

Order continuity of Arens extensions of regular multilinear operators

Geraldo Botelho

Faculdade de Matemática, Universidade Federal de Uberlândia,
 38.400-902 Uberlândia, Brazil (botelho@ufu.br)

Luis Alberto Garcia

Instituto de Matemática e Estatística, Universidade de São Paulo,
 05.508-090 São Paulo, Brazil (luisgarcia@ime.usp.br)

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First we give a counterexample showing that recent results on separate order continuity of Arens extensions of multilinear operators cannot be improved to get separate order continuity on the product of the whole of the biduals. Then we establish conditions on the operators and/or on the underlying Riesz spaces/Banach lattices so that the extensions are order continuous on the product of the whole biduals. We also prove that all Arens extensions of any regular multilinear operator are order continuous in at least one variable and we study when Arens extensions of regular homogeneous polynomials on a Banach lattice E are order continuous on E^{**} .

Keywords: Riesz spaces; Banach lattices; Arens extension; separate order continuity

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1. Introduction

The second adjoint u^{**} of a linear operator u , which is a bidual extension of u , is a powerful tool in several areas of mathematics. For multilinear operators, this role has been played by Arens extensions. This has been extensively studied over the past 70 years since its introduction by Arens in his seminal paper, [3]. In order to state the two recent results that have motivated our work, let us fix some notation. By E^\sim we denote the order dual of a Riesz space E , hence $E^{\sim\sim} = (E^\sim)^\sim$ denotes its second order dual. For a Banach lattice E , E^* denotes its topological dual, hence E^{**} stands for its bidual. The symbols $(E^\sim)_n^\sim$ and $(E^*)_n^*$ stand for the corresponding subspaces formed by the order continuous functionals. The results that motivated our research are the following:

- Buskes and Roberts (2019) [10, Theorem 3.4]: If $A: E_1 \times \cdots \times E_m \longrightarrow F$ is an m -linear operator of order bounded variation between Riesz spaces, then its Arens extension $A^{*[m+1]}: E_1^{\sim\sim} \times \cdots \times E_m^{\sim\sim} \longrightarrow F^{\sim\sim}$ is separately order continuous on $(E_1^\sim)_n^\sim \times \cdots \times (E_m^\sim)_n^\sim$.

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- Boyd, Ryan and Snigireva (2021) [7, Theorem 1]: If $A: E_1 \times \cdots \times E_m \longrightarrow F$ is a regular m -linear operator between Banach lattices, with F Dedekind complete, then its Arens extension $A^{*[m+1]}: E_1^{**} \times \cdots \times E_m^{**} \longrightarrow F^{**}$ is separately order continuous on $(E_1^*)^*_n \times \cdots \times (E_m^*)^*_n$.

The obvious question is whether or not these results can be improved to get order continuity on $E_1^{\sim\sim} \times \cdots \times E_m^{\sim\sim}$ and $E_1^{**} \times \cdots \times E_m^{**}$, respectively. By means of a simple but efficient counterexample we show that this is not the case (cf. Section 3). Actually the counterexample, which is the bilinear form A on $\ell_1 \times c_0$ given by the duality $c_0^* = \ell_1$, discloses an interesting phenomenon: its Arens extension A^{***} is not separately order continuous on $\ell_1^{**} \times c_0^{**}$ (more precisely, it is order continuous in the first variable but not in the second one), while the other Arens extension of A is. We then proceed to find conditions on the operator and/or on the underlying spaces so that all Arens extensions are separately order continuous on the product of the whole of the biduals. In § 4 we prove that this holds for finite sums of multiplicative operators from Riesz spaces to Archimedean f -algebras, in particular for operators of finite type between arbitrary Riesz spaces. The main result of § 4 (Theorem 4.6): (i) implies that all Arens extensions of any regular multilinear operator between Riesz spaces are order continuous in at least one variable, (ii) implies that all Arens extensions of a regular homogeneous polynomial from the Riesz space E to a Riesz space F are order continuous at the origin on $E^{\sim\sim}$, (iii) improves the results of Boyd, Ryan and Snigireva and of Buskes and Roberts for regular operators (see Remark 4.7). In the final § 5 we give sufficient conditions on the Banach lattices E_1, \dots, E_m so that Arens extensions of any regular m -linear operator from $E_1 \times \cdots \times E_m$ to an arbitrary Banach lattice F are separately order continuous on $E_1^{**} \times \cdots \times E_m^{**}$; and conditions so that Arens extensions of regular homogeneous polynomials on a Banach lattice E are order continuous on E^{**} .

In § 2 we discuss briefly the notion of order continuity of linear operators and recall the characterization of the Arens extensions of regular multilinear operators between Riesz spaces that shall fit our purposes. Although these extensions are usually called Aron-Berner extensions in the case of operators between Banach spaces (see [4, 13]), for simplicity we shall refer to Arens extensions even for operators between Banach lattices.

2. Background

Our references to Riesz spaces, Banach lattices and regular linear operators are the canonical ones [2, 17, 19].

The following three definitions of order convergence can be found in the literature (see [1]). A net $(x_\alpha)_{\alpha \in \Omega}$ in a Riesz space E is said to be:

- *order convergent* to $x \in E$ if there is a net $(y_\alpha)_{\alpha \in \Omega}$ in E such that $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leq y_\alpha$ for every $\alpha \in \Omega$.
- *1-convergent* to $x \in E$ if there are a net $(y_\alpha)_{\alpha \in \Omega}$ in E and $\alpha_0 \in \Omega$ such that $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leq y_\alpha$ for every $\alpha \geq \alpha_0$.

- 2-convergent to $x \in E$ is there are a net $(y_\beta)_{\beta \in \Gamma}$ in E such that $y_\beta \downarrow 0$ and for every $\beta \in \Gamma$ there exists $\alpha_0 \in \Omega$ such that $|x_\alpha - x| \leq y_\beta$ for any $\alpha \geq \alpha_0$.

If E is Dedekind complete, then the notions of 1-convergence and 2-convergence coincide [1], but otherwise they may be different (see [1, Example 1.4]).

Order continuity can be considered with respect to any of these three notions of order convergence: a linear operator $T: E \rightarrow F$ between Riesz spaces is said to be *order continuous* (*1-order continuous*, *2-order continuous*, respectively) if $(T(x_\alpha))_{\alpha \in \Omega}$ is order convergent (1-convergent, 2-convergent, respectively) to zero in F whenever $(x_\alpha)_{\alpha \in \Omega}$ is order convergent (1-convergent, 2-convergent, respectively) to zero in E .

The following coincidences are known (see [2, Theorem 1.56] or [17, Proposition 1.3.9]):

- If F is Dedekind complete, then T is order continuous if and only if T is 1-order continuous.
- If E and F are Dedekind complete, then T is order continuous if and only if T is 1-order continuous if and only if T is 2-order continuous.

In most cases in this paper we will investigate the order continuity of regular linear operators from $E^{\sim\sim}$ to $F^{\sim\sim}$, where E and F are Riesz spaces, or from E^{**} to F^{**} , where E and F are Banach lattices. Since all these spaces are Dedekind complete, we are free to use any of the three notions of order continuity. We shall denote the order convergence by $x_\alpha \xrightarrow{o} x$.

For the theory of regular multilinear operators and regular homogeneous polynomials we refer to [9, 10, 16]. An m -linear operator $A: E_1 \times \dots \times E_m \rightarrow F$ is *separately order continuous* (*separately 1-order continuous*, *separately 2-order continuous*) if for all $j \in \{1, \dots, m\}$ and $x_k \in E_k, k = 1, \dots, m, k \neq j$, the linear operator

$$x_j \in E_j \mapsto A(x_1, \dots, x_m) \in F$$

is order continuous (1-order continuous, 2-order continuous). For the definition of joint order continuity see [7, p.234]. If A is a regular operator between Banach lattices with F Dedekind complete, then A is separately order continuous if and only if A is jointly order continuous [7, Theorem 2].

Now we recall the description of the Arens extensions of regular multilinear operators between Riesz spaces as presented in [6]. By $J_E: E \rightarrow E^{\sim\sim}$ we denote the canonical operator ($J_E(x)(x'') = x''(x)$), which happens to be a Riesz homomorphism.

Given Riesz spaces E_1, \dots, E_m, F , the space of regular m -linear operators from $E_1 \times \dots \times E_m$ to F is denoted by $\mathcal{L}_r(E_1, \dots, E_m; F)$. When F is the scalar field we write $\mathcal{L}_r(E_1, \dots, E_m)$. We use S_m to denote the set of permutations of $\{1, \dots, m\}$.

Let $m \in \mathbb{N}$ and $\rho \in S_m$ be given. For an m -linear form $C: E_1 \times \dots \times E_m \rightarrow \mathbb{R}$, define the m -linear form

$$C_\rho: E_{\rho(1)} \times \dots \times E_{\rho(m)} \rightarrow \mathbb{R}, C_\rho(x_1, \dots, x_m) = C(x_{\rho^{-1}(1)}, \dots, x_{\rho^{-1}(m)}),$$

where $x_j \in E_{\rho(j)}$, $j = 1, \dots, m$. Furthermore, for each $i = 1, \dots, m - 1$, and $x''_{\rho(i)} \in E_{\rho(i)}$, consider the regular linear operator

$$\overline{x''_{\rho(i)}}: \mathcal{L}_r(E_{\rho(i)}, \dots, E_{\rho(m)}) \longrightarrow \mathcal{L}_r(E_{\rho(i+1)}, \dots, E_{\rho(m)}), \quad \overline{x''_{\rho(i)}}(B) = x''_{\rho(i)} \circ B^i, \tag{2.1}$$

where $B^i: E_{\rho(i+1)} \times \dots \times E_{\rho(m)} \longrightarrow E_{\rho(i)}$ is given by

$$B^i(x_{i+1}, \dots, x_m)(x_i) = B(x_i, x_{i+1}, \dots, x_m), \text{ for all } x_j \in E_{\rho(j)}, \quad j = i, \dots, m.$$

In the case $i = m$ we define $\overline{x''_{\rho(m)}}: E_{\rho(m)} \longrightarrow \mathbb{R}$ by $\overline{x''_{\rho(m)}}(x_{\rho(m)}) = x''_{\rho(m)}(x_{\rho(m)})$.

The Arens extension of the regular m -linear operator $A: E_1 \times \dots \times E_m \longrightarrow F$ is the regular m -linear operator $AR_m^\rho(A): E_1^{\sim} \times \dots \times E_m^{\sim} \longrightarrow F^{\sim}$ given by

$$AR_m^\rho(A)(x''_1, \dots, x''_m)(y') = (\overline{x''_{\rho(m)}} \circ \dots \circ \overline{x''_{\rho(1)}})((y' \circ A)_\rho),$$

for all $x''_1 \in E_1^{\sim}, \dots, x''_m \in E_m^{\sim}$ and every $y' \in F^{\sim}$. Although the notation is a bit different, these are exactly the Arens extensions studied in [6].

According to [6, Theorem 2.2], $AR_m^\rho(A)$ is a regular m -linear operator that extends A in the sense that $AR_m^\rho(A) \circ (J_{E_1}, \dots, J_{E_m}) = J_F \circ A$. Moreover, $AR_m^\rho(A)$ is positive for positive A .

The extension $A^{*[m+1]}$ from [7, 10] is recovered by considering the permutation $\theta(m) = 1, \theta(m - 1) = 2, \dots, \theta(2) = m - 1, \theta(1) = m$, that is, $AR_m^\theta(A) = A^{*[m+1]}$. In particular, $AR_2^\theta(A) = A^{***}$ in the bilinear case $m = 2$.

3. The counterexample

Consider the positive bilinear form given by the duality bracket (c_0, ℓ_1) :

$$A: \ell_1 \times c_0 \longrightarrow \mathbb{R}, A((x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty) = \sum_{n=1}^\infty x_n y_n.$$

The two Arens extensions of A shall be denoted by $A^{***} = AR_2^\theta(A)$ and $AR_2^{\text{id}}(A)$, where id is the identity permutation. As announced, we shall prove that $AR_2^{\text{id}}(A)$ is separately order continuous on $\ell_1^{**} \times c_0^{**}$ and that A^{***} is order continuous in the first variable but not in the second one. Although everything can be proved directly to this bilinear form, to avoid unnecessary repetitions we shall apply some results that will be proved later.

From Theorem 4.6 we know that $A^{***}: \ell_1^{**} \times c_0^{**} \longrightarrow \mathbb{R}$ is order continuous in the first variable. Suppose that A^{***} is order continuous in the second variable, that is, for every $x^{**} \in \ell_1^{**}$, the linear functional $A^{***}(x^{**}, \bullet): c_0^{**} \longrightarrow \mathbb{R}$, defined by $A^{***}(x^{**}, \bullet)(y^{**}) = A^{***}(x^{**}, y^{**})$, is order continuous. Denoting by $\psi: \ell_1 \longrightarrow c_0^*$ the canonical isometric isomorphism, note that ψ and $\psi^{-1}: c_0^* \longrightarrow \ell_1$, $\psi^{-1}(\varphi) = (\varphi(e_n))_{n=1}^\infty$, are positive operators, hence ψ is a Riesz homomorphism [2, Theorem 2.15]. Moreover, $\psi(x) = A_\theta(\bullet, x)$ for every $x \in \ell_1$, where $A_\theta(\bullet, x): c_0 \longrightarrow \mathbb{R}$ is given by $A_\theta(\bullet, x)(y) = A_\theta(y, x)$.

Claim 1. $\psi^{**}(x^{**}) = A^{***}(x^{**}, \bullet)$ for every $x^{**} \in \ell_1^{**}$.

Indeed, given $y^{**} \in c_0^{**}$ and $x \in \ell_1$, noting that $A_\theta^1: \ell_1 \rightarrow c_0^*$ is given by $A_\theta^1(x)(y) = A_\theta(y, x)$, we have $A_\theta^1(x) = A_\theta(\bullet, x)$, hence

$$\begin{aligned} \psi^*(y^{**})(x) &= y^{**}(\psi(x)) \\ &= y^{**}(A_\theta(\bullet, x)) = y^{**}(A_\theta^1(x)) = (y^{**} \circ A_\theta^1)(x) = \overline{y^{**}}(A_\theta)(x), \end{aligned}$$

proving that $\psi^*(y^{**}) = \overline{y^{**}}(A_\theta)$. Therefore, for $x^{**} \in \ell_1^{**}$ and $y^{**} \in c_0^{**}$, bearing in mind that $A^{***} = AR_2^\theta(A)$,

$$\begin{aligned} \psi^{**}(x^{**})(y^{**}) &= x^{**}(\psi^*(y^{**})) \\ &= x^{**}(\overline{y^{**}}(A_\theta)) = AR_2^\theta(A)(x^{**}, y^{**}) = A^{***}(x^{**}, \bullet)(y^{**}). \end{aligned}$$

Claim 2. $x^{**} \in \ell_1^{**}$ is order continuous on ℓ_1^* if and only if $\psi^{**}(x^{**}) \in c_0^{***}$ is order continuous on c_0^{**} .

Let $x^{**} \in \ell_1^{**}$ be such that $\psi^{**}(x^{**}) \in c_0^{***}$ is order continuous on c_0^{**} . Supposing that x^{**} fails to be order continuous on ℓ_1^* , the positive functional $|x^{**}| \in \ell_1^{**}$ is not order continuous on ℓ_1^* either [2, Theorem 1.56]. Then there is a net $(x_\alpha^*)_{\alpha \in \Omega}$ in ℓ_1^* such that $x_\alpha^* \downarrow 0$ but $\inf_{\alpha \in \Omega} |x^{**}|(x_\alpha^*) > 0$. For each $\alpha \in \Omega$ let $y_\alpha^{**} \in c_0^{**}$ be such that $\psi^*(y_\alpha^{**}) = x_\alpha^*$. Thus

$$y_\alpha^{**} = (\psi^*)^{-1}(x_\alpha^*) = (\psi^{-1})^*(x_\alpha^*) \downarrow 0 \text{ in } c_0^{**}$$

because $(\psi^{-1})^*$ is positive and order continuous [2, Theorem 1.73]. By assumption $\psi^{**}(x^{**})$ is order continuous on c_0^{**} , so is $|\psi^{**}(x^{**})|$ [2, Theorem 1.56]. Since ψ is a Riesz homomorphism, ψ^{**} is as well, so $\psi^{**}(|x^{**}|)(y_\alpha^{**}) = |\psi^{**}(x^{**})|(y_\alpha^{**}) \downarrow 0$, from which it follows that

$$0 = \inf_{\alpha \in \Omega} \psi^{**}(|x^{**}|)(y_\alpha^{**}) = \inf_{\alpha \in \Omega} |x^{**}|(\psi^*(y_\alpha^{**})) = \inf_{\alpha \in \Omega} |x^{**}|(x_\alpha^*) > 0.$$

This contradiction proves that $x^{**} \in \ell_1^{**}$ is order continuous on ℓ_1^* . The reverse implication is straightforward.

Claim 3. ℓ_1^{**} contains a functional that fails to be order continuous on ℓ_1^* .

Let c be the space of convergent real sequences and consider the positive linear functional $\varphi \in c^*$ given by $\varphi((x_n)_{n=1}^\infty) = \lim_{n \rightarrow \infty} x_n$. Since c is a majorizing subspace of ℓ_∞ , φ admits a positive extension $\tilde{\varphi} \in \ell_\infty^*$ [2, Theorem 1.32]. Suppose that $\tilde{\varphi}$ is σ -order continuous on ℓ_∞ . For each $n \in \mathbb{N}$ let $x_n = (1, \dots, 1, 0, \dots) = e_1 + \dots + e_n \in c$ and $y = (1, 1, \dots) \in c$. Note that $0 \leq x_n \uparrow y$ and, since $\tilde{\varphi}$ is a positive σ -order continuous operator, $0 \leq \tilde{\varphi}(x_n) \uparrow \tilde{\varphi}(y)$ (see [2, p. 46]). So,

$$1 = \tilde{\varphi}(y) = \sup_{n \in \mathbb{N}} \tilde{\varphi}(x_n) = 0,$$

which proves that $\tilde{\varphi} \in \ell_\infty^*$ is not σ -order continuous on ℓ_∞ . Considering the canonical Riesz isomorphism $\phi: \ell_\infty \rightarrow \ell_1^*$, there is $z^{**} \in \ell_1^{**}$ such that $\phi^*(z^{**}) = \tilde{\varphi}$. Since $\tilde{\varphi} \in \ell_\infty^*$ fails to be σ -order continuous, there is a sequence $(z_n)_{n=1}^\infty$ in ℓ_∞ such that $z_n \downarrow 0$ and $\inf_{n \in \mathbb{N}} \tilde{\varphi}(z_n) > 0$. Furthermore, for each $n \in \mathbb{N}$ there is $y_n^* \in \ell_1^*$ such that $\phi^{-1}(y_n^*) = z_n$, hence $y_n^* = \phi(z_n) \downarrow$ because ϕ is positive, so $0 \leq y_n^* \downarrow$. Suppose that

there exists $y^* \in \ell_1^*$ such that $0 < y^* \leq y_n^*$ for every $n \in \mathbb{N}$. On the one hand, as ϕ^{-1} is positive,

$$0 \leq \phi^{-1}(y^*) \leq \phi^{-1}(y_n^*) = z_n \text{ for every } n,$$

from which we conclude that $\phi^{-1}(y^*) = 0$, and so $y^* = 0$ because ϕ^{-1} is injective. This shows that $y_n^* \downarrow 0$ in ℓ_1^* . On the other hand,

$$\begin{aligned} \inf_{n \in \mathbb{N}} z^{**}(y_n^*) &= \inf_{n \in \mathbb{N}} (\phi^*)^{-1}(\tilde{\varphi})(y_n^*) \\ &= \inf_{n \in \mathbb{N}} (\phi^{-1})^*(\tilde{\varphi})(y_n^*) = \inf_{n \in \mathbb{N}} \tilde{\varphi}(\phi^{-1}(y_n^*)) = \inf_{n \in \mathbb{N}} \tilde{\varphi}(z_n) > 0, \end{aligned}$$

proving that z^{**} is not order continuous on ℓ_∞ , as claimed.

Finally, combining claims 1 and 2 we have that $A^{***}(z^{**}, \bullet) = \psi^{**}(z^{**})$ is not order continuous on c_0^{**} . We have established that A^{***} is order continuous in the first variable and fails to be order continuous in the second variable.

As to the other Arens extension of A , namely $AR_2^{id}(A): \ell_1^{**} \times c_0^{**} \rightarrow \mathbb{R}$, since $c_0^* = \ell_1$ has order continuous norm, Corollary 5.1 guarantees that $AR_2^{id}(A)$ is separately order continuous, hence jointly order continuous by [7, Theorem 2].

Since the bilinear form A is regular and of bounded order variation, this example shows that the results of Buskes and Roberts and of Boyd, Ryan and Snigireva quoted in the introduction cannot be improved to get separate order continuity on the product of the whole biduals.

4. Operators between Riesz spaces

In this section we present our results on order continuity to the whole of the bidual of Arens extensions of multilinear operators on Riesz spaces. The main result of the section, namely Theorem 4.6, is a multipurpose result: in this section it will be used to prove that Arens extensions of regular homogeneous polynomials are always order continuous at the origin on the whole of the bidual of the domain space, to extend [7, Theorem 1] and to show that Arens extensions are always order continuous in at least one variable. Furthermore, Theorem 4.6 will be helpful a couple of times in the next section.

Recall that a *Riesz algebra* \mathcal{A} is a Riesz space which is an associative algebra with respect to a product $*$ such that $x * y \geq 0$ for all $x, y \in \mathcal{A}^+$. And that a Riesz algebra $(\mathcal{A}, *)$ is an *f-algebra* if $x \wedge y = 0$ in \mathcal{A} implies that $(x * z) \wedge y = (z * x) \wedge y = 0$ for every $z \in \mathcal{A}^+$. The following Riesz spaces are *f*-algebras: \mathbb{R} , $C(X)$ and $C_b(X)$ -spaces, where X is a topological space, $\ell_\infty(X)$ and \mathbb{R}^X -spaces, where X is a nonempty set, $L_\infty(\mu)$ -spaces, and spaces of measurable functions on measure spaces (see [2, p.126]). If $(\mathcal{A}, *)$ is an *f*-algebra, then the Arens product \odot , defined as follows, makes $\mathcal{A}^{\sim\sim}$ an *f*-algebra [20]: for $x \in \mathcal{A}$, $y' \in \mathcal{A}^{\sim}$ and $x'', y'' \in \mathcal{A}^{\sim\sim}$,

$$\begin{aligned} y' \cdot x: \mathcal{A} &\longrightarrow \mathbb{R}, & (y' \cdot x)(y) &= y'(x * y). \\ x'' \diamond y': \mathcal{A} &\longrightarrow \mathbb{R}, & (x'' \diamond y')(y) &= x''(y' \cdot y). \\ x'' \odot y'': \mathcal{A}^{\sim} &\longrightarrow \mathbb{R}, & (x'' \odot y'')(z') &= x''(y'' \diamond z'). \end{aligned}$$

An operator $A \in \mathcal{L}_r(E_1, \dots, E_m; \mathcal{A})$ is *multiplicative* if there are regular linear operators $T_i: E_i \rightarrow \mathcal{A}$, $i = 1, \dots, m$, such that $A(x_1, \dots, x_m) = T_1(x_1) * \dots * T_m(x_m)$ for all $x_1 \in E_1, \dots, x_m \in E_m$.

Since Arens extensions $AR_m^\rho(A)$ of multilinear operators A are mappings between Dedekind complete spaces, we can use any of the three notions of order continuous linear operators to investigate the separate order continuity of $AR_m^\rho(A)$.

PROPOSITION 4.1. *Let E_1, \dots, E_m be Riesz spaces and $(\mathcal{A}, *)$ be an Archimedean f -algebra. If $A \in \mathcal{L}_r(E_1, \dots, E_m; \mathcal{A})$ is a finite sum of multiplicative operators, then all Arens extensions of A , $AR_m^\rho(A)$, $\rho \in S_m$, coincide and are separately order continuous.*

Proof. Given a multiplicative operator $B \in \mathcal{L}_r(E_1, \dots, E_m; \mathcal{A})$, let $T_i: E_i \rightarrow \mathcal{A}$, $i = 1, \dots, m$, be such that $B(x_1, \dots, x_m) = T_1(x_1) * \dots * T_m(x_m)$ for all $x_1 \in E_1, \dots, x_m \in E_m$. By [6, Remark 3.3 and the proof of Theorem 3.2] we have that, for each $\rho \in S_m$ and all $x''_1 \in E_1^{\sim\sim}, \dots, x''_m \in E_m^{\sim\sim}$,

$$AR_m^\rho(B)(x''_1, \dots, x''_m) = T''_{\rho(m)}(x''_{\rho(m)}) \odot \dots \odot T''_{\rho(1)}(x''_{\rho(1)}).$$

The Arens product \odot makes $\mathcal{A}^{\sim\sim}$ a Dedekind complete, hence Archimedean, commutative f -algebra [20, Corollaries 3.5 and 3.6], so

$$AR_m^\rho(B)(x''_1, \dots, x''_m) = T''_1(x''_1) \odot \dots \odot T''_m(x''_m),$$

which gives, in particular, that all Arens extensions of A coincide. In order to check that $AR_m^\rho(B)$ is separately order continuous, let $j \in \{1, \dots, m\}$, $x''_i \in E_i^{\sim\sim}$, $i = 1, \dots, m$, with $i \neq j$ be given and let $(x''_{\alpha_j})_{\alpha_j \in \Omega_j}$ be a net in $E_j^{\sim\sim}$ such that $x''_{\alpha_j} \xrightarrow{o} 0$. There exists a net $(z''_{\alpha_j})_{\alpha_j \in \Omega_j}$ such that $z''_{\alpha_j} \downarrow 0$ and $|x''_{\alpha_j}| \leq z''_{\alpha_j}$ for every $\alpha_j \in \Omega_j$. The functional

$$\varphi := |T''_1(x''_1)| \odot \dots \odot |T''_{j-1}(x''_{j-1})| \odot |T''_{j+1}(x''_{j+1})| \odot \dots \odot |T''_m(x''_m)| \in \mathcal{A}^{\sim\sim}$$

is positive. Using again that the product \odot is commutative and [2, Exercise 12, p. 131],

$$\begin{aligned} |AR_m^\rho(B)(x''_1, \dots, x''_{\alpha_j}, \dots, x''_m)| &= |T''_1(x''_1) \odot \dots \odot T''_j(x''_{\alpha_j}) \odot \dots \odot T''_m(x''_m)| \\ &= |T''_1(x''_1)| \odot \dots \odot |T''_{j-1}(x''_{j-1})| \odot |T''_j(x''_{\alpha_j})| \odot |T''_{j+1}(x''_{j+1})| \odot \dots \odot |T''_m(x''_m)| \\ &= |T''_j(x''_{\alpha_j})| \odot (|T''_1(x''_1)| \odot \dots \odot |T''_{j-1}(x''_{j-1})| \odot |T''_{j+1}(x''_{j+1})| \odot \dots \odot |T''_m(x''_m)|) \\ &= |T''_j(x''_{\alpha_j})| \odot \varphi \leq |T''_j|(x''_{\alpha_j}) \odot \varphi \leq |T''_j|(z''_{\alpha_j}) \odot \varphi. \end{aligned}$$

Now it is enough to prove that $|T''_j|(z''_{\alpha_j}) \odot \varphi \downarrow 0$. Let $0 \leq y' \in \mathcal{A}^{\sim}$ be given. Then $\varphi \diamond y'$ is positive and, since $|T''_j|$ is order continuous and positive [2, Theorems 1.56 and 1.73],

$$(|T''_j|(z''_{\alpha_j}) \odot \varphi)(y') = |T''_j|(z''_{\alpha_j})(\varphi \diamond y') \downarrow 0,$$

from which it follows that $(|T''_j|(z''_{\alpha_j}) \odot \varphi) \downarrow 0$ [2, Theorem 1.18] and gives the separate order continuity of $AR_m^\rho(B)$.

The linearity of the correspondence $A \mapsto AR_m^\rho(A)$ gives the result for finite sums of multiplicative operators. \square

Since scalar-valued Riesz multimorphisms are multiplicative [14, Theorem 6], Proposition 4.1 yields the following.

COROLLARY 4.2. *All Arens extensions of a finite sum of scalar-valued Riesz multimorphisms coincide and are separately order continuous.*

An operator $A \in \mathcal{L}_r(E_1, \dots, E_m; F)$ is of *finite type* if there are $n \in \mathbb{N}$, functionals $\varphi_j^i \in E_i^\sim$ and vectors $y_j \in F, j = 1, \dots, n, i = 1, \dots, m$, such that

$$A(x_1, \dots, x_m) = \sum_{j=1}^n \varphi_j^1(x_1) \cdots \varphi_j^m(x_m) y_j \text{ for all } x_i \in E_i, \quad i = 1, \dots, m.$$

COROLLARY 4.3. *All Arens extensions of a multilinear operator of finite type coincide, are of finite type and are separately order continuous.*

Proof. It is not difficult to check that if $A \in \mathcal{L}_r(E_1, \dots, E_m)$ is separately order continuous and $y \in F$, then the operator

$$(x_1, \dots, x_m) \in E_1 \times \cdots \times E_m \mapsto A(x_1, \dots, x_m)y \in F,$$

is separately order continuous as well. Now the result follows from Proposition 4.1 and from its proof. \square

To proceed to the main results of the section we need some preparation.

LEMMA 4.4. *Let E_1, \dots, E_m, F be Riesz spaces with F Dedekind complete and $(B_\alpha)_\alpha$ be a net in $\mathcal{L}_r(E_1, \dots, E_m; F)$. Then $B_\alpha \downarrow 0$ if and only if $B_\alpha(x_1, \dots, x_m) \downarrow 0$ in F for all $x_1 \in E_1^+, \dots, x_m \in E_m^+$.*

Proof. It is straightforward that $B_\alpha \downarrow 0$ if $B_\alpha(x_1, \dots, x_m) \downarrow 0$ in F for all $x_1 \in E_1^+, \dots, x_m \in E_m^+$. We prove the other implication by induction on m . The case $m = 1$ follows from the Riesz-Kantorovich theorem [2, Theorem 1.18]. Assume that the result holds for n and let $(B_\alpha)_\alpha$ be a net in $\mathcal{L}_r(E_1, \dots, E_{n+1}; F)$ such that $B_\alpha \downarrow 0$. Consider the canonical Riesz isomorphism

$$\psi: \mathcal{L}_r(E_1, \dots, E_{n+1}; F) \longrightarrow \mathcal{L}_r(E_1; \mathcal{L}_r(E_2, \dots, E_{n+1}; F)).$$

We have $0 \leq \psi(B_\alpha) \downarrow$ because ψ is positive. Let $T \in \mathcal{L}_r(E_1; \mathcal{L}_r(E_2, \dots, E_{n+1}; F))$ be such that $0 \leq T \leq \psi(B_\alpha)$ for every α . Since ψ^{-1} is positive, $0 \leq \psi^{-1}(T) \leq B_\alpha \downarrow 0$, hence $0 \leq \psi^{-1}(T) \leq 0$, which proves that $\psi(B_\alpha) \downarrow 0$ in $\mathcal{L}_r(E_1; \mathcal{L}_r(E_2, \dots, E_{n+1}; F))$. The linear case of the result gives that $\psi(B_\alpha)(x_1) \downarrow 0$ in $\mathcal{L}_r(E_2, \dots, E_{n+1}; F)$ for every $x_1 \in E_1^+$. The induction hypothesis gives that, regardless of the $x_2 \in E_2^+, \dots, x_{n+1} \in E_{n+1}^+$,

$$B_\alpha(x_1, x_2, \dots, x_{n+1}) = \psi(B_\alpha)(x_1)(x_2, \dots, x_{n+1}) \downarrow 0,$$

completing the proof. \square

LEMMA 4.5. Let E_1, \dots, E_m be Riesz spaces, $\rho \in S_m$, $k \in \{1, \dots, m-1\}$ and $x''_{\rho(k)} \in (E_{\rho(k)})_{\tilde{n}}$. Then the operator

$$\overline{x''_{\rho(k)}}: \mathcal{L}_r(E_{\rho(k)}, \dots, E_{\rho(m)}) \longrightarrow \mathcal{L}_r(E_{\rho(k+1)}, \dots, E_{\rho(m)}),$$

defined in (2.1), is order continuous.

Proof. We already know that $\overline{x''_{\rho(k)}}$ is a regular linear operator, $|\overline{x''_{\rho(k)}}| \leq \overline{|x''_{\rho(k)}|}$ because B^k is positive for every positive $B \in \mathcal{L}_r(E_{\rho(k)}, \dots, E_{\rho(m)})$, so

$$\overline{|x''_{\rho(k)}|}(B) = |x''_{\rho(k)}| \circ B^k \geq \pm x''_{\rho(k)} \circ B^k = \pm (x''_{\rho(k)} \circ B^k) = \pm \overline{|x''_{\rho(k)}|}(B).$$

Let $(A_\alpha)_{\alpha \in \Omega}$ be a net in $\mathcal{L}_r(E_{\rho(k)}, \dots, E_{\rho(m)})$ such that $A_\alpha \xrightarrow{o} 0$. Then there are a net $(B_\alpha)_{\alpha \in \Omega}$ in $\mathcal{L}_r(E_{\rho(k)}, \dots, E_{\rho(m)})$ and $\alpha_0 \in \Omega$ such that $B_\alpha \downarrow 0$ and $|A_\alpha| \leq B_\alpha$ for every $\alpha \geq \alpha_0$. Thus,

$$|\overline{|x''_{\rho(k)}|}(A_\alpha)| \leq |\overline{|x''_{\rho(k)}|}(|A_\alpha|) \leq \overline{|x''_{\rho(k)}|}(|A_\alpha|) \leq \overline{|x''_{\rho(k)}|}(B_\alpha) \text{ for every } \alpha \geq \alpha_0.$$

For $x_i \in E_{\rho(i)}^+$, $i \in \{k, \dots, m\}$, Lemma 4.4 gives

$$B_\alpha^k(x_{k+1}, \dots, x_m)(x_k) = B_\alpha(x_k, x_{k+1}, \dots, x_m) \downarrow 0.$$

By [2, Theorem 1.18] it follows that $B_\alpha^k(x_{k+1}, \dots, x_m) \downarrow 0$. Since $x''_{\rho(k)}$ is an order continuous functional, $|x''_{\rho(k)}|$ is a positive order continuous operator [2, Theorem 1.56], so $|x''_{\rho(k)}|(B_\alpha^k(x_{k+1}, \dots, x_m)) \downarrow 0$, that is, $\overline{|x''_{\rho(k)}|}(B_\alpha)(x_{k+1}, \dots, x_m) \downarrow 0$. Calling on Lemma 4.4 once again it follows that $\overline{|x''_{\rho(k)}|}(B_\alpha) \downarrow 0$, proving that $\overline{x''_{\rho(k)}}$ is order continuous. \square

THEOREM 4.6. Let E_1, \dots, E_m, F be Riesz spaces, $\rho \in S_m$ and $A \in \mathcal{L}_r(E_1, \dots, E_m; F)$.

- (a) For all $j \in \{1, \dots, m\}$, $x''_{\rho(i)} \in E_{\rho(i)}^{\sim\sim}$, $i = 1, \dots, j-1$, and $x''_{\rho(i)} \in (E_{\rho(i)}^{\sim})_{\tilde{n}}$, $i = j+1, \dots, m$, the operator

$$x''_{\rho(j)} \in E_{\rho(j)}^{\sim\sim} \mapsto AR_m^\rho(A)(x''_1, \dots, x''_{\rho(j)}, \dots, x''_m) \in F^{\sim\sim} \tag{4.1}$$

is order continuous on $E_{\rho(j)}^{\sim\sim}$.

- (b) $AR_m^\rho(A)$ is separately order continuous on $(E_1^{\sim})_{\tilde{n}} \times \dots \times (E_m^{\sim})_{\tilde{n}}$.
- (c) $AR_m^\rho(A)$ is order continuous in the $\rho(m)$ -th variable on the whole of $E_{\rho(m)}^{\sim\sim}$.

Proof. It is clear that (b) and (c) follow from (a) (for (c) just take $j = m$ in (a)). To prove (a), take $j \in \{1, \dots, m\}$, $x''_{\rho(i)} \in E_{\rho(i)}^{\sim\sim}$, $i = 1, \dots, j-1$, and $x''_{\rho(i)} \in (E_{\rho(i)}^{\sim})_{\tilde{n}}$, $i = j+1, \dots, m$. Given a net $(x''_{\alpha_{\rho(j)}})_{\alpha_{\rho(j)} \in \Omega_{\rho(j)}}$ in $E_{\rho(j)}^{\sim\sim}$ such that $x''_{\alpha_{\rho(j)}} \xrightarrow{o} 0$, there is a net $(z''_{\alpha_{\rho(j)}})_{\alpha_{\rho(j)} \in \Omega_{\rho(j)}}$ in $E_{\rho(j)}^{\sim\sim}$ and $\alpha_{\rho(j)0}$ such that $z''_{\alpha_{\rho(j)}} \downarrow 0$ and $|x''_{\alpha_{\rho(j)}}| \leq z''_{\alpha_{\rho(j)}}$ for every $\alpha_{\rho(j)} \geq \alpha_{\rho(j)0}$. Let $A_1, A_2 \in \mathcal{L}_r(E_1, \dots, E_m; F)$ be positive operators such that $A = A_1 - A_2$ and put $B := A_1 + A_2$. Of course B is

positive. Denoting the operator in (4.1) by $AR_m^\rho(A)_{x''_{\rho(1)}, \dots, x''_{\rho(j-1)}, x''_{\rho(j+1)}, \dots, x''_{\rho(m)}}$, for every $\alpha_{\rho(j)} \geq \alpha_{\rho(j)_0}$,

$$\begin{aligned} & |AR_m^\rho(A)_{x''_{\rho(1)}, \dots, x''_{\rho(j-1)}, x''_{\rho(j+1)}, \dots, x''_{\rho(m)}}(x''_{\alpha_{\rho(j)}})| = |AR_m^\rho(A)(x''_1, \dots, x''_{\alpha_{\rho(j)}}, \dots, x''_m)| \\ & \leq |AR_m^\rho(A)|(|x''_1|, \dots, |x''_{\alpha_{\rho(j)}}|, \dots, |x''_m|) \\ & = |AR_m^\rho(A_1) - AR_m^\rho(A_2)|(|x''_1|, \dots, |x''_{\alpha_{\rho(j)}}|, \dots, |x''_m|) \\ & \leq (AR_m^\rho(A_1) + AR_m^\rho(A_2))(|x''_1|, \dots, |x''_{\alpha_{\rho(j)}}|, \dots, |x''_m|) \\ & = AR_m^\rho(B)(|x''_1|, \dots, |x''_{\alpha_{\rho(j)}}|, \dots, |x''_m|) \\ & = AR_m^\rho(B)_{|x''_{\rho(1)}|, \dots, |x''_{\rho(j-1)}|, |x''_{\rho(j+1)}|, \dots, |x''_{\rho(m)}|}(|x''_{\alpha_{\rho(j)}}|) \\ & \leq AR_m^\rho(B)_{|x''_{\rho(1)}|, \dots, |x''_{\rho(j-1)}|, |x''_{\rho(j+1)}|, \dots, |x''_{\rho(m)}|}(z''_{\alpha_{\rho(j)}}). \end{aligned}$$

As Arens extensions of positive operators are positive, it follows that

$$0 \leq AR_m^\rho(B)_{|x''_{\rho(1)}|, \dots, |x''_{\rho(j-1)}|, |x''_{\rho(j+1)}|, \dots, |x''_{\rho(m)}|}(z''_{\alpha_{\rho(j)}}) \downarrow.$$

Setting $T := \overline{|x''_{\rho(m)}|} \circ \dots \circ \overline{|x''_{\rho(j+1)}|}$, since each $|x''_{\rho(i)}|, i = j + 1, \dots, m$, is order continuous, by Lemma 4.5 it follows that $\overline{|x''_{\rho(i)}|}$ is order continuous, so T is order continuous and positive. On the other hand, it is clear that, for every positive $y' \in F^\sim$,

$$S := (\overline{|x''_{\rho(j-1)}|} \circ \dots \circ \overline{|x''_{\rho(1)}|})(y' \circ B)_\rho \in \mathcal{L}_r(E_{\rho(j)}, \dots, E_{\rho(m)})$$

is positive. From $z''_{\alpha_{\rho(j)}} \downarrow 0$ we conclude that $\overline{z''_{\alpha_{\rho(j)}}}(S) \downarrow 0$, therefore $T(\overline{z''_{\alpha_{\rho(j)}}}(S)) \downarrow 0$. It follows that, for every positive $y' \in F^\sim$,

$$\begin{aligned} & AR_m^\rho(B)_{|x''_{\rho(1)}|, \dots, |x''_{\rho(j-1)}|, |x''_{\rho(j+1)}|, \dots, |x''_{\rho(m)}|}(z''_{\alpha_{\rho(j)}})(y') \\ & = AR_m^\rho(B)(|x''_1|, \dots, z''_{\alpha_{\rho(j)}}, \dots, |x''_m|)(y') \\ & = (\overline{|x''_{\rho(m)}|} \circ \dots \circ \overline{|x''_{\rho(j+1)}|} \circ \overline{z''_{\alpha_{\rho(j)}}} \circ \overline{|x''_{\rho(j-1)}|} \circ \dots \circ \overline{|x''_{\rho(1)}|})(y' \circ B)_\rho \\ & = (T \circ \overline{z''_{\alpha_{\rho(j)}}} \circ \overline{|x''_{\rho(j-1)}|} \circ \dots \circ \overline{|x''_{\rho(1)}|})(y' \circ B)_\rho \\ & = T((\overline{z''_{\alpha_{\rho(j)}}} \circ \overline{|x''_{\rho(j-1)}|} \circ \dots \circ \overline{|x''_{\rho(1)}|})(y' \circ B)_\rho) \\ & = T(\overline{z''_{\alpha_{\rho(j)}}}((\overline{|x''_{\rho(j-1)}|} \circ \dots \circ \overline{|x''_{\rho(1)}|})(y' \circ B)_\rho)) = T(\overline{z''_{\alpha_{\rho(j)}}}(S)) \downarrow 0. \end{aligned}$$

Lemma 4.4 gives that $AR_m^\rho(B)_{|x''_{\rho(1)}|, \dots, |x''_{\rho(j-1)}|, |x''_{\rho(j+1)}|, \dots, |x''_{\rho(m)}|}(z''_{\alpha_{\rho(j)}}) \downarrow 0$, and this allows us to conclude that $AR_m^\rho(A)_{x''_{\rho(1)}, \dots, x''_{\rho(j-1)}, x''_{\rho(j+1)}, \dots, x''_{\rho(m)}}$ is order continuous. \square

REMARK 4.7. Theorem 4.6 improves [7, Theorem 1] in the sense that it holds for all Arens extensions, it holds for operators between Riesz spaces, it drops the assumption of F being Dedekind complete and it assures the order continuity on the whole bidual in one of the variables. For regular operators, it improves [10,

Theorem 3.4] by taking into account all Arens extensions and by assuring the order continuity on the whole bidual in one of the variables. In particular, Theorem 4.6(b) provides an alternative proof of [7, Theorem 1] and of [10, Theorem 3.4] for regular operators between Riesz spaces and (c) shows that $A^{*[m+1]} = AR_m^\theta(A)$ is order continuous in the first variable on the whole of $E_1^{\sim\sim}$.

Recall that an m -homogeneous polynomial $P: E \rightarrow F$ between Riesz spaces is positive if the corresponding symmetric m -linear operator \check{P} is positive, and that P is regular, in symbols $P \in \mathcal{P}_r(mE; F)$, if P can be written as the difference of two positive polynomials.

The Arens extensions of a regular polynomial $P \in \mathcal{P}_r(mE; F)$ are the polynomials associated to the Arens extensions of \check{P} , that is: for $\rho \in S_m$, the Arens extension of P with respect to ρ is the polynomial

$$AR_m^\rho(P): E^{\sim\sim} \rightarrow F^{\sim\sim}, \quad AR_m^\rho(P)(x'') = AR_m^\rho(\check{P})(x'', \dots, x'').$$

In [10, Theorem 3.5] it is proved that $AR_m^\theta(P)$ is order continuous on $(E^\sim)_n^{\sim}$. We can go a bit further at the origin:

PROPOSITION 4.8. *All Arens extensions of a polynomial $P \in \mathcal{P}_r(mE; F)$ are order continuous at the origin on $E^{\sim\sim}$, meaning that $AR_m^\rho(P)(x''_\alpha) \xrightarrow{o} 0$ in $F^{\sim\sim}$ for every $\rho \in S_m$ and any $(x''_\alpha)_{\alpha \in \Omega}$ in $E^{\sim\sim}$ such that $x''_\alpha \xrightarrow{o} 0$ in $E^{\sim\sim}$.*

Proof. Write $P = P_1 - P_2$, where P_1 and P_2 are positive m -homogeneous polynomials, and let $\check{P}_1, \check{P}_2: E^m \rightarrow F$ be the positive symmetric m -linear operators associated to P_1 and P_2 , respectively. Let $(x''_\alpha)_{\alpha \in \Omega}$ be a net in $E^{\sim\sim}$ such that $x''_\alpha \xrightarrow{o} 0$. There are a net $(z''_\alpha)_{\alpha \in \Omega}$ in $E^{\sim\sim}$ and $\alpha_0 \in \Omega$ such that $z''_\alpha \downarrow 0$ and $|x''_\alpha| \leq z''_\alpha$ for every $\alpha \geq \alpha_0$. For a permutation $\rho \in S_m$, we know from Theorem 4.6 that the operator

$$x'' \in E^{\sim\sim} \mapsto AR_m^\rho(\check{P}_1 + \check{P}_2)(z''_{\alpha_0}, \dots, z''_{\alpha_0}, x'', z''_{\alpha_0}, \dots, z''_{\alpha_0}),$$

where x'' is placed at the $\rho(m)$ -th coordinate, is order continuous. For $\alpha \geq \alpha_0$ we have $z''_\alpha \leq z''_{\alpha_0}$, so, using that $AR_m^\rho(\check{P}_1 + \check{P}_2)$ is positive,

$$\begin{aligned} |AR_m^\rho(P)(x''_\alpha)| &= |AR_m^\rho(P_1 - P_2)(x''_\alpha)| = |AR_m^\rho((P_1 - P_2)^\vee)(x''_\alpha, \dots, x''_\alpha)| \\ &= |AR_m^\rho(\check{P}_1 - \check{P}_2)(x''_\alpha, \dots, x''_\alpha)| \leq |AR_m^\rho(\check{P}_1 - \check{P}_2)|(|x''_\alpha|, \dots, |x''_\alpha|) \\ &\leq |AR_m^\rho(\check{P}_1 - \check{P}_2)|(z''_\alpha, \dots, z''_\alpha) \leq AR_m^\rho(\check{P}_1 + \check{P}_2)(z''_\alpha, \dots, z''_\alpha) \\ &\leq AR_m^\rho(\check{P}_1 + \check{P}_2)(z''_{\alpha_0}, \dots, z''_{\alpha_0}, z''_\alpha, z''_{\alpha_0}, \dots, z''_{\alpha_0}) \downarrow 0. \end{aligned}$$

This proves that $AR_m^\rho(P)(x''_\alpha) \xrightarrow{o} 0$. □

In [8] it is proved that, for a regular homogeneous polynomial, order continuity at one point does not imply order continuity at every point in general. The result above shall be useful later.

5. Operators between Banach lattices

In this section we give conditions on the Banach lattices E_1, \dots, E_m so that, for every Banach lattice F , all Arens extensions of any regular m -linear operator from $E_1 \times \dots \times E_m$ to F are separately order continuous on $E_1^{**} \times \dots \times E_m^{**}$. Consequences on order continuity of extensions of regular homogeneous polynomials shall also be obtained.

If the dual E^* of a Banach lattice E has order continuous norm, then $E^{**} = (E^*)^*$ [17, Theorem 2.4.2]. So, the following is immediate from Theorem 4.6 .

COROLLARY 5.1. *Let E_1, \dots, E_m, F be Banach lattices, $A \in \mathcal{L}_r(E_1, \dots, E_m; F)$ and $\rho \in S_m$. If E_j^* has order continuous norm for $j = 1, \dots, m, j \neq \rho(1)$, then the Arens extension $AR_m^\rho(A)$ of A is separately order continuous on $E_1^{**} \times \dots \times E_m^{**}$.*

The next result makes clear what type of condition should be asked to get order continuity of Arens extensions on the product of the whole of the biduals.

PROPOSITION 5.2. *Let $m \geq 2$ and E_1, \dots, E_m be Banach lattices such that the Arens extension $A^{*[m+1]}$ of any form $A \in \mathcal{L}_r(E_1, \dots, E_m)$ is separately order continuous on $E_1^{**} \times \dots \times E_m^{**}$. Then, for every operator $T \in \mathcal{L}_r(E_i; E_j^*)$, $i, j = 1, \dots, m, i \neq j$, the functional $T^{**}(x_i^{**})$ is order continuous on E_j^{**} for every $x_i^{**} \in E_i^{**}$.*

Proof. Let $i, j = 1, \dots, m, i \neq j$, and $T \in \mathcal{L}_r(E_i; E_j^*)$ be given. For $k = 1, \dots, m, i \neq k \neq j$, choose $0 \neq \varphi_k \in E_k^*$ and consider the regular m -linear form

$$A: E_1 \times \dots \times E_m \longrightarrow \mathbb{R}, A(x_1, \dots, x_m) = \left(\prod_{\substack{k=1 \\ k \neq i, j}}^m \varphi_k(x_k) \right) T(x_i)(x_j).$$

Of course we can assume $i < j$. Using the Davie–Gamelin description of the Arens extensions [12], for $x_l^{**} \in E_l^{**}$ and nets $(x_{\alpha_l})_{\alpha_l \in \Omega_l}$ in E_l such that $x_l^{**} = \omega^* - \lim_{\alpha_l} J_{E_l}(x_{\alpha_l})$, $l = 1, \dots, m$, we have

$$\begin{aligned} A^{*[m+1]}(x_1^{**}, \dots, x_i^{**}, \dots, x_j^{**}, \dots, x_m^{**}) &= \lim_{\alpha_1} \dots \lim_{\alpha_i} \dots \lim_{\alpha_j} \dots \lim_{\alpha_m} A(x_{\alpha_1}, \dots, x_{\alpha_m}) \\ &= \lim_{\alpha_1} \dots \lim_{\alpha_i} \dots \lim_{\alpha_j} \dots \lim_{\alpha_m} \left(\prod_{\substack{k=1 \\ k \neq i, j}}^m \varphi_k(x_{\alpha_k}) \right) T(x_{\alpha_i})(x_{\alpha_j}) \\ &= \lim_{\alpha_1} \dots \lim_{\alpha_i} \dots \lim_{\alpha_j} \dots \lim_{\alpha_{m-1}} \left(\prod_{\substack{k=1 \\ k \neq i, j}}^{m-1} \varphi_k(x_{\alpha_k}) \right) T(x_{\alpha_i})(x_{\alpha_j}) \lim_{\alpha_m} J_{E_m}(x_{\alpha_m})(\varphi_m) \end{aligned}$$

$$\begin{aligned}
 &= \lim_{\alpha_1} \cdots \lim_{\alpha_i} \cdots \lim_{\alpha_j} \cdots \lim_{\alpha_{m-1}} \left(\prod_{\substack{k=1 \\ k \neq i, j}}^{m-1} \varphi_k(x_{\alpha_k}) \right) T(x_{\alpha_i})(x_{\alpha_j})x_m^{**}(\varphi_m) \\
 &\vdots \\
 &= \prod_{\substack{k=i+1 \\ k \neq j}}^m x_k^{**}(\varphi_k) \lim_{\alpha_1} \cdots \lim_{\alpha_i} \left(\prod_{k=1}^{i-1} \varphi_k(x_{\alpha_k}) \right) x_j^{**}(T(x_{\alpha_i})) \\
 &= \prod_{\substack{k=i+1 \\ k \neq j}}^m x_k^{**}(\varphi_k) \lim_{\alpha_1} \cdots \lim_{\alpha_{i-1}} \left(\prod_{k=1}^{i-1} \varphi_k(x_{\alpha_k}) \right) \lim_{\alpha_i} T^*(x_j^{**})(x_{\alpha_i}) \\
 &= \prod_{\substack{k=1 \\ k \neq i, j}}^m x_k^{**}(\varphi_k) \lim_{\alpha_i} J_{E_i}(x_{\alpha_i})(T^*(x_j^{**})) = \left(\prod_{\substack{k=1 \\ k \neq i, j}}^m x_k^{**}(\varphi_k) \right) x_i^{**}(T^*(x_j^{**})) \\
 &= \left(\prod_{\substack{k=1 \\ k \neq i, j}}^m x_k^{**}(\varphi_k) \right) T^{**}(x_i^{**})(x_j^{**}).
 \end{aligned}$$

Choosing $x_k \in E_k$ so that $\varphi_k(x_k) = 1$, $i \neq k \neq j$, we get

$$A^{*[m+1]}(J_{E_1}(x_1), \dots, x_i^{**}, \dots, x_j^{**}, \dots, J_{E_m}(x_m)) = T^{**}(x_i^{**})(x_j^{**}).$$

Since $A^{*[m+1]}$ is separately order continuous by assumption, the functional $T^{**}(x_i^{**})$ is order continuous for every $x_i^{**} \in E_i^{**}$. \square

Although the next results hold, with the obvious modifications, for all Arens extensions $AR_m^\rho(A)$ of a regular m -linear operator A , to make the proofs more readable we shall restrict ourselves to the extension $A^{*[m+1]} = AR_m^\theta(A)$.

LEMMA 5.3. *Let E_1, \dots, E_m be Banach lattices, $A \in \mathcal{L}_r(E_1, \dots, E_m)$ and $i \in \{1, \dots, m\}$. If $x_1 \in E_1, \dots, x_{i-1} \in E_{i-1}$, and $x_{i+1}^{**} \in E_{i+1}^{**}, \dots, x_m^{**} \in E_m^{**}$, then the operator*

$$A^{*[m+1]}(J_{E_1}(x_1), \dots, J_{E_{i-1}}(x_{i-1}), \bullet, x_{i+1}^{**}, \dots, x_m^{**}): E_i^{**} \longrightarrow \mathbb{R}$$

is ω^* -continuous and

$$\begin{aligned}
 &A^{*[m+1]}(J_{E_1}(x_1), \dots, J_{E_{i-1}}(x_{i-1}), x_i^{**}, \dots, x_m^{**}) \\
 &= (\overline{x_i^{**}} \circ \cdots \circ \overline{x_m^{**}})(A_\theta)(x_{i-1}, \dots, x_1).
 \end{aligned}$$

Proof. Let $(x_{\alpha_i}^{**})_{\alpha_i \in \Omega_i}$ be a net in E_i^{**} such that $x_{\alpha_i}^{**} \xrightarrow{\omega^*} x_i^{**} \in E_i^{**}$. For every $x_i^* \in E_i^*$ we have $x_i^{**}(x_i^*) = \lim_{\alpha_i} x_{\alpha_i}^{**}(x_i^*)$. Given $x_1 \in E_1, \dots, x_{i-1} \in E_{i-1}$ and $x_j^{**} \in$

$E_j^{**}, j = i + 1, \dots, m,$

$$\begin{aligned}
 & A^{*[m+1]}(J_{E_1}(x_1), \dots, J_{E_{i-1}}(x_{i-1}), x_i^{**}, \dots, x_m^{**}) \\
 &= (\overline{J_{E_1}(x_1)} \circ \dots \circ \overline{J_{E_{i-1}}(x_{i-1})} \circ \overline{x_i^{**}} \circ \dots \circ \overline{x_m^{**}})(A_\theta) \\
 &= \overline{J_{E_1}(x_1)}((\overline{J_{E_2}(x_2)} \circ \dots \circ \overline{J_{E_{i-1}}(x_{i-1})} \circ \overline{x_i^{**}} \circ \dots \circ \overline{x_m^{**}})(A_\theta)) \\
 &= J_{E_1}(x_1)((\overline{J_{E_2}(x_2)} \circ \dots \circ \overline{J_{E_{i-1}}(x_{i-1})} \circ \overline{x_i^{**}} \circ \dots \circ \overline{x_m^{**}})(A_\theta)) \\
 &= (\overline{J_{E_2}(x_2)} \circ \dots \circ \overline{J_{E_{i-1}}(x_{i-1})} \circ \overline{x_i^{**}} \circ \dots \circ \overline{x_m^{**}})(A_\theta)(x_1) \\
 &= \overline{J_{E_2}(x_2)}((\overline{J_{E_3}(x_3)} \circ \dots \circ \overline{J_{E_{i-1}}(x_{i-1})} \circ \overline{x_i^{**}} \circ \dots \circ \overline{x_m^{**}})(A_\theta))(x_1) \\
 &= J_{E_2}(x_2)((\overline{J_{E_3}(x_3)} \circ \dots \circ \overline{J_{E_{i-1}}(x_{i-1})} \circ \overline{x_i^{**}} \circ \dots \circ \overline{x_m^{**}})(A_\theta))^{m-1}(x_1) \\
 &= ((\overline{J_{E_3}(x_3)} \circ \dots \circ \overline{J_{E_{i-1}}(x_{i-1})} \circ \overline{x_i^{**}} \circ \dots \circ \overline{x_m^{**}})(A_\theta))^{m-1}(x_1)(x_2) \\
 &= (\overline{J_{E_3}(x_3)} \circ \dots \circ \overline{J_{E_{i-1}}(x_{i-1})} \circ \overline{x_i^{**}} \circ \dots \circ \overline{x_m^{**}})(A_\theta)(x_2, x_1) \\
 &\vdots \\
 &= (\overline{x_i^{**}} \circ \dots \circ \overline{x_m^{**}})(A_\theta)(x_{i-1}, \dots, x_1) \\
 &= \overline{x_i^{**}}((\overline{x_{i+1}^{**}} \circ \dots \circ \overline{x_m^{**}})(A_\theta))(x_{i-1}, \dots, x_1) \\
 &= x_i^{**}(((\overline{x_{i+1}^{**}} \circ \dots \circ \overline{x_m^{**}})(A_\theta))^{m-i+1}(x_{i-1}, \dots, x_1)) \\
 &\stackrel{(\Delta)}{=} \lim_{\alpha_i} x_{\alpha_i}^{**} (((\overline{x_{i+1}^{**}} \circ \dots \circ \overline{x_m^{**}})(A_\theta))^{m-i+1}(x_{i-1}, \dots, x_1)) \\
 &= \lim_{\alpha_i} A^{*[m+1]}(J_{E_1}(x_1), \dots, J_{E_{i-1}}(x_{i-1}), x_{\alpha_i}^{**}, \dots, x_m^{**}),
 \end{aligned}$$

where, in (Δ) , we used that $((\overline{x_{i+1}^{**}} \circ \dots \circ \overline{x_m^{**}})(A_\theta))^{m-i+1}(x_{i-1}, \dots, x_1) \in E_i^*$. \square

DEFINITION 5.4. Let \mathcal{P} be a property of linear functionals on Banach lattices. We say that:

- A form $A: E_1^{**} \times \dots \times E_m^{**} \rightarrow \mathbb{R}$, where E_1, \dots, E_m are Banach lattices, has \mathcal{P} -separately if for all $j \in \{1, \dots, m\}$ and $x_i^{**} \in E_i^{**}, i = 1, \dots, m, i \neq j$, the functional

$$A_{x_1^{**}, \dots, x_{j-1}^{**}, x_{j+1}^{**}, \dots, x_m^{**}} : E_j^{**} \rightarrow \mathbb{R}, x_j^{**} \mapsto A(x_1^{**}, \dots, x_m^{**}),$$

has property \mathcal{P} .

- \mathcal{P} is an Arens property if, regardless of positive $m \geq 2$, the Banach lattices E_1, \dots, E_m and the form $A \in \mathcal{L}_r(E_1, \dots, E_m)$, the Arens extension $A^{*[m+1]}$ of A has \mathcal{P} in the first variable, in the sense that the operator $A_{x_2^{**}, \dots, x_m^{**}}^{*[m+1]} : E_1^{**} \rightarrow \mathbb{R}$ has \mathcal{P} for all $x_2^{**} \in E_2^{**}, \dots, x_m^{**} \in E_m^{**}$.

EXAMPLE 5.5. Order continuity (Theorem 4.6(c)) and ω^* -continuity [13, p.413] are Arens properties.

THEOREM 5.6. Let \mathcal{P} be an Arens property, $m \geq 2$ and E_1, \dots, E_m be Banach lattices. Suppose that:

- (a) For $j = 2, \dots, m - 1$, and $i = 1, \dots, m - j$, every regular linear operator from E_j to E_{j+i}^* is weakly compact;
- (b) For all $k = 2, \dots, m$, $x_1^{**} \in E_1^{**}$ and $T \in \mathcal{L}_r(E_1; E_k^*)$, the functional $T^{**}(x_1^{**}) \in E_k^{***}$ has property \mathcal{P} .

Then, for every form $A \in \mathcal{L}_r(E_1, \dots, E_m)$, the Arens extension $A^{*[m+1]}: E_1^{**} \times \dots \times E_m^{**} \rightarrow \mathbb{R}$ has \mathcal{P} -separately.

Proof. We shall proceed by induction on m . Given $A \in \mathcal{L}_r(E_1, E_2)$, A^{***} has property \mathcal{P} in the first variable because \mathcal{P} is an Arens property. Let us prove that, for every $x_1^{**} \in E_1^{**}$, $A^{***}(x_1^{**}, \bullet) \in E_2^{***}$ has property \mathcal{P} . Recall that $A_\theta: E_2 \times E_1 \rightarrow \mathbb{R}$ and consider the regular linear operator $T := A_\theta^1: E_1 \rightarrow E_2^*$, $T(x_2)(x_1) = A_\theta(x_1, x_2)$. For all $x_2^{**} \in E_2^{**}$ and $x_2 \in E_1$,

$$T^*(x_2^{**})(x_2) = x_2^{**}(T(x_2)) = x_2^{**}(A_\theta^1(x_2)) = (x_2^{**} \circ A_\theta^1)(x_2) = \overline{x_2^{**}}(A_\theta)(x_2),$$

that is, $T^*(x_2^{**}) = \overline{x_2^{**}}(A_\theta)$. So, for all $x_1^{**} \in E_1^{**}$, $x_2^{**} \in E_2^{**}$,

$$T^{**}(x_1^{**})(x_2^{**}) = x_1^{**}(T^*(x_2^{**})) = x_1^{**}(\overline{x_2^{**}}(A_\theta)) = (\overline{x_1^{**}} \circ \overline{x_2^{**}})(A_\theta) = A^{***}(x_1^{**}, x_2^{**}).$$

Since $T^{**}(x_1^{**})$ has property \mathcal{P} by assumption, it follows that $A^{***}(x_1^{**}, \bullet)$ has property \mathcal{P} . This shows that the result holds for $m = 2$.

Assume now that the result holds for n and let us prove it holds for $n + 1$. To do so we suppose that conditions (a) and (b) hold for $n + 1$. Let $A \in \mathcal{L}_r(E_1, \dots, E_{n+1})$ be given. For every $x_i^{**} \in E_i^{**}$, $i = 2, \dots, n + 1$, we have

$$\overline{x_{n-i+2}^{**}}: \mathcal{L}_r(E_{n-i+2}, \dots, E_1) \rightarrow \mathcal{L}_r(E_{n-i+1}, \dots, E_1), \quad \overline{x_{n-i+2}^{**}}(B) = x_{n-i+2}^{**} \circ B^i,$$

where $B^i: E_{n-i+1} \times \dots \times E_1 \rightarrow E_{n-i+2}^*$, $B^i(x_{i+1}, \dots, x_{n+1})(x_i) = B(x_i, x_{i+1}, \dots, x_{n+1})$. And for each $x_1^{**} \in E_1^{**}$, the functional $\overline{x_1^{**}}: E_1^* \rightarrow \mathbb{R}$ is given by $\overline{x_1^{**}} = x_1^{**}$. Moreover,

$$\begin{aligned} A^{*[n+2]}(x_1^{**}, \dots, x_{n+1}^{**}) &= (\overline{x_1^{**}} \circ \dots \circ \overline{x_{n+1}^{**}})(A_\theta) = (\overline{x_1^{**}} \circ \dots \circ \overline{x_n^{**}})(\overline{x_{n+1}^{**}}(A_\theta)) \\ &= (\overline{x_{n+1}^{**}}(A_\theta))^{*[n+1]}(x_1^{**}, \dots, x_n^{**}). \end{aligned}$$

Since $\overline{x_{n+1}^{**}}(A_\theta) \in \mathcal{L}_r(E_n, \dots, E_1)$, by the induction hypothesis we have that $(\overline{x_{n+1}^{**}}(A_\theta))^{*[n+1]}$ has \mathcal{P} -separately, so $A^{*[n+2]}$ has property \mathcal{P} in the first n variables. To prove that $A^{*[n+2]}$ has property \mathcal{P} in the $(n + 1)$ -th variable, let $x_i^{**} \in E_i^{**}$, $i = 1, \dots, n$, be given. Our job is to show that $A^{*[n+2]}(x_1^{**}, \dots, x_n^{**}, \bullet): E_{n+1}^{**} \rightarrow \mathbb{R}$ has property \mathcal{P} . Recall that $A_\theta: E_{n+1} \times \dots \times E_1 \rightarrow \mathbb{R}$ is given by $A_\theta(x_1, \dots, x_{n+1}) = A(x_{n+1}, \dots, x_1)$ and $A_\theta^1: E_n \times \dots \times E_1 \rightarrow E_{n+1}^*$, $A_\theta^1(x_2, \dots, x_{n+1})(x_1) = A_\theta(x_1, x_2, \dots, x_{n+1})$.

Given $x_3 \in E_{n-1}, \dots, x_{n+1} \in E_1$, consider the regular linear operator

$$A_{x_3, \dots, x_{n+1}} : E_n \longrightarrow E_{n+1}^*, \quad A_{x_3, \dots, x_{n+1}}(x_2) = A_\theta^1(x_2, x_3, \dots, x_{n+1}).$$

Given $x_{n+1}^{**} \in E_{n+1}^{**}$, take a net $(x_{\alpha_2})_{\alpha_2}$ in E_n such that $J_{E_n}(x_{\alpha_2}) \xrightarrow{\omega^*} x_n^{**}$ and apply the ω^* - ω^* -continuity of $[A_{x_3, \dots, x_{n+1}}]^{**}$ and Lemma 5.3 for $i = n + 1$ and $i = n$ to obtain

$$\begin{aligned} & [A_{x_3, \dots, x_{n+1}}]^{**}(x_n^{**})(x_{n+1}^{**}) \\ &= \lim_{\alpha_2} [A_{x_3, \dots, x_{n+1}}]^{**}(J_{E_n}(x_{\alpha_2}))(x_{n+1}^{**}) \\ &= \lim_{\alpha_2} J_{E_{n+1}^*}(A_{x_3, \dots, x_{n+1}}(x_{\alpha_2}))(x_{n+1}^{**}) \\ &= \lim_{\alpha_2} x_{n+1}^{**}(A_{x_3, \dots, x_{n+1}}(x_{\alpha_2})) = \lim_{\alpha_2} x_{n+1}^{**}(A_\theta^1(x_{\alpha_2}, x_3, \dots, x_{n+1})) \\ &= \lim_{\alpha_2} \overline{x_{n+1}^{**}}(A_\theta)(x_{\alpha_2}, x_3, \dots, x_{n+1}) \\ &= \lim_{\alpha_2} A^{*[n+2]}(J_{E_1}(x_{n+1}), \dots, J_{E_{n-1}}(x_3), J_{E_n}(x_{\alpha_2}), x_{n+1}^{**}) \\ &= A^{*[n+2]}(J_{E_1}(x_{n+1}), \dots, J_{E_{n-1}}(x_3), x_n^{**}, x_{n+1}^{**}). \end{aligned} \tag{5.1}$$

For $x_n^{**} \in E_n^{**}$ and $x_4 \in E_{n-2}, \dots, x_{n+1} \in E_1$, consider the regular linear operator $A_{x_4, \dots, x_{n+1}, x_n^{**}} : E_{n-1} \longrightarrow E_{n+1}^*$ given by

$$A_{x_4, \dots, x_{n+1}, x_n^{**}}(x_3)(x_1) = A^{*[n+2]}(J_{E_1}(x_{n+1}), \dots, J_{E_{n-1}}(x_3), x_n^{**}, J_{E_{n+1}}(x_1)).$$

On the one hand, for every $x_3 \in E_{n-1}$ the functional $[A_{x_4, \dots, x_{n+1}, x_n^{**}}(x_3)]^{**}$ is a ω^* -continuous extension of $A_{x_4, \dots, x_{n+1}, x_n^{**}}(x_3)$. On the other hand, since $A_{x_3, \dots, x_{n+1}}$ is weakly compact by assumption, for every $x_n^{**} \in E_n^{**}$ the functional $[A_{x_3, \dots, x_{n+1}}]^{**}(x_n^{**})$ is ω^* -continuous. Taking a net $(x_{\alpha_1})_{\alpha_1}$ in E_{n+1} such that $J_{E_{n+1}}(x_{\alpha_1}) \xrightarrow{\omega^*} x_{n+1}^{**}$,

$$\begin{aligned} & [A_{x_4, \dots, x_{n+1}, x_n^{**}}(x_3)]^{**}(x_{n+1}^{**}) = \lim_{\alpha_1} [A_{x_4, \dots, x_{n+1}, x_n^{**}}(x_3)]^{**}(J_{E_{n+1}}(x_{\alpha_1})) \\ &= \lim_{\alpha_1} J_{E_{n+1}}(x_{\alpha_1})(A_{x_4, \dots, x_{n+1}, x_n^{**}}(x_3)) \\ &= \lim_{\alpha_1} A_{x_4, \dots, x_{n+1}, x_n^{**}}(x_3)(x_{\alpha_1}) \\ &= \lim_{\alpha_1} A^{*[n+2]}(J_{E_1}(x_{n+1}), \dots, J_{E_{n-1}}(x_3), x_n^{**}, J_{E_{n+1}}(x_{\alpha_1})) \\ &= \lim_{\alpha_1} [A_{x_3, \dots, x_{n+1}}]^{**}(x_n^{**})(J_{E_{n+1}}(x_{\alpha_1})) = [A_{x_3, \dots, x_{n+1}}]^{**}(x_n^{**})(x_{n+1}^{**}) \\ &\stackrel{(5.1)}{=} A^{*[n+2]}(J_{E_1}(x_{n+1}), \dots, J_{E_{n-1}}(x_3), x_n^{**}, x_{n+1}^{**}). \end{aligned} \tag{5.2}$$

Take a net $(x_{\alpha_3})_{\alpha_3}$ in E_{n-1} such that $J_{E_{n-1}}(x_{\alpha_3}) \xrightarrow{\omega^*} x_{n-1}^{**}$. Using that $[A_{x_4, \dots, x_{n+1}, x_n^{**}}]^{**}$ is ω^* - ω^* -continuous and calling on Lemma 5.3 for $i = n - 1$,

for each $x_{n+1}^{**} \in E_{n+1}^{**}$ we have

$$\begin{aligned}
 [A_{x_4, \dots, x_{n+1}, x_n^{**}}]^{**}(x_{n-1}^{**})(x_{n+1}^{**}) &= \lim_{\alpha_3} [A_{x_4, \dots, x_{n+1}, x_n^{**}}]^{**}(J_{E_{n-1}}(x_{\alpha_3}))(x_{n+1}^{**}) \\
 &= \lim_{\alpha_3} J_{E_{n+1}^*}(A_{x_4, \dots, x_{n+1}, x_n^{**}}(x_{\alpha_3}))(x_{n+1}^{**}) \\
 &= \lim_{\alpha_3} x_{n+1}^{**}(A_{x_4, \dots, x_{n+1}, x_n^{**}}(x_{\alpha_3})) = \lim_{\alpha_3} [A_{x_4, \dots, x_{n+1}, x_n^{**}}(x_{\alpha_3})]^{**}(x_{n+1}^{**}) \\
 &\stackrel{(5.2)}{=} \lim_{\alpha_3} A^{*[n+2]}(J_{E_1}(x_{n+1}), \dots, J_{E_{n-1}}(x_{\alpha_3}), x_n^{**}, x_{n+1}^{**}) \\
 &= A^{*[n+2]}(J_{E_1}(x_{n+1}), \dots, J_{E_{n-2}}(x_4), x_{n-1}^{**}, x_n^{**}, x_{n+1}^{**}). \tag{5.3}
 \end{aligned}$$

For $x_{n-1}^{**} \in E_{n-1}^{**}$, $x_n^{**} \in E_n^{**}$ and $x_5 \in E_{n-3}, \dots, x_{n+1} \in E_1$, consider the regular linear operator $A_{x_5, \dots, x_{n+1}, x_{n-1}^{**}, x_n^{**}} : E_{n-2} \rightarrow E_{n+1}^*$ given by

$$\begin{aligned}
 A_{x_5, \dots, x_{n+1}, x_{n-1}^{**}, x_n^{**}}(x_4)(x_1) \\
 = A^{*[n+2]}(J_{E_1}(x_{n+1}), \dots, J_{E_{n-2}}(x_4), x_{n-1}^{**}, x_n^{**}, J_{E_{n+1}}(x_1)).
 \end{aligned}$$

For every $x_4 \in E_{n-2}$ the functional $[A_{x_5, \dots, x_{n+1}, x_{n-1}^{**}, x_n^{**}}(x_4)]^{**}$ is a ω^* -continuous extension of $A_{x_5, \dots, x_{n+1}, x_{n-1}^{**}, x_n^{**}}(x_4)$. On the other hand, since $A_{x_4, \dots, x_{n+1}, x_n^{**}}$ is weakly compact by assumption, for every $x_{n-1}^{**} \in E_{n-1}^{**}$ the functional $[A_{x_4, \dots, x_{n+1}, x_n^{**}}]^{**}(x_{n-1}^{**})$ is ω^* -continuous on E_{n-2} . So, repeating the procedure using (5.3) we get

$$\begin{aligned}
 [A_{x_5, \dots, x_{n+1}, x_{n-1}^{**}, x_n^{**}}(x_4)]^{**}(x_{n+1}^{**}) \\
 = A^{*[n+2]}(J_{E_1}(x_{n+1}), \dots, J_{E_{n-2}}(x_4), x_{n-1}^{**}, x_n^{**}, x_{n+1}^{**}). \tag{5.4}
 \end{aligned}$$

Since the operator $[A_{x_5, \dots, x_{n+1}, x_{n-1}^{**}, x_n^{**}}]^{**}$ is ω^* - ω^* -continuous, for every $x_{n+1}^{**} \in E_{n+1}^{**}$, taking a net $(x_{\alpha_4})_{\alpha_4}$ in E_{n-2} such that $J_{E_{n-2}}(x_{\alpha_4}) \xrightarrow{\omega^*} x_{n-2}^{**}$, using Lemma 5.3 for $i = n - 2$ and (5.4) we have

$$\begin{aligned}
 [A_{x_5, \dots, x_{n+1}, x_{n-1}^{**}, x_n^{**}}]^{**}(x_{n-2}^{**})(x_{n+1}^{**}) \\
 = A^{*[n+2]}(J_{E_1}(x_{n+1}), \dots, J_{E_{n-3}}(x_5), x_{n-2}^{**}, x_{n-1}^{**}, x_n^{**}, x_{n+1}^{**}).
 \end{aligned}$$

Repeating the procedure $(n - 3)$ times, we end up with

$$[A_{x_{n+1}, x_3^{**}, \dots, x_n^{**}}]^{**}(x_2^{**})(x_{n+1}^{**}) = A^{*[n+2]}(J_{E_1}(x_{n+1}), x_2^{**}, \dots, x_{n+1}^{**}) \tag{5.5}$$

for every $x_{n+1}^{**} \in E_{n+1}^{**}$, where, for each $x_{n+1} \in E_1$ and $x_i^{**} \in E_i^{**}$, $i = 3, \dots, n$, $A_{x_{n+1}, x_3^{**}, \dots, x_n^{**}} : E_2 \rightarrow E_{n+1}^*$ is the regular linear operator given by

$$A_{x_{n+1}, x_3^{**}, \dots, x_n^{**}}(x_n)(x_1) = A^{*[n+2]}(J_{E_1}(x_{n+1}), J_{E_2}(x_n), x_3^{**}, \dots, x_n^{**}, J_{E_{n+1}}(x_1)).$$

Finally, given $x_i^{**} \in E_i^{**}$, $i = 2, \dots, n$, the regular linear operator $A_{x_2^{**}, \dots, x_n^{**}} : E_1 \rightarrow E_{n+1}^*$ defined by

$$A_{x_2^{**}, \dots, x_n^{**}}(x_{n+1})(x_1) = A^{*[n+2]}(J_{E_1}(x_{n+1}), x_2^{**}, \dots, x_n^{**}, J_{E_{n+1}}(x_1)),$$

is weakly compact by condition (ii) for $n + 1$. So, for every $x_2^{**} \in E_2^{**}$, $[A_{x_{n+1}, x_3^{**}, \dots, x_n^{**}}]^{**}(x_2^{**})$ is ω^* -continuous, therefore

$$\begin{aligned} [A_{x_2^{**}, \dots, x_n^{**}}(x_{n+1})]^{**}(x_{n+1}^{**}) &= \lim_{\alpha_1} [A_{x_2^{**}, \dots, x_n^{**}}(x_{n+1})]^{**}(J_{E_{n+1}}(x_{\alpha_1})) \\ &= \lim_{\alpha_1} J_{E_{n+1}}(x_{\alpha_1})(A_{x_2^{**}, \dots, x_n^{**}}(x_{n+1})) = \lim_{\alpha_1} A_{x_2^{**}, \dots, x_n^{**}}(x_{n+1})(x_{\alpha_1}) \\ &= \lim_{\alpha_1} A^{*[n+2]}(J_{E_1}(x_{n+1}), x_2^{**}, \dots, x_n^{**}, J_{E_{n+1}}(x_{\alpha_1})) \\ &= \lim_{\alpha_1} [A_{x_{n+1}, x_3^{**}, \dots, x_n^{**}}]^{**}(x_2^{**})(J_{E_{n+1}}(x_{\alpha_1})) = [A_{x_{n+1}, x_3^{**}, \dots, x_n^{**}}]^{**}(x_2^{**})(x_{n+1}^{**}) \\ &\stackrel{(5.5)}{=} A^{*[n+2]}(J_{E_1}(x_{n+1}), x_2^{**}, \dots, x_{n+1}^{**}). \end{aligned} \tag{5.6}$$

For the final time, taking a net $(x_{\alpha_{n+1}})_{\alpha_{n+1}}$ in E_1 such that $J_{E_1}(x_{\alpha_{n+1}}) \xrightarrow{\omega^*} x_1^{**}$, the ω^* - ω^* continuity of $[A_{x_2^{**}, \dots, x_n^{**}}]^{**}$ and Lemma 5.3 for $i = 1$ give, for every $x_{n+1}^{**} \in E_{n+1}^{**}$,

$$\begin{aligned} [A_{x_2^{**}, \dots, x_n^{**}}]^{**}(x_1^{**})(x_{n+1}^{**}) &= \lim_{\alpha_{n+1}} [A_{x_2^{**}, \dots, x_n^{**}}]^{**}(J_{E_1}(x_{\alpha_{n+1}}))(x_{n+1}^{**}) \\ &= \lim_{\alpha_{n+1}} J_{E_{n+1}^*}(A_{x_2^{**}, \dots, x_n^{**}}(x_{\alpha_{n+1}}))(x_{n+1}^{**}) \\ &= \lim_{\alpha_{n+1}} x_{n+1}^{**}(A_{x_2^{**}, \dots, x_n^{**}}(x_{\alpha_{n+1}})) = \lim_{\alpha_{n+1}} [A_{x_2^{**}, \dots, x_n^{**}}(x_{\alpha_{n+1}})]^{**}(x_{n+1}^{**}) \\ &\stackrel{(5.6)}{=} \lim_{\alpha_{n+1}} A^{*[n+2]}(J_{E_1}(x_{\alpha_{n+1}}), x_2^{**}, \dots, x_{n+1}^{**}) = A^{*[n+2]}(x_1^{**}, \dots, x_{n+1}^{**}). \end{aligned}$$

This proves that $[A_{x_2^{**}, \dots, x_n^{**}}]^{**}(x_1^{**}) = A^{*[n+2]}(x_1^{**}, \dots, x_n^{**}, \bullet)$. By condition (b) for $n + 1$ we know that $[A_{x_2^{**}, \dots, x_n^{**}}]^{**}(x_1^{**})$ has property \mathcal{P} , so $A^{*[n+2]}(x_1^{**}, \dots, x_n^{**}, \bullet)$ has property \mathcal{P} , which completes the proof. \square

Theorem 5.6 gives sufficient conditions for Arens extensions of regular multilinear forms to be separately order continuous on the product of the whole of the biduals. Now we derive the case of vector-valued regular multilinear operators.

THEOREM 5.7. *Let $m \geq 2$ and E_1, \dots, E_m be Banach lattices such that:*

- (a) *For $j = 2, \dots, m - 1$, and $i = 1, \dots, m - j$, every regular linear operator from E_j to E_{j+i}^* is weakly compact;*
- (b) *For all $k = 2, \dots, m$, $x_1^{**} \in E_1^{**}$ and $T \in \mathcal{L}_r(E_1; E_k^*)$, the functional $T^{**}(x_1^{**})$ is order continuous on E_k^{***} .*

Then, for every Banach lattice F and any $A \in \mathcal{L}_r(E_1, \dots, E_m; F)$, the Arens extension $A^{[m+1]}$ is separately order continuous on $E_1^{**} \times \dots \times E_m^{**}$.*

Proof. Let $A \in \mathcal{L}_r(E_1, \dots, E_m; F)$ and $y^* \in F^*$ be given. Since $y^* \circ A \in \mathcal{L}_r(E_1, \dots, E_m)$ and order continuity is an Arens property, by Theorem 5.6 the extension $(y^* \circ A)^{*[m+1]}$ is separately order continuous. For all $x_i^{**} \in E_i^{**}$, $i =$

1, \dots, m,

$$A^{*[m+1]}(x_1^{**}, \dots, x_m^{**})(y^*) = (y^* \circ A)^{*[m+1]}(x_1^{**}, \dots, x_m^{**}).$$

For each $j \in \{1, \dots, m\}$ let $x_j^{**} \in E_j^{**}$ and let $(x_{\alpha_j}^{**})_{\alpha_j \in \Omega_j}$ be a net E_j^{**} such that $x_{\alpha_j}^{**} \xrightarrow{o} 0$. There exists a net $(y_{\alpha_j}^{**})_{\alpha_j \in \Omega_j}$ in E_j^{**} and $\alpha_{j_0} \in \Omega_j$ so that $y_{\alpha_j}^{**} \downarrow 0$ and $|x_{\alpha_j}^{**}| \leq y_{\alpha_j}^{**}$ for every $\alpha_j \geq \alpha_{j_0}$. Without loss of generality, assume that A and y^* are positive. Since $(y^* \circ A)^{*[m+1]}(|x_1^{**}|, \dots, \bullet, \dots, |x_m^{**}|): E_j^{**} \rightarrow \mathbb{R}$ is positive and order continuous,

$$\begin{aligned} &A^{*[m+1]}(|x_1^{**}|, \dots, y_{\alpha_j}^{**}, \dots, |x_m^{**}|)(y^*) \\ &= (y^* \circ A)^{*[m+1]}(|x_1^{**}|, \dots, \bullet, \dots, |x_m^{**}|)(y_{\alpha_j}^{**}) \downarrow 0. \end{aligned}$$

It follows that $A^{*[m+1]}(|x_1^{**}|, \dots, y_{\alpha_j}^{**}, \dots, |x_m^{**}|) \downarrow 0$ [2, Theorem 1.18] and, for every $\alpha_j \geq \alpha_{j_0}$,

$$\begin{aligned} |A^{*[m+1]}(x_1^{**}, \dots, x_{\alpha_j}^{**}, \dots, x_m^{**})| &\leq A^{*[m+1]}(|x_1^{**}|, \dots, |x_{\alpha_j}^{**}|, \dots, |x_m^{**}|) \\ &\leq A^{*[m+1]}(|x_1^{**}|, \dots, y_{\alpha_j}^{**}, \dots, |x_m^{**}|) \downarrow 0. \end{aligned}$$

This shows that $A^{*[m+1]}(x_1^{**}, \dots, x_{\alpha_j}^{**}, \dots, x_m^{**}) \xrightarrow{o} 0$ and proves that $A^{*[m+1]}$ is separately order continuous. \square

Recall that a Banach lattice E is said to be a *Kantorovich-Banach space* (or briefly a *KB-space*) if increasing positive bounded sequences of E are norm convergent.

EXAMPLE 5.8. As to condition (a) above, we have the following examples between nonreflexive Banach lattices:

- (a) Every operator from c_0 to $c_0^* = \ell_1$ is compact, hence weakly compact (this is Pitt’s theorem).
- (b) Every operator from $C(K)$, where K is a compact Hausdorff space, to a KB-space is weakly compact. Just recall that KB-spaces do not contain a copy of c_0 [2, Theorem 4.60] and apply [18, Theorem 5].
- (c) Since any AM-space with order unity is order isometric to a $C(K)$ -space [2, Theorem 4.29], from (b) it follows that every operator from an AM-space with order unity to a KB-space is weakly compact. And since the dual of an AM-space is a KB-space, every operator from an AM-space with order unity to its dual is weakly compact. In particular, every operator from ℓ_∞ to ℓ_∞^* is weakly compact.

COROLLARY 5.9. Let $m \geq 2$ and E_1, \dots, E_m, F be Banach lattices such that every regular operator from E_j to E_{j+i}^* is weakly compact, $j = 2, \dots, m - 1, i = 1, \dots, m - j$. If E_1^* has order continuous norm, then the Arens extension $A^{*[m+1]}$ of any operator $A \in \mathcal{L}_r(E_1, \dots, E_m; F)$ is separately order continuous on $E_1^{**} \times \dots \times E_m^{**}$.

Proof. Condition (a) of Theorem 5.7 is given by assumption. For $x_1^{**} \in E_1^{**}$ and $T \in \mathcal{L}_r(E_1; E_k^*)$, x_1^{**} is order continuous because the norm of E_1^* is order continuous [17, Theorem 2.4.2]. Since T^* is order continuous [2, Theorem 1.73], $T^{**}(x_1^{**}) = x_1^{**} \circ T^*$ is order continuous as well, so condition (b) is fulfilled too. \square

Recall that a Banach space E is *Arens regular* if every bounded linear operator from E to E^* is weakly compact (see, e.g., [13]). The Banach lattices c_0 , ℓ_∞ and $C(K)$, where K is a compact Hausdorff space, in particular AM-spaces with order unit, are Arens regular (cf. Example 5.8).

COROLLARY 5.10. *Let E be an Arens regular Banach lattice. Then, for every Banach lattice F , the Arens extension $A^{*[m+1]}$ of any regular m -linear operator $A: E^m \rightarrow F$ is separately order continuous on $(E^{**})^m$.*

Proof. The Arens regularity of E immediately gives condition (a) of Theorem 5.7 and implies that, for every $T \in \mathcal{L}_r(E; E^*)$, $T^{**}(E^{**}) \subseteq J_{E^*}(E^*) \subseteq (E^{**})_n^*$, which gives condition (b). \square

We conclude this paper with one more result on order continuity of Arens extensions of homogeneous polynomials. For a polynomial $P \in \mathcal{P}_r(^m E; F)$, we write $P^{*[m+1]} := AR_m^\theta(P)$. Recall that P is *orthogonally additive* if $P(x + y) = P(x) + P(y)$ whenever x and y are disjoint. The literature on orthogonally additive polynomials is extensive.

A linear operator $u: E \rightarrow E^*$ is *symmetric* if $u(x)(y) = u(y)(x)$ for all $x, y \in E$. A Banach space E is *symmetrically Arens regular* if every symmetric operator from E to E^* is weakly compact. Of course, Arens regular spaces are symmetrically Arens regular, but there are symmetrically Arens regular spaces that fail to be Arens regular [15].

PROPOSITION 5.11. *Let E, F be Banach lattices and $P \in \mathcal{P}_r(^m E; F)$. If either P is orthogonally additive and $F = \mathbb{R}$ or E is symmetrically Arens regular, then the Arens extension $P^{*[m+1]}: E^{**} \rightarrow F^{**}$ of P is order continuous on E^{**} .*

Proof. Assume first that P is orthogonally additive and $F = \mathbb{R}$. By Proposition 4.8 we know that $P^{*[m+1]}$ is order continuous at the origin on E^{**} , therefore it is order continuous at every point of E^{**} by [8, Proposition 8].

Suppose now that E is symmetrically Arens regular. It is clear that we can assume that P is positive. We know that $(\check{P})^{*[m+1]}$ is order continuous in the first variable on E^{**} (Theorem 4.6) and positive because \check{P} is positive. In order to check that it is symmetric, let $\rho \in S_m$ be given. For every $\varphi \in F^*$, since E is symmetrically Arens regular and $\varphi \circ \check{P}$ is symmetric, by [5, Theorem 8.3] (or [11, Corollary 6])

we know that $AR_m^\theta(\varphi \circ \check{P})$ is symmetric as well. So, for $x_1^{**}, \dots, x_m^{**} \in E^{**}$,

$$\begin{aligned} (\check{P})^{*[m+1]}(x_1^{**}, \dots, x_m^{**})(\varphi) &= AR_m^\theta(\check{P})(x_1^{**}, \dots, x_m^{**})(\varphi) \\ &= AR_m^\theta(\varphi \circ \check{P})(x_1^{**}, \dots, x_m^{**}) \\ &= AR_m^\theta(\varphi \circ \check{P})(x_{\rho(1)}^{**}, \dots, x_{\rho(m)}^{**}) \\ &= AR_m^\theta(\check{P})(x_{\rho(1)}^{**}, \dots, x_{\rho(m)}^{**})(\varphi) \\ &= (\check{P})^{*[m+1]}(x_{\rho(1)}^{**}, \dots, x_{\rho(m)}^{**})(\varphi), \end{aligned}$$

proving that $(\check{P})^{*[m+1]}$ is symmetric. The order continuity in the first variable and the symmetry yield that the positive m -linear operator $(\check{P})^{*[m+1]}$ is separately order continuous on $(E^{**})^m$. By [10, Lemma 2.6] it follows that $(\check{P})^{*[m+1]}$ is jointly order continuous on $(E^{**})^m$. So, if $x_\alpha^{**} \xrightarrow{o} x^{**}$ in E^{**} , then

$$\begin{aligned} P^{*[m+1]}(x_\alpha^{**}) &= (\check{P})^{*[m+1]}(x_\alpha^{**}, \dots, x_\alpha^{**}) \xrightarrow{o} (\check{P})^{*[m+1]}(x^{**}, \dots, x^{**}) \\ &= P^{*[m+1]}(x^{**}). \end{aligned}$$

□

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