Can. J. Math. Vol. **44** (2), 1992 pp. 298–307

THE STRUCTURE OF THE SEQUENCE SPACES OF MADDOX

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ABSTRACT. The sequence of spaces of Maddox, $c_0(p)$, $c(p)$ and $l_\infty(p)$, are investigated. Here, $p = (p_k)$ is a bounded sequence of strictly positive numbers. It is observed that $c_0(p)$ is an echelon space of order 0 and that $l_\infty(p)$ is a co-echelon space of order ∞, while clearly $c(p) = c_0(p) \oplus \langle (1, 1, 1, \ldots) \rangle$. This sheds a new light on the topological and sequence space structure of these spaces: Based on the highly developed theory of (co-) echelon spaces all known and various new structural properties are derived.

1. **Introduction.** Let $p = (p_k)$ be a bounded sequence of strictly positive numbers. Then the sequence spaces of Maddox are defined as

$$c_0(p) = \{ (x_k) : \lim_k |x_k|^{p_k} = 0 \},$$
$$c(p) = \{ (x_k) : \lim_k |x_k - l|^{p_k} = 0 \text{ for some } l \in \mathbb{C} \},$$
$$l_\infty(p) = \{ (x_k) : \sup_k |x_k|^{p_k} < \infty \},$$
$$l(p) = \{ (x_k) : \sum_k |x_k|^{p_k} < \infty \}.$$

These spaces were introduced by Nakano [17], Simons [19] and Maddox [9], [10]; a detailed study was undertaken by I. J. Maddox and his students. For a recent survey article see Luh [8].

Various papers have dealt with the topological and sequence space structure of these spaces, see [6–17, 19]. It seems to have gone unnoticed, however, that $c_0(p)$ and $l_\infty(p)$ belong to well-known and by now well-studied classes of sequence spaces, the echelon and co-echelon spaces. We have as a key result and starting point of our investigations:

THEOREM 0. *Let $p = (p_k)$ be a bounded sequence of strictly positive numbers. Then*
(i) $c_0(p) = \bigcap_{n=1}^\infty \{ (x_k) : \lim_k |x_k| n^{1/p_k} = 0 \}$. *Hence $c_0(p)$ is an echelon space of order 0;*
(ii) $l_\infty(p) = \bigcup_{n=1}^\infty \{ (x_k) : \sup_k |x_k| n^{-1/p_k} < \infty \}$. *Hence $l_\infty(p)$ is a co-echelon space (of order ∞).*

Moreover we obviously have $c(p) = c_0(p) \oplus \langle e \rangle$, where $e = (1, 1, 1, \ldots)$. These observations are not entirely new: In [18] Rolewicz effectively showed (i) if $p_k \longrightarrow 0$, while (ii) is an immediate consequence of two results of Lascarides ([6], Theorems 3, 5). The interpretation of $c_0(p)$ and $l_\infty(p)$ as (co-) echelon spaces sheds a new light on the structure theory of these spaces. Applying the theory of (co-) echelon spaces as developed,

Received by the editors November 26, 1990.
AMS subject classification: 46A45.

for example, in [5], [20] and [1] we derive here the fundamental properties of the spaces $c_0(p)$, $c(p)$ and $l_\infty(p)$ including, as it seems, all previously known results. In contrast, the space $l(p)$ does not seem to be reducible to simpler spaces; it is best studied within the framework of modular sequence spaces (cf. [17]).

This paper can be read independently of earlier work on the sequence spaces of Maddox. See the Remark in Section 4 for a comparison with Maddox's own approach within the theory of paranormed sequence spaces. Instead, we rely heavily on the theory of (co-) echelon spaces, in particular on the study of Bierstedt, Meise and Summers [1]. Their results are used throughout.

2. **Notation and definitions.** A sequence space is a linear subspace of the space ω of all complex sequences. Along with the sequence spaces c_0 and $l_{\mathcal{P}}$ ($1 \leq \mathcal{P} \leq \infty$; we use a script $\mathcal{P}$ to distinguish it from the sequence $p = (p_k)$) we will need the following sequence spaces:

$$\mathrm{cs} = \{x \in \omega : \sum_k x_k \text{ converges}\},$$

$$\mathrm{bs} = \{x \in \omega : (\sum_{k=1}^n x_k)_n \text{ is bounded}\},$$

$$\mathrm{bv} = \{x \in \omega : \sum_k |x_k - x_{k+1}| < \infty\}.$$

A sequence space E is called normal if $y \in E$ and $|x_k| \leq |y_k|$ for $k \in \mathbb{N}$ implies $x \in E$. Let e denote the sequence $(1, 1, 1, \ldots)$ and e_k the sequence $(0, \ldots, 0, 1, 0, \ldots)$ with 1 in the kth position, for $k \in \mathbb{N}$.

The α-, β- and γ-dual of a sequence space E are defined as

$$E^\zeta = \{x \in \omega : (x_k \cdot y_k) \in X_\zeta \text{ for every } y \in E\}$$

for $\zeta = \alpha, \beta, \gamma$, where $X_\alpha = l_1, X_\beta = \mathrm{cs}$ and $X_\gamma = \mathrm{bs}$. Note that E^α is often written $E^\times$, E^β sometimes $E^\dagger$. The space E is called ζ-perfect ($\zeta = \alpha, \beta, \gamma$) if $E^{\zeta\zeta} := (E^\zeta)^\zeta = E$.

Now assume that the sequence space E carries a linear topology. Then E' denotes its topological dual; E'_b denotes the space E' endowed with the strong topology. If E contains every sequence e_k ($k \in \mathbb{N}$), then the f-dual of E is defined as

$$E^f = \left\{x \in \omega : x = \big(f(e_k)\big)_k \text{ for some } f \in E'\right\}$$

If every functional $x = (x_k) \mapsto x_n$ ($n \in \mathbb{N}$) is continuous on E, then E is called a K-space. A K-space that is a Fréchet (Banach, LB-, LF-) space is called an FK-(BK-, LBK-, LFK-) space. For the theory of FK-spaces see [22]. LFK-spaces were introduced (as lFK-spaces) and studied by Boos [2].

Write $X_0 := c_0$ and $X_{\mathcal{P}} := l_{\mathcal{P}}$ for $1 \leq \mathcal{P} \leq \infty$. Let $A = (a_k^{(n)})_{n,k}$ be a Köthe matrix, i.e. a matrix with $a_k^{(n+1)} \geq a_k^{(n)} > 0$ for $n, k \in \mathbb{N}$ ([1], 1.2), and let $V = (1/a_k^{(n)})_{n,k}$ be the associated matrix. Then the spaces

$$\lambda_{\mathcal{P}}(A) = \{x \in \omega : (x_k a_k^{(n)})_k \in X_{\mathcal{P}} \text{ for all } n \in \mathbb{N}\}$$
$$= \bigcap_{n=1}^{\infty} \{x \in \omega : (x_k a_k^{(n)})_k \in X_{\mathcal{P}}\}$$

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and

$$\begin{aligned} k_{\mathcal{P}}(V) &= \{x \in \omega : (x_k / a_k^{(n)})_k \in X_{\mathcal{P}} \text{ for some } n \in \mathbb{N}\} \\ &= \bigcup_{n=1}^{\infty} \{x \in \omega : (x_k / a_k^{(n)})_k \in X_{\mathcal{P}}\} \end{aligned}$$

are called *echelon* and *co-echelon spaces*, respectively, of order $\mathcal{P}$. Echelon spaces of order 1 and co-echelon spaces of order ∞ are known as the echelon and co-echelon spaces of Köthe (cp. [5], § 30.8). $\lambda_{\mathcal{P}}(A)$ is endowed with the projective limit topology that is induced by the seminorms $q_n(x) = \|(x_k a_k^{(n)})_k\|_{\mathcal{P}}$ for $n \in \mathbb{N}$ with $\|\cdot\|_{\mathcal{P}}$ being the norm in $X_{\mathcal{P}} \cdot k_{\mathcal{P}}(V)$ is endowed with the obvious inductive limit topology. Thus $\lambda_{\mathcal{P}}(A)$ becomes an FK-space, $k_{\mathcal{P}}(V)$ an LBK-space.

A Köthe matrix $A = (a_k^{(n)})$ is said to satisfy condition (M) if for each infinite subset K of $\mathbb{N}$ and each $n \in \mathbb{N}$ there is some $m > n$ with $\inf_{k \in K} a_k^{(n)} / a_k^{(m)} = 0$. It satisfies conditions (S) and (N) if for each $n \in \mathbb{N}$ there is some $m > n$ with $(a_k^{(n)} / a_k^{(m)}) \in c_0$ or l_1, respectively. Its associated matrix V is said to be *regularly decreasing* if for each $n \in \mathbb{N}$ there is some $m \geq n$ such that for all subsets K of $\mathbb{N}$, $\inf_{k \in K} a_k^{(n)} / a_k^{(m)} \neq 0$ implies $\inf_{k \in K} a_k^{(n)} / a_k^{(m')} \neq 0$ for $m' \geq m$ (see [1], 3.1).

Throughout this paper $p = (p_k)$ stands for a bounded sequence of strictly positive numbers.

3. **Maddox spaces as (co-) echelon spaces.** We start with the

PROOF OF THEOREM 0. (i) If $x \in c_0(p)$, then for every $n \in \mathbb{N}$ we have $|x_k|^{p_k} \cdot n := \delta_k \longrightarrow 0$; hence $|x_k| \cdot n^{1/p_k} = \delta_k^{1/p_k} \longrightarrow 0$ as $k \longrightarrow \infty$. Here we have used that $p \in l_\infty$. Conversely, assume that $\lim_k |x_k| \cdot n^{1/p_k} = 0$ for every $n \in \mathbb{N}$. Then for $n \in \mathbb{N}$ we have $|x_k|^{p_k} \leq \frac{1}{n}$ for large k, hence $x \in c_0(p)$.

(ii) If $x \in l_\infty(p)$, then there is some $n \in \mathbb{N}$ with $|x_k|^{p_k} \leq n$; hence $|x_k| n^{-1/p_k} \leq 1$ for $k \in \mathbb{N}$. Conversely, if $|x_k| n^{-1/p_k} \leq M$, then $|x_k|^{p_k} \leq M^{p_k} \cdot n$ for all k, which is bounded because $p \in l_\infty$. Hence $x \in l_\infty(p)$. ∎

Thereom 0 shows in particular that $c_0(p)$, $c(p)$ and $l_\infty(p)$ are indeed sequence spaces, i.e linear. It is not difficult to see that, conversely, boundedness of p is also necessary for the linearity of these spaces.

For the sake of completeness we introduce some further sequence spaces that will be considered in this paper:

$$M_0(p) = \bigcup_{n=1}^{\infty} \{x \in \omega : \sum_k |x_k| n^{-1/p_k} < \infty\},$$

$$M_\infty(p) = \bigcap_{n=1}^{\infty} \{x \in \omega : \sum_k |x_k| n^{1/p_k} < \infty\},$$

$$\begin{aligned} \underline{l}_\infty(p) &= \bigcap_{n=1}^{\infty} \{x \in \omega : \sup_k |x_k| n^{1/p_k} < \infty\} \\ &= \{x \in \omega : \lim_k |\delta_k x_k|^{p_k} = 0 \text{ for each } \delta \in c_0\}, \end{aligned}$$

$$\underline{c}_0(p) = \bigcup_{n=1}^{\infty} \{x \in \omega : \lim_k |x_k| n^{-1/p_k} = 0\}$$
$$= \{x \in \omega : \sup_k |x_k/\delta_k| p_k < \infty \text{ for some } \delta \in c_0 \text{ with } \delta_k \neq 0 \text{ for all } k\}$$

and

$$\underline{c}(p) = \underline{c}_0(p) + \langle e \rangle$$
$$= \{x \in \omega : \sup_k |(x_k - l)/\delta_k|^{p_k} < \infty \text{ for some } l \in \mathbb{C}$$
$$\text{and some } \delta \in c_0 \text{ with } \delta_k \neq 0 \text{ for all } k\}.$$

The spaces $M_0(p)$, $M_\infty(p)$ and $\underline{l}_\infty(p)$ were introduced in [11], [7] and [6]; they appear as α-duals of $c_0(p)$ and $l_\infty(p)$. Note also that the spaces $M_\infty(p)$ are essentially the power series spaces $\Lambda_\infty(\alpha)$ of infinite type (see [4], 10.6): If $p_k \longrightarrow 0$ monotonically, then $M_\infty(p) = \Lambda_\infty(\alpha)$ with $\alpha = (1/p_k)$.

The equality of the different representations of $\underline{l}_\infty(p)$ and $\underline{c}_0(p)$ is easily established. As for $\underline{c}(p)$ it has to be noted that $e \in \underline{c}_0(p)$ if (and only if) $p_k \longrightarrow 0$, so that in this case the space $\underline{c}(p)$ collapses to $\underline{c}_0(p)$.

Let $A(p)$ denote the Köthe matrix $(n^{1/p_k})_{n,k}$ and $V(p)$ the associated matrix $(n^{1/p_k})_{n,k}$. Then we have:

$$c_0(p) = \lambda_0\big(A(p)\big), \quad M_\infty(p) = \lambda_1\big(A(p)\big), \quad \underline{l}_\infty(p) = \lambda_\infty\big(A(p)\big),$$
$$\underline{c}_0(p) = k_0\big(V(p)\big), \quad M_0(p) = k_1\big(V(p)\big), \quad l_\infty(p) = k_\infty\big(V(p)\big),$$

as well as

$$c(p) = \lambda_0\big(A(p)\big) \oplus \langle e \rangle \text{ and } \underline{c}(p) = k_0\big(V(p)\big) \oplus \langle e \rangle \text{ (if } p \notin c_0).$$

Thus in order to apply the theory of (co-) echelon spaces we have to determine the properties of the matrix $A(p)$ (or equivalently of $V(p)$). The following is easily obtained:

THEOREM 1. *(i) $A(p)$ (or rather $V(p)$) is regularly decreasing for every p.*

(ii) $A(p)$ satisfies condition (M) if and only if it satisfies condition (S), and if and only if $p_k \longrightarrow 0$.

(iii) $A(p)$ satisfies condition (N) if and only if $\sum_k n^{-1/p_k} < \infty$ for some $n \in \mathbb{N}$.

4. **Topologies.** The spaces $c_0(p), M_\infty(p), \underline{l}_\infty(p)$ are echelon spaces, the spaces $\underline{c}_0(p)$, $M_0(p)$, $l_\infty(p)$ are co-echelon spaces. We consider them endowed with their projective and inductive limit topologies, respectively. The spaces $c(p)$ and $\underline{c}(p)$ (if $p \notin c_0$) are given direct sum topologies. A different way of topologising these spaces is by considering the projective limit topology for $c(p) = \bigcap_{n=1}^{\infty}(\{x \in \omega : \lim_k |x_k| n^{1/p_k} = 0\} \oplus \langle e \rangle)$ and the inductive limit topology for $\underline{c}(p) = \bigcup_{n=1}^{\infty}(\{x \in \omega : \lim_k |x_k| n^{-1/p_k} = 0\} \oplus \langle e \rangle)$ if $p \notin c_0$. It is left to the reader to verify that the respective topologies coincide. Obviously $c_0(p)$ and $\underline{c}_0(p)$ are (isomorphic to) closed subspaces of $c(p)$ and $\underline{c}(p)$, respectively.

For the following result note that for each of the sequence spaces considered here the condition $\inf_k p_k > 0$ holds if and only if the space reduces to its corresponding classical space, i.e., $c_0(p) = c_0$, $M_\infty(p) = l_1$, etc.

THEOREM 2. *(i)* $c_0(p)$, $c(p)$, $M_\infty(p)$ *and* $\underline{l}_\infty(p)$ *are* FK-*spaces. Each of them is normable if and only if* $\inf_k p_k > 0$.

(ii) $\underline{c}_0(p)$, $\underline{c}(p)$, $M_0(p)$ *and* $l_\infty(p)$ *are complete* LBK-*spaces. Each of them is metrisable if and only if* $\inf_k p_k > 0$.

PROOF. (i) Assume that $c_0(p) = \bigcap_{n=1}^\infty E_n$, $E_n = \{x \in \omega : \lim_k |x_k| n^{1/p_k} = 0\}$, is normable with norm $\|\cdot\|$. Then there are $n \in \mathbb{N}$ and $M > 0$ with $\|x\| \leq M \cdot \sup_k |x_k| n^{1/p_k}$ for $x \in c_0(p)$. By continuity of the inclusion map $c_0(p) \longrightarrow E_{n+1}$ there is $\tilde{M} > 0$ with $\sup_k |x_k|(n+1)^{1/p_k} \leq \tilde{M} \cdot \|x\| \leq M\tilde{M} \cdot \sup_k |x_k| n^{1/p_k}$ for $x \in c_0(p)$. Taking $x = e_k$ ($k \in \mathbb{N}$) shows that $\big((n+1)/n\big)^{1/p_k}$ is bounded in k, hence that $\inf_k p_k > 0$. Conversely, $\inf_k p_k > 0$ implies that $c_0(p) = c_0$ is normable. Similar arguments work for $M_\infty(p)$ and $\underline{l}_\infty(p)$; for $c(p)$ apply the result for $c_0(p)$.

(ii) Completeness of the topologies follows from [1], 3.7(3), 2.3(a) and 2.8(e). Now assume that $\underline{c}_0(p) = \bigcup_{n=1}^\infty E_n$, $E_n = \{x \in \omega : \lim_k |x_k| n^{-1/p_k} = 0\}$, is metrisable, hence an FK-space. Then by a result of Zeller ([23], Satz 4.6) there is some $n \in \mathbb{N}$ with $\underline{c}_0(p) = E_n$. This forces $\inf_k p_k > 0$. Conversely, if $\inf_k p_k > 0$, then $\underline{c}_0(p) = c_0$ is metrisable. A similar argument applies to the other three spaces. ∎

The result for $c_0(p)$ was obtained in [12], p. 318 with [15], Theorem 2 and [16], Theorem 8.

Note that as LBK-spaces the spaces $\underline{c}_0(p)$, $\underline{c}(p)$, $M_0(p)$ and $l_\infty(p)$ are barrelled and bornological (DF)-spaces.

REMARK. Maddox studies his sequence spaces within the framework of paranormed sequence spaces (for the notion of a paranorm cp. [21], 2-1). In each of the spaces $l_\infty(p)$, $c_0(p)$ and $c(p)$ he considered the function $g(x) = \sup_k |x_k|^{p_k/M}$ with $M = \max(1, \sup_k p_k)$ and introduced a topology τ_g via the corresponding metric $d(x,y) = g(x-y)$—with varying success:

In $l_\infty(p)$, g is a paranorm (and τ_g a linear topology) only in the trivial case $\inf_k p_k > 0$, when $l_\infty(p) = l_\infty$ ([19], Theorem 9). Indeed, by the result above, the natural topology of $l_\infty(p)$ is not metrisable hence not paranormable unless $l_\infty(p) = l_\infty$.

In $c_0(p)$, g is a paranorm and τ_g is an FK-topology ([10], Theorem 1, [12], p. 318 and [15], Theorem 2), so that by the uniqueness of FK-topologies ([22], 4.2.4) τ_g coincides with the projective limit for $c_0(p)$.

In $c(p)$, again g is a paranorm (and τ_g a linear topology) only if $\inf_k p_k > 0$, when $c(p) = c$, by an argument as in [19], Theorem 9. But, in contrast to $l_\infty(p)$, the natural topology of $c(p)$ can be induced by a paranorm. A convenient one is $\tilde{g}(x) = \sup_k |x_k - l|^{p_k/M} + |l|$ where l is the unique number with $x - l \cdot e \in c_0(p)$.

Maddox considered his sequence spaces as particular cases of spaces $[A,p]_0$, $[A,p]$ and $[A,p]_\infty$ where A is some matrix (see, e.g., [9], [10]). In the light of the remarks above it seems that at least the question of topologising the space $[A,p]_\infty$ has to be reexamined.

A locally convex inductive limit $E = \varinjlim E_n$ of an increasing sequence of subspaces E_n of some space is called *boundedly retractive* ([1], 1.9) if each bounded subset B of E is contained in some E_n and the topologies of E_n and E coincide on B. By [1], 3.7 and

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3.4 and Theorem 1 the spaces $\underline{c}_0(p)$, $M_0(p)$ and $l_\infty(p)$ enjoy this property for every p. As a consequence we have:

THEOREM 3. *(i) A subset B of $\underline{c}_0(p)$ is bounded if and only if there are $\rho > 0$ and $M > 0$ such that for each $x \in B$ there is some $\delta \in c_0$, $0 < \delta_k \leqq \rho$ for all k, with $|x_k/\delta_k|^{p_k} \leq M$ for $k \in \mathbb{N}$. In that case the topology of $\underline{c}_0(p)$ is metrisable on B with $d(x,y) = \sup_k |x_k - y_k| \cdot M^{-1/p_k}$.*

(ii) A subset B of $l_\infty(p)$ is bounded if and only if there is some $M > 0$ such that $|x_k|^{p_k} \leq M$ for $k \in \mathbb{N}$ and $x \in B$. In that case the topology of $l_\infty(p)$ is metrisable on B with $d(x,y) = \sup_k |x_k - y_k| \cdot M^{-1/p_k}$.

5. **Duality.** We determine the various duals of our spaces.

THEOREM 4. *(i) For $\zeta = \alpha, \beta, \gamma, f$ we have*

$$c_0(p)^\zeta = M_0(p), \quad c_0(p)'_b \cong M_0(p);$$
$$\underline{c}_0(p)^\zeta = M_\infty(p), \quad \underline{c}_0(p)'_b \cong M_\infty(p);$$
$$\underline{c}(p)^\zeta = M_\infty(p), \quad \underline{c}(p)'_b \cong M_\infty(p) \times \mathbb{C} \ (p \notin c_0);$$

$$M_0(p)^\zeta = \underline{l}_\infty(p), \quad M_0(p)'_b \cong \underline{l}_\infty(p);$$
$$M_\infty(p)^\zeta = l_\infty(p), \quad M_\infty(p)'_b \cong l_\infty(p);$$

$$\underline{l}_\infty(p)^\zeta = M_0(p),$$
$$\underline{l}_\infty(p)'_b \supset M_0(p) \text{ with } \underline{l}_\infty(p)' = M_0(p) \text{ iff } p_k \longrightarrow 0; \text{ in that case } \underline{l}_\infty(p)'_b \cong M_0(p)$$

$$l_\infty(p)^\zeta = M_\infty(p),$$
$$l_\infty(p)'_b \supset M_\infty(p) \text{ with } l_\infty(p)' = M_\infty(p) \text{ iff } p_k \longrightarrow 0; \text{ in that case } l_\infty(p)'_b \cong M_\infty(p).$$

(ii) We have

$$c(p)^\times = l_1, \quad c(p)^\beta = M_0(p) \cap \mathrm{cs}, \quad c(p)^\gamma = M_0(p) \cap \mathrm{bs}, \quad c(p)^f = M_0(p);$$
$$c(p)^{\times\times} = l_\infty, \quad c(p)^{\beta\beta} = c(p)^{\gamma\gamma} = \underline{l}_\infty(p) + \mathrm{bv};$$
$$c(p)'_b \cong M_0(p) \times \mathbb{C}, \quad \left(c(p)'_b\right)'_b \cong \underline{l}_\infty(p) \times \mathbb{C}.$$

PROOF. (i) excluding $\underline{c}(p)$: For the α-duals see [20], Chapter 2; the β- and γ-duals coincide with the α-duals because each of the spaces considered here is normal. For the topological duals of $c_0(p)$, $\underline{c}_0(p)$, $M_0(p)$ and $M_\infty(p)$; apply [1], 2.8(a), 2.7 and 3.5(b), from which their f-duals follow. Now consider $\underline{l}_\infty(p)$. Since $\underline{l}_\infty(p)$ is a barrelled K-space, the Banach-Steinhaus theorem implies that every element $y \in M_0(p) = \underline{l}_\infty(p)^\beta$ defines an element of $\underline{l}_\infty(p)'$ via $f(x) = \sum_k x_k y_k$. Hence $M_0(p) \subset \underline{l}_\infty(p)'$. Since $M_0(p)'_b \cong \underline{l}_\infty(p)$, we have that $\underline{l}_\infty(p)' = M_0(p)$ is equivalent to $M_0(p)$ being (semi-) reflexive which in turn is equivalent to $p \in c_0$ by [1], 4.7. For the f-dual consider the subspace φ of $\underline{l}_\infty(p)$ of all finite sequences. Since the closure of φ is $c_0(p)$ ([1], p. 32), we have that $\underline{l}_\infty(p)^f = c_0(p)^f = M_0(p)$ by [22], 7.2.4. The arguments for $l_\infty(p)$ are similar. There the closure of φ is $\underline{c}_0(p)$ (use Theorem 3 and [1], 2.4.).

(ii) First employ the fact that for any sequence spaces E, F one has $(E+F)^\zeta = E^\zeta \cap F^\zeta$ for $\zeta = \alpha, \beta, \gamma$ and use the results for $c_0(p)$. Since the closure of φ in $c(p)$ is $c_0(p)$, we have $c(p)^f = c_0(p)^f = M_0(p)$ as in (i). For the second duals note that $M_0(p)$ and cs have the AK-property i.e. $\sum_{k=1}^n x_k e_k \longrightarrow x$ for every sequence x in these spaces. Hence a result of Goes ([3], Satz 2.3(a), which also holds for arbitrary barrelled K-spaces) implies that $c(p)^{\beta\beta} = M_0(p)^\beta + \mathrm{cs}^\beta$; moreover we have $c(p)^{\gamma\gamma} = c(p)^{\beta\gamma} = c(p)^{\beta\beta}$ by the AK-property for $c(p)^\beta$.

(iii) For $\underline{c}(p)$ $(p \notin c_0)$ follows as for $c(p)$. The results simplify due to the fact that $M_\infty(p) \subset l_1$ for every p. ∎

The β-duals of $c_0(p)$, $c(p)$, $l_\infty(p)$, $M_0(p)$ and $M_\infty(p)$ and the topological dual of $c_0(p)$ have been obtained before (see [11], [7] and [6]).

REMARK. There is a further natural way to introduce a topology in the space $l_\infty(p)$. Since $M_\infty(p)' = l_\infty(p)$, one may consider the strong topology $\beta\big(l_\infty(p), M_\infty(p)\big)$. Because of $M_\infty(p)'_b \cong l_\infty(p)$, however, this topology coincides with the inductive limit topology on $l_\infty(p)$.

THEOREM 5. *I. (i) $M_0(p)$, $M_\infty(p)$ and $l_\infty(p)$ are α-, β- and γ-perfect for any p.*

(ii) Each of the spaces $c_0(p)$, $\underline{c}_0(p)$ and $\underline{c}(p)$ is α-, β-, or γ-perfect if and only if $p_k \longrightarrow 0$.

(iii) $c(p)$ is α-, β- or γ-perfect for no p.

II. Each of the spaces $c_0(p)$, $c(p)$, $M_\infty(p)$, $\underline{l}_\infty(p)$, $\underline{c}_0(p)$, $\underline{c}(p)$, $M_0(p)$ and $l_\infty(p)$ is reflexive if and only if $p_k \longrightarrow 0$.

PROOF. I. (i) is clear from Theorem 4. (ii): For $c_0(p)$ and $\underline{c}_0(p)$ apply [1], 4.7(4) and 4.9(IV). For $\underline{c}(p)$ note that if $p_k \longrightarrow 0$, then $\underline{c}(p) = \underline{c}_0(p)$. Conversely, if $\underline{c}(p)$ is ζ-perfect, $\zeta = \alpha, \beta, \gamma$, then $\underline{c}(p) = l_\infty(p)$. This is easily seen to imply $p_k \longrightarrow 0$. (iii) is known if $\inf_k p_k > 0$, when $c(p) = c$. If $\inf_k p_k = 0$, choose k_n with $p_{k_n} \leq 1/n$ for $n \in \mathbb{N}$ and consider the sequence x with $x_k = 1/n^2$ if $k = k_n$, $x_k = 0$ else. Then x belongs to $\mathrm{bv} \subset c(p)^{\zeta\zeta}$ but not to $c(p)$ so that $c(p)$ cannot be ζ-perfect.

II. This is by [1], 4.7, 4.9 and [5], 23.3.(5), (6). ∎

For $c_0(p)$ and $l_\infty(p)$ assertion I was obtained in [6].

6. **Montel, Schwartz, nuclearity.** We characterise when our spaces are Montel spaces, Schwartz spaces or nuclear spaces in terms of a) properties of the sequence p and b) reversibility of the (trivial) inclusion relations $M_\infty(p) \subset c_0(p) \subset \underline{l}_\infty(p)$ and $M_0(p) \subset \underline{c}_0(p) \subset l_\infty(p)$.

THEOREM 6. *Let E be one of the spaces $c_0(p)$, $c(p)$, $M_\infty(p)$, $\underline{l}_\infty(p)$ and F one of the spaces $\underline{c}_0(p)$, $\underline{c}(p)$, $M_0(p)$, $l_\infty(p)$. Then the following assertions are equivalent:*

(a) E is a Montel space;
(a′) F is a Montel (= Schwartz) space;
(b) E is a Schwartz space;
(c) $c_0(p) = \underline{l}_\infty(p)$;
(c′) $\underline{c}_0(p) = l_\infty(p)$;
(d) $p_k \longrightarrow 0$.

PROOF. Use [1], 4.7, 4.9 with Theorem 1 and, for $c(p)$ and $\underline{c}(p)$, the permanence properties of (semi-) Montel and Schwartz space ([4], 11.5.4, 21.1.7). ∎

In (a′) note that a barrelled (DF)-space is a Montel space if and only if it is a Schwartz space ([4], 11.5.3, 12.4.7).

THEOREM 7. *Let E be one of the spaces* $c_0(p)$, $c(p)$, $M_\infty(p)$, $\underline{l}_\infty(p)$, $\underline{c}_0(p)$, $\underline{c}(p)$, $M_0(p)$ *and* $l_\infty(p)$. *Then the following assertions are equivalent:*
(a) E is nuclear;
(b) $M_\infty(p) = c_0(p)$;
(b′) $M_0(p) = \underline{c}_0(p)$;
(c) $M_\infty(p) = \underline{l}_\infty(p)$;
(c′) $M_0(p) = l_\infty(p)$;
(d) $\sum_k n^{-1/p_k} < \infty$ *for some* $n \in \mathbb{N}$.

PROOF. The spaces $c_0(p)$ and $M_\infty(p)$ are nuclear if and only if (N) holds ([20], 2.4.4.(1), (3) and 2.2.3.(16)) which is equivalent to (d) by Theorem 1. Now, if E is a Fréchet or (DF)-space, then E is nuclear if and only if E'_b is ([4], 21.5.3). Using Theorem 4(i) the equivalence of (a) and (d) now follows for $M_0(p)$, $\underline{l}_\infty(p)$, $\underline{c}_0(p)$ and $l_\infty(p)$; for $c(p)$ and $\underline{c}(p)$ use the permanence properties of nuclearity ([4], 21.2.3).

Now assume (d) : $\sum_k N^{-1/p_k} < \infty$ for some $N \in \mathbb{N}$, and let $x \in \underline{l}_\infty(p)$ and $n \in \mathbb{N}$. Then there is $M > 0$ with $|x_k| \leq M \cdot (n \cdot N)^{-1/p_k}$ for all k, hence $\sum_k |x_k| n^{-1/p_k} \leq M \cdot \sum_k N^{-1/p_k} < \infty$. This implies $\underline{l}_\infty(p) \subset M_\infty(p)$, hence (c).

(c) trivially implies (b), (b) implies (c′) by α-duality and (c′) trivially implies (b′).

Finally assume that (b′) holds, i.e. $\underline{c}_0(p) \subset M_0(p)$. Then we have in particular: If $|x_k| 2^{-1/p_k} \longrightarrow 0$, then $\sum_k |x_k| n^{-1/p_k} < \infty$ for some $n \in \mathbb{N}$. This forces $p_k \longrightarrow 0$ and, taking $x = e$, we obtain (d). ∎

7. **Weak sequential completeness and the Schur property.** In [6] and [14] Maddox and Lascarides have characterised for which p the space $c_0(p)$ is weakly sequentially complete and for which it has the Schur property. In this section we consider this problem for general (co-) echelon spaces.

A topological vector space E is said to have the Schur property if every weakly convergent sequence in E is convergent. We will use the following permanence properties that are easily established: The Schur property is preserved under the formation of arbitrary products and subspaces, weak sequential completeness is preserved under the formation of arbitrary products and closed subspaces. Hence both properties are preserved under the formation of projective limits.

THEOREM 8. *Let A be a Köthe matrix, V the associated matrix, Then (i) $\lambda_{\mathcal{P}}(A)$ and $\mathfrak{k}_{\mathcal{P}}(V)$ are weakly sequentially complete for any $\mathcal{P}$, $1 \leq \mathcal{P} < \infty$.*

(ii) Each of the spaces $\lambda_0(A)$, $\lambda_\infty(A)$ and $\mathfrak{k}_\infty(V)$ is weakly sequentially complete if and only if condition (M) holds.

(iii) $\mathfrak{k}_0(V)$ is weakly sequentially complete if and only if condition (S) holds.

PROOF. (i) Being reflexive the spaces $l_{\mathcal{P}}$, $l < \mathcal{P} < \infty$, are weakly sequentially complete by [21], 10–2–4, as is l_1 ([5], 22.4.2). Since $\lambda_{\mathcal{P}}(A)$ and $\mathfrak{k}_{\mathcal{P}}(V)$ (for $l \leq \mathcal{P} < \infty$, using [1], 2.3(a)) are projective limits of weighted $l_{\mathcal{P}}$-spaces, the assertion follows.

(ii) Condition (M) implies that $\lambda_0(A)$, $\lambda_\infty(A)$ and $\mathfrak{k}_\infty(V)$ are reflexive ([1], 4.7), hence weakly sequentially complete (see (i)). Conversely, if (M) does not hold, then these spaces have diagonal transforms of c_0, l_∞ and l_∞, respectively, as sectional subspaces. And neither c_0 nor l_∞ is weakly sequentially complete ([21], 12–5–102).

(iii) Condition (S) implies that $\mathfrak{k}_0(V)$ is reflexive ([1], 4.9), hence weakly sequentially complete as in (i). Conversely assume that $\mathfrak{k}_0(V)$ has this property. Consider $x \in \mathfrak{k}_\infty(V)$ and let $x^{[n]} := (x_1, \ldots, x_n, 0, 0, \ldots)$ for $n \in \mathbb{N}$. Then $(x^{[n]})$ is a sequence in $\mathfrak{k}_0(V)$. Since $\mathfrak{k}_0(V)' = \lambda_1(A) = \mathfrak{k}_\infty(V)^\times$ by [1], 2.7 and [5], 30.8.(1), we have that $\sum_k x_k y_k$ converges for every $f = (y_k) \in \mathfrak{k}_0(V)'$. Consequently $(x^{[n]})$ is a weak Cauchy sequence in $\mathfrak{k}_0(V)$, hence weakly convergent to some $\xi \in \mathfrak{k}_0(V)$. Since necessarily $\xi = x$, we find $\mathfrak{k}_\infty(V) = \mathfrak{k}_0(V)$, which implies (S) by [1], 4.9. ∎

THEOREM 9. *Let A be a Köthe matrix, V the associated matrix. Then (i) $\lambda_1(A)$ and $\mathfrak{k}_1(V)$ have the Schur property.*

(ii) Each of the spaces $\lambda_{\mathcal{P}}(A)$ and $\mathfrak{k}_{\mathcal{P}}(V)$ with $\mathcal{P} = 0$ or $1 < \mathcal{P} \leq \infty$ has the Schur property if and only if condition (M) holds.

PROOF. (i) It is well known that l_1 has the Schur property ([5], 22.4.2). Since $\lambda_1(A)$ and $\mathfrak{k}_1(V)$ (using [1], 2.3(a)) are projective limits of weighted l_1-spaces, the assertion follows.

(ii) Condition (M) implies that $\lambda_{\mathcal{P}}(A)$ ($\mathcal{P} = 0$ or $1 < \mathcal{P} \leq \infty$) and $\mathfrak{k}_{\mathcal{P}}(V)$ ($1 < \mathcal{P} \leq \infty$) are Montel spaces ([1], 4.7, 2.3(a)). Hence they have the Schur property by [5], 27.2.(3). The same holds for $\mathfrak{k}_0(V)$ ([1], 2.4). Conversely, if (M) does not hold, then $\lambda_{\mathcal{P}}(A)$ and $\mathfrak{k}_{\mathcal{P}}(V)$ have sectional subspaces that are diagonal transforms of c_0 (if $\mathcal{P} = 0$) and $l_{\mathcal{P}}$ (if $1 < \mathcal{P} \leq \infty$). But none of the latter spaces has the Schur property as can be seen from considering the sequence (e_k) in each case. This implies (ii). ∎

As a consequence of Theorems 8 and 9 and, for $c(p)$ and $\underline{c}(p)$, the permanence properties stated above we have:

THEOREM 10. *(i) $M_0(p)$ and $M_\infty(p)$ are weakly sequentially complete and have the Schur property for every p.*

(ii) Each of spaces $c_0(p)$, $c(p)$, $\underline{l}_\infty(p)$, $\underline{c}_0(p)$, $\underline{c}(p)$ and $l_\infty(p)$ is weakly sequentially complete if and only if it has the Schur property, and if and only if $p_k \longrightarrow 0$.

For the space $c_0(p)$ this was proved in [6] and [14].

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