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Error bounds for the modified Newton's method

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A strengthened form of the Kantorovich convergence theorem for the modified Newton's method is proved. The result is compared with previously known results.

1. Introduction

Let F be a continuously Fréchet differentiable mapping of an open subset Ω of a Banach space X into a Banach space Y. This note concerns the numerical solution of

$$F(x) = 0 \tag{1}$$

by the modified Newton's method, that is by the iteration

$$x_{n+1} = x_n - \Gamma_0 F(x_n) , \tag{2}$$

where $\Gamma_0 = [F'(x_0)]^{-1}$, the inverse of the Fréchet derivative of F at x_0. Dennis [1] has strengthened Kantorovich's convergence theorem [2, Theorem 6 (1.XVIII)] for both (2) and the original Newton's method. Theorem 1 below further strengthens Dennis's result for (2). It proves existence and local uniqueness of the solution of (1) and the convergence of (2) to this solution under weaker conditions on F, and for a given F establishes a rate of convergence for (2) which is at least as fast as, and generally faster than, that proved by Dennis. Ways in which Theorem 1 may be further strengthened are noted. Theorem 1 may be proved by modifying Dennis's proof, but in this case the proof given below, which follows [2] even more closely, is simpler. For further references, see [3].

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2. Error bounds

THEOREM 1. *Let* $\Omega_0 = \{x : \|x-x_0\| \le r\} \subset \Omega$. *Let* Γ_0 *exist in* $L(Y, X)$, *the set of bounded linear mappings from* Y *into* X , *and let* $\eta = \|\Gamma_0 F(x_0)\|$. *Let*

(3) $$\|I-\Gamma_0 F'(x)\| \le K\|x_0-x\| \quad \text{for all } x \text{ in } \Omega_0 ,$$

where I *is the identity operator. Let* $0 < h = K\eta \le \frac{1}{2}$ *and* $r \ge r_-$ *where*

$$r_\pm = \left[1\pm(1-2h)^{\frac{1}{2}}\right]\eta/h .$$

Then all elements of the sequence defined by (2) *lie in* Ω_0 , *and the sequence converges to a solution* x^* *of* (1) *with*

(4) $$\|x^*-x_n\| \le \frac{\eta}{2h}\left[1-(1-2h)^{\frac{1}{2}}\right]^{n+1} , \quad n = 1, 2, \ldots .$$

The inequality is strict if $n > 1$. *If also either* $r < r_+$ *or* $r_- = r = r_+$, *then* x^* *is the only solution of* (1) *in* Ω_0 .

Note that if $h = 0$, then either (1) is linear or $F(x_0) = 0$, so that in neither case does the question of convergence arise. The proof of Theorem 1 uses the following two lemmas.

LEMMA 1. *Let* $\psi \in C^1[0, r]$ *be a real valued function with* $c_0 = [-\psi'(0)]^{-1} > 0$ *and* $1 + c_0\psi'(t) \ge 0$ *for all* $t \ge 0$. *Let*

(5) $$\psi(t) = 0$$

have a root in the interval $[0, r]$ *and let* t^* *be the smallest root of* (5) *in* $[0, r]$. *Let* $\Gamma_0 \in L(Y, X)$ *and, for some nonnegative integer* p , *let*

(i) $\|\Gamma_0 F(x_i)\| \le c_0\psi(t_i)$, $i = 0, \ldots, p$, *and*

(ii) $\|I-\Gamma_0 F'(x)\| \le 1 + c_0\psi'(t)$ *whenever*

$$\|x-x_p\| \le t - t_p \le r - t_p ,$$

where $t_0 = 0$ *and* $t_{n+1} = t_n - \psi(t_n)/\psi'(t_0)$,

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$n = 0, 1, \ldots$.

Then (1) *has a solution* x^* *in* $\Omega_p = \{x : \|x-x_p\| \le r-t_p\}$, *all members of the sequence defined by* (2) *lie in* Ω_0 , *and*

$$\|x^*-x_n\| \le t^* - t_n , \quad n = 0, 1, \ldots .$$

If also $\psi'(r) \le 0$ *and* (5) *has a unique solution in* $[0, r]$ *then* (1) *has a unique solution in* Ω_p .

The proof is omitted since it requires only minor changes in the proofs of Theorems 1, 2, 3, and 4 (1.XVIII) of [2]. Note that $\Omega_{p+1} \subset \Omega_p$ and the proof of Theorem 1 (1.XVIII) of [2] shows that if conditions *(i)* and *(ii)* of Lemma 1 are satisfied for some integer p , they are also satisfied for $p + 1$.

LEMMA 2. *Let* $f(t) = Kt^2 - 2t + 2\eta$ *where* K, η *are the constants defined in Theorem* 1 *and* $0 < h = K\eta \le \frac{1}{2}$. *Let* $t_0 = 0$ *and* $t_{n+1} = t_n - f(t_n)/f'(t_0)$, $n = 0, 1, \ldots$. *Then* $t_n \to r_-$ *as* $n \to \infty$ *and, for* $n \ge 1$, $t_{n-1} < t_n < r_-$ *and*

$$(6) \qquad r_- - t_n = r_- \prod_{i=0}^{n-1} \frac{K(r_-+t_i)}{2} \le \frac{\eta}{2h} \left[1-(1-2h)^{\frac{1}{2}}\right]^{n+1} ,$$

with strict inequality in (6) *when* $n > 1$.

The proof, which uses the fact that $f(r_\pm) = 0$, is by induction.

It is readily verified that f has the properties required of ψ in Lemma 1 with $r_- = t^*$ so that Theorem 1 follows.

The proof of Theorem 1 uses only the case $p = 0$ in Lemma 1. The case $p = 1$ yields the following result.

THEOREM 2. *Let all the conditions of Theorem* 1 *be satisfied except that* K *is not required to satisfy* (3). *Let* $\|\Gamma_0 F(x_1)\| \le K\eta^2/2$ *and let*

$$\|I-\Gamma_0 F'(x)\| \le K(\|x-x_1\|+\eta) \quad \textit{for all} \quad x \quad \textit{in} \quad \Omega_1 .$$

Then all the conclusions of Theorem 1 *follow except possibly the uniqueness in* Ω_0 *of the solution of* (1). *The solution is however unique in* Ω_1 .

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The proofs of Theorems 1 and 2 show that sharper but more complicated bounds for $\|x^*-x_n\|$ may easily be obtained by using the equality instead of the inequality in (6).

3. Comparison with previously known results

Theorem 1 strengthens Corollary 4.1 of [1] in two respects. The result in [1] proves only weak inequality in (4) and, more importantly, it replaces (3) with the stronger condition

(7) $\|\Gamma_0 F'(x)-\Gamma_0 F'(y)\| \le K\|x-y\|$ for all x and y in Ω_0 .

Otherwise the results are identical.

Let h_1, h_2, h_3 be the smallest possible values of $h = K\eta$ when K is defined as in Theorem 1, Theorem 2, and (7), respectively. Clearly $h_3 \ge h_1$, and the one dimensional example

(8) $$F(x) = x - 1 + a \sin e^{bx} , \quad x_0 = 0 ,$$

with a and b constants, b large and a very small, shows that the ratio h_3/h_1 may be arbitrarily large. Also the remark following Lemma 1 shows that $h_1 \ge h_2$ and the one dimensional example

$$F(x) = x - 1 + c^{-1} \cos(cx/5) , \quad x_0 = 0 ,$$

where $c = 10n\pi + 1$ and n is a large integer, shows that h_1/h_2 may be arbitrarily large. Since r_- increases with h in $(0, \frac{1}{2})$, it is clear that decreasing h in Theorem 1 sharpens the error bounds (4) and weakens the conditions required for convergence. Moreover it is sometimes easier to calculate h_1 or h_2 , or upper bounds for them that are smaller than h_3 , than it is to calculate h_3 . As shown in [1], Kantorovich's result is still weaker than the results in [1].

As well as giving *a priori* error bounds, Theorem 1 may also be used to calculate more accurate *a posteriori* error bounds. Let y be an approximation to x^* obtained by (2) or by any other means. Set $x_0 = y$ and calculate x_1 by (2). Then (4) with $n = 1$ gives an error bound for

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x_1. Generally this will be better than can be obtained from known bounds for Newton's method as such bounds involve h_3 instead of h_1. However since the difference between h_3 and h_1 is generally smaller when x_0 is nearer x^*, the gain from using (4) will be less than is the case with *a priori* bounds.

Both *a priori* and *a posteriori* error bounds for (2) are sometimes obtained from the following special case of the contraction mapping theorem [2, 4].

THEOREM 3. *Using the same notation as before, let* Γ_0 *exist and let*

$$\alpha = \sup_{x\in\Omega_0} \|I-\Gamma_0 F'(x)\| < 1 ,$$

where $\Omega_0 \subset \Omega$ *and* $r = \eta/(1-\alpha)$. *Then* (1) *has a unique solution* x^* *in* Ω_0, *and, for* $n \geq 1$, $x_n \in \Omega_0$ *and*

$$\|x_n - x^*\| \leq \frac{\alpha}{1-\alpha}\|x_n - x_{n-1}\| \leq \frac{\alpha^n}{1-\alpha}\|x_1 - x_0\| = \frac{\alpha^n \eta}{1-\alpha} .$$

Note that each of Theorems 1, 2, and 3 proves convergence of (2) for (8) when, say, $a = 10^{-4}$, $b = 5$, although the theorems of Kantorovich and Dennis both fail in this case.

In their important book on shooting methods, Roberts and Shipman [4, p. 126] showed that if $F \in C^2(\Omega_0)$, then

$$\alpha^2 - \alpha + h_4 \geq 0 , \tag{9}$$

whenever $h_4 \geq \|\Gamma_0 F''(x)\|\eta$ for all x in Ω_0. Clearly $h_4 \geq h_3 \geq h_1$. From (9) they deduced, erroneously as (8) shows, that

$$\alpha \geq \left[1-(1-4h_4)^{\frac{1}{2}}\right]/2 ,$$

and hence that Kantorovich's Theorem always gave sharper error bounds than does Theorem 3. In fact the one dimensional example

$$F(x) = 10 - x + c\left[\min(0, x^2-1)\right]^2 , \quad x_0 = 0 ,$$

where c is a constant, shows that sometimes Theorem 3 also gives sharper error bounds than Theorem 1. However Theorem 4 below shows that for an important class of functions Theorem 1 gives sharper error bounds than Theorem 3. For many, but not all, of these functions, Kantorovich's Theorem also proves sharper error bounds than Theorem 3.

THEOREM 4. *Let F have the properties required in Theorems* 1 *and* 3 *and in addition let the maximum value of*

$$\|I-\Gamma_0 F'(x)\|/\|x_0-x\|$$

for x satisfying $0 < \|x_0-x\| \le r$ be attained on the boundary $\|x_0-x\| = r$. Let the bounds for $\|x^*-x_n\|$ given by Theorem 1 *with $h = h_1$ and by Theorem* 3 *be A_n and B_n respectively. Then Theorem* 3 *proves convergence of* (2) *only if $h_1 \le \frac{1}{4}$. Also B_n/A_n increases with h and n and is always greater than* 2 .

Proof. A simplification of the proof of (9) [4] shows that in this case

$$\alpha^2 - \alpha + h_1 = 0 .$$

It follows that, since α must be real, Theorem 3 is applicable only when $h_1 \le \frac{1}{4}$ and that in this case

$$\alpha \ge \left[1-\left(1-4h_1\right)^{\frac{1}{2}}\right]/2 .$$

Hence for $0 < h_1 \le \frac{1}{4}$ and $n \ge 1$,

$$\frac{B_n}{A_n} \ge \frac{4h_1}{\left[1+\left(1-4h_1\right)^{\frac{1}{2}}\right]\left[1-\left(1-2h_1\right)^{\frac{1}{2}}\right]}\left[\frac{1-\left(1-4h_1\right)^{\frac{1}{2}}}{2\left[1-\left(1-2h_1\right)^{\frac{1}{2}}\right]}\right]^n$$

$$= 2\left[g\left(h_1\right)\right]^{n+1} ,$$

where $g(h) = \left[1+(1-2h)^{\frac{1}{2}}\right]/\left[1+(1-4h)^{\frac{1}{2}}\right]$. Clearly $g(h) > 1$ and $g'(h) > 0$ for $0 < h \le \frac{1}{4}$. The result follows.

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