K(X)$ in terms of $C(X)$.

We have the following result, relating relation containment to relation composition. Recall from Proposition 1.6 that $M(X)$ is the 0-minimal ideal of $C(X)$.

LEMMA 4.2. *If R and S are any continuous relations on X , a 0-dimensional Hausdorff space, then the following are equivalent:*

- (i) $R \subseteq S$
- (ii) $P \circ R \circ Q \neq \emptyset$ implies $P \circ S \circ Q \neq \emptyset$ for all $P, Q \in C(X)$
- (iii) $P \circ R \circ Q \neq \emptyset$ implies $P \circ S \circ Q \neq \emptyset$ for all $P, Q \in M(X)$.

PROOF. The implications (i) \Rightarrow (ii) \Rightarrow (iii) hold trivially in any space X for any relations, so there remains only the implication (iii) \Rightarrow (i). This is proved in essentially the same way as Lemma 2 of Bednarek and Magill (1973). Since that result may not be readily available to the reader, we give a proof here. Let $(x, y) \in R$ and let $A \times B$, be any clopen neighborhood of (x, y) . Then $A \times \{x\}$ and $B \times \{y\}$ belong to $M(X)$ and

$$A \times \{x\} \circ R \circ B \times \{y\} \neq \emptyset.$$

Therefore there is a point (z, w) in

$$A \times \{x\} \circ S \circ B \times \{y\}.$$

Hence there are points u, v for which $(z, u) \in A \times \{x\}$, $(u, v) \in S$ and $(v, w) \in B \times \{y\}$. This implies $(u, v) \in A \times B \cap S$, i.e. (x, y) is in the closure of S . Now by Lemma 4.1, S is closed and hence (i) is established.

We may note in passing that the hypothesis of 0-dimensionality in this lemma is essential: the result cannot be carried to 1-dimensional spaces, e.g. the unit interval. Indeed for any connected space X , $C(X)$ is just $CW(X)$ with a zero adjoined; every nontrivial composition is nonvoid.

Lemma 4.2 holds of course for the full relation semigroup $\mathcal{B}(X)$ on any discrete space X . It is of considerable interest to study the class of relation semigroups for which containment is so characterized.

We use Lemma 4.2 to obtain the next easy but interesting result.

PROPOSITION 4.3. *Let X be any space. If Y is a 0-dimensional Hausdorff space then every isomorphism $\Phi: C(X) \rightarrow C(Y)$ is order-preserving.*

PROOF. Suppose P, Q, R and S are in $C(X)$ and $R \subseteq S$ and suppose further that $\Phi(P) \circ \Phi(R) \circ \Phi(Q) \neq \emptyset$; since Φ is a one-to-one homomorphism, $P \circ R \circ Q \neq \emptyset$ and therefore $P \circ S \circ Q \neq \emptyset$. Applying Φ , we see that $\Phi(P) \circ \Phi(S) \circ \Phi(Q) \neq \emptyset$. Since Φ maps $C(X)$ onto $C(Y)$, Lemma 4.2 applies to allow the conclusion $\Phi(R) \subseteq \Phi(S)$.

The next lemma is elementary and involves only the fact—true on any space X —that the union of two continuous relations is continuous. We recall the elementary fact that any order-preserving bijection between join semilattices is join-preserving if its inverse is order-preserving.

COROLLARY 4.4. *Let X and Y be 0-dimensional Hausdorff spaces. Every isomorphism $\Phi: C(X) \rightarrow C(Y)$ preserves finite unions.*

We will now locate $CW(X)$ algebraically in $C(X)$.

PROPOSITION 4.5. *Suppose X is any space and let $R \in C(X)$. The following assertions are equivalent:*

- (i) $R \in CW(X)$
- (ii) $S \circ R \neq \emptyset$ for all $S \in C(X)$, $S \neq \emptyset$
- (iii) $S \circ R \neq \emptyset$ for all $S \in M(X)$, $S \neq \emptyset$

PROOF. The implications (i) \Rightarrow (ii) \Rightarrow (iii) are clear, so we need only verify (iii) \Rightarrow (i). If R satisfies (iii) and $R \notin CW(X)$, then there is a point x not in RX . The rectangular relation $X \times \{x\}$ is in $M(X)$, but $X \times \{x\} \circ R = X \times xR = \emptyset$, contrary to (iii).

COROLLARY 4.6. *If X and Y are spaces, each isomorphism $\Phi: C(X) \rightarrow C(Y)$ carries $CW(X)$ isomorphically onto $CW(Y)$.*

The aim of this sequence of propositions is the next result.

THEOREM 4.7. *Let X and Y be 0-dimensional Hausdorff spaces. Then X and Y are homeomorphic if and only if $C(X)$ and $C(Y)$ are isomorphic.*

PROOF. As is usually the case, it is easy to see that homeomorphism of X and Y implies isomorphism of $C(X)$ and $C(Y)$. To see the converse, let $\Phi: C(X) \rightarrow C(Y)$ be any isomorphism; Φ is necessarily finite-union-preserving by Corollary 4.4 and maps $CW(X)$ onto $CW(Y)$ by Corollary 4.6. Since a 0-dimensional Hausdorff space is a CW -space, Theorem 2.3 applies to yield a homeomorphism $h: X \rightarrow Y$, proving the theorem.

We remark that the homeomorphism h satisfies the condition $\Phi(R) = h^{-1} \circ R \circ h$ for all $R \in CW(X)$. That this equation holds for all $R \in C(X)$ is a highly attractive conjecture.

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Department of Mathematics,
University of Florida,
Gainesville, Florida 32601,
U.S.A.

and

Department of Mathematics and Computer Science,
University of South Carolina,
Columbia, South Carolina 29208,
U.S.A.