A THEOREM ON NETS

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It is well-known that Tychonoff's theorem on the product of compact spaces may be proved, for the special case of a countable number of metric spaces $X_1, X_2, \ldots, X_n, \ldots,$ in the following simple manner.

Let $P = (p^i)$ be any sequence in $X = \prod_{n=1}^{\infty} X_n$. Choose a subsequence $P^{(1)} = (p^{i1})$ of P whose projection onto X_1 converges, then a subsequence $P^{(2)} = (p^{i2})$ of $P^{(1)}$ whose projection onto X_2 converges, and so on by induction. We obtain a sequence of sequences $P^{(j)} = (p^{ij})_{i \geq 1}$ such that, for any fixed j, the projection of $P^{(j)}$ onto each X_k $(k \leq j)$ converges. Consider the diagonal sequence $P^* = (p^{ii})_{i \geq 1}$. Since P^* is, for any fixed j, essentially a subsequence of $P^{(j)}$, its projection onto any $X^{(j)}$ converges, and so P^* converges in X. Thus we have shown that any sequence in X has a convergent subsequence, which, for metric spaces, means compactness. The word "essentially" above means "from a certain index on", and this will be the key to the generalization of the method.

If we wish to turn our attention to general compact spaces we have to replace sequences by, for example, nets (in the sense of Moore-Smith convergence).

Consider now a well-ordered system of compact spaces X_α of any length. We will be able to prove the compactness of

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ΠX_α by an extension of the above method, using the following definition and theorem. The definition generalizes the concept of "essentially a subsequence"; the theorem might be regarded as a substitute for the diagonal process, applicable to any number of nets (rather than a sequence of sequences). We shall use the terminology of Kelly [1].

Definition. The net S is subequivalent to the net T (not necessarily defined on the same directed set), if some tail of S (i.e., the net consisting of all $S(a)$ with $a \geq a_0$ for a fixed a_0) is a subnet of T.

Example. The sequence (special case of net) $1, 3, 5, 7, \ldots$ is subequivalent to $5, 6, 7, 8, \ldots$.

THEOREM 1. Let X be a fixed (abstract) set, D a directed set, and, for each $d \in D$, let $S^{(d)}$ be a net in X such that, whenever $d' \geq d$, the net $S^{(d')}$ is subequivalent to $S^{(d)}$. Then there is a net $\mathcal{S}^*$ in X which is subequivalent to each $S^{(d)}$. (The system $\{S^{(d)}\}_{d \in D}$ may be described as a directed system of nets in X which is monotone decreasing in the sense of subequivalence.)

Assuming Theorem 1 we now have

THEOREM 2. Let $\{X_\alpha\}_{\alpha \leq \alpha_0}$ be an indexed system of compact topological spaces, where α_0 is a fixed ordinal. Then the product space $X = \Pi_{\alpha \leq \alpha_0} X_\alpha$ is compact.

Proof. It will be sufficient to show that every net in X has a convergent subnet. Let S be any net in X. Assume that for a certain ordinal $\alpha_1 \leq \alpha_0$, a system of nets $\mathcal{S} = \{S^{(\alpha)}\}_{\alpha < \alpha_1}$ exists such that

(i) $\mathcal{S}$ is monotone decreasing in the sense of subequivalence;

(ii) the projection of each $S^{(\alpha)}$ into X_α converges.

By our theorem there is a net $\mathcal{S}^*$ subequivalent to all the $S^{(\alpha)}$ of $\mathcal{S}$. (In the case $\alpha_1 = 0$, set $\mathcal{S}^* = S$.) Now we define $S^{(\alpha_1)}$ as a subnet of $\mathcal{S}^*$ whose projection into X_{α_1} converges. Since

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X_{α_1} is a compact, such an $S^{(\alpha_1)}$ exists. In this manner the system $\mathscr{S}$ is expanded by transfinite induction until we obtain the net $S^{(\alpha_0)}$, whose projection converges on each X_α, and which, therefore, converges in X. A restriction of it is a subnet of the original S.

On the assumption of the well-ordering theorem, we have therefore proved the Theorem of Tychonoff.

We now proceed to prove Theorem 1.

Each $S^{(d)}$ is defined on a directed set $E^{(d)}$. We assume the $E^{(d)}$'s are mutually disjoint and denote their union by E. Consider subsets F of E ("F-sets") having the following property: For every d in some tail of D, F contains a tail of $E^{(d)}$. Let Φ be the collection of all these F-sets, and let us regard Φ as directed by inclusion, i.e., $F_1 \geq F_2$ if and only if $F_1 \subset F_2$.

We define a net S^* with domain Φ by choosing in each F an arbitrary but fixed element $\alpha(F) \in F$ and then setting

$$S^*(F) = S \circ \alpha(F),$$

where S is the union of all the mappings $S^{(d)}$, $d \in D$.

We shall now prove that S^* is subequivalent to each $S^{(d)}$. Let us fix d up to the end of the proof. To establish our assertion we have to find a mapping μ of some tail of Φ into $E^{(d)}$ such that

(a) every tail of $E^{(d)}$ contains the μ-map of some tail of Φ;

(b) $S^* = S \circ \mu$, wherever the latter is defined.

By hypothesis each $S^{(d')}$ with $d' \geq d$ is subequivalent to $S^{(d)}$, i.e., there is a mapping $\gamma^{(d,d')}$ of some tail, $T^{(d')}$, of $E^{(d')}$ into $E^{(d)}$ such that

(a') every tail of $E^{(d)}$ contains the map of some tail of $E^{(d')}$;

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(b') $S^{(d')} = S^{(d)} \circ \gamma^{(d, d')}$ on $T^{(d')}$.

Let $\gamma = \bigcup_{d' \geq d} (\gamma^{(d, d')})$, (d is fixed). Now define the mapping μ by

$$\mu = \gamma \circ \alpha.$$

μ is defined on that tail of Φ determined by the F-set

$$\bigcup_{d' \geq d} (T^{(d')}).$$

Given any element e_1 in $E^{(d)}$, set

$$F_1 = \gamma^{-1}(e_1^{+})$$

where e_1^{+} is the tail of $E^{(d)}$ determined by e_1. It is easily seen that $F_1 \in \Phi$. Whenever $F \geq F_1$ i.e., $F \subset F_1$ we have

$$\alpha(F) \in F_1 ,$$

$$\mu(F) = \gamma \circ \alpha(F) \in \gamma(F_1) = \gamma \circ \gamma^{-1}(e_1^{+}) \subset e_1^{+}$$

and so (a) is satisfied. By (b') we have, on $F = \bigcup_{d' \geq d} T^{(d')}$, $S = S \circ \gamma$. Thus for every $F \geq F_0$

$$S^*(F) = S \circ \alpha(F) = S \circ \gamma \circ \alpha(F) = S \circ \mu(F),$$

which proves (b).

REFERENCES

1. J. L. Kelley, General Topology, New York; Van Nostrand 1965.

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