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NOTE ON AUTOMORPHISMS OF A FREE ABELIAN GROUP

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Let F be a free group. Denote by $\bar{F}=F/F'$ the quotient group by the commutator subgroup which is a free abelian group. The fact that the natural map from $\mathrm{Aut}(F)$ into $\mathrm{Aut}(\bar{F})$ is an epimorphism, in case when F is finitely generated, was known as a consequence of the theory of Nielsen transformations ([2]) Proposition 4.4 and [3] Corollary 3.5.1).

This fact was proved recently by R. G. Swan in [1] for any free group F. We give here a much simpler proof of the theroem for F countably generated with the use of Nielsen transformations. The general case follows from the countable case in the same way as in [1] (except for misprints).

Let x_i, $i=1,2,\ldots$ generate F freely, then the cosets $\bar{x}_i=x_iF'$, $i=1,2,\ldots$ constitute an abelian base in $\bar{F}$.

THEOREM. *Every automorphism of $\bar{F}$ is induced by some automorphism of F.*

Proof. Let $\bar{\alpha}$ be an automorphism of $\bar{F}$. Denote $\bar{\alpha}(\bar{x}_i)=\bar{a}_i$, $i=1,2,\ldots$. The cosets $\bar{a}_i$, $i=1,2,\ldots$ give us another abelian base in $\bar{F}$. To show that $\bar{\alpha}$ is induced by some automorphism α of F we shall find a set of representatives $a_i\in\bar{a}_i$, $i=1,2,\ldots$ which freely generate F. Denote

(1) $$\bar{X}_n=gp(\bar{x}_1,\ldots,\bar{x}_n),$$

(2) $$\bar{A}_n=gp(\bar{a}_1,\ldots,\bar{a}_n).$$

Let ℓ_i, L_i, $(\ell_1=1)$ be successively defined as the minimal numbers satisfying

$$\bar{X}_{\ell_1}\subset\bar{A}_{L_1}\subset\bar{X}_{\ell_2}\subset\cdots\subset\bar{X}_{\ell_k}\subset\bar{A}_{L_k}\subset\bar{X}_{\ell_{k+1}}\subset\cdots$$

We complete each set $\{\bar{x}_1,\ldots,\bar{x}_{\ell_k}\}$ to an abelian base in $\bar{A}_{L_k}$ and each set $\{\bar{a}_i,\ldots,\bar{a}_{L_k}\}$ to an abelian base in $\bar{X}_{\ell_{k+1}}$, $k\geq 1$. Let these bases be fixed. Then

(3) $$\bar{A}_{L_k}=gp(\bar{x}_1,\ldots,\bar{x}_{\ell_k},\bar{u}_{\ell_k+1},\ldots,\bar{u}_{L_k}),$$

(4) $$\bar{X}_{\ell_{k+1}}=gp(\bar{a}_1,\ldots,\bar{a}_{L_k},\bar{v}_{L_k+1},\ldots,\bar{v}_{\ell_{k+1}}),$$

(5) $$\bar{X}_{\ell_{k+1}}=gp(\bar{x}_1,\ldots,\bar{x}_{\ell_k},\bar{u}_{\ell_k+1},\ldots,\bar{u}_{L_k},\bar{v}_{L_k+1},\ldots,\bar{v}_{\ell_{k+1}}).$$

Consider that automorphism of $\bar{X}_{\ell_{k+1}}$ which maps $\bar{x}_i$, $i=1,2,\ldots,\ell_{k+1}$ into the successive generators in (5). By [3] Corollary 3.5.1 there exists a Nielsen

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transformation N which induces this automorphism and is identical for elements with indices $i \le \ell_k$. Then

$$N(x_1, \ldots, x_{\ell_{k+1}}) = (x_1, \ldots, x_{\ell_k}, u_{\ell_k+1}, \ldots, u_{L_k}, v_{L_k+1}, \ldots, v_{\ell_{k+1}}) \tag{6}$$

for some representatives $u_i \in \bar{u}_i$, $\ell_k + 1 \le i \le L_k$, and $v_i \in \bar{v}_i$, $L_k + 1 \le i \le \ell_{k+1}$. Now (3) and (6) suggest the following definition of inverse-image subgroups for $\bar{X}_{\ell_k}$ and $\bar{A}_{L_k}$, $k \ge 1$ in F

$$X_{\ell_k} = gp(x_1, \ldots, x_{\ell_k}), \tag{7}$$

$$A_{L_k} = gp(x_1, \ldots, x_{\ell_k}, u_{\ell_k+1}, \ldots, u_{L_k}). \tag{8}$$

We then have

$$X_{\ell_1} \subset A_{L_1} \subset X_{\ell_2} \subset \cdots \subset X_{\ell_k} \subset A_{L_k} \subset X_{\ell_{k+1}} \subset \cdots. \tag{9}$$

To prove the Theorem we will find a set of representatives $a_i \in \bar{a}_i$, $i = 1, 2, \ldots$ such that its subset $\{a_i, i \le L_k\}$ freely generates A_{L_k}, $k \ge 1$. We proceed by induction on k. For $k = 1$ we have by (3) $\bar{A}_{L_1} = gp(\bar{a}_1, \ldots, \bar{a}_{L_1}) = gp(\bar{x}_1, \bar{u}_2, \ldots, \bar{u}_{L_1})$. Let N_1 be a Nielsen transformation such that

$$N_1(\bar{x}_1, \bar{u}_2, \ldots, \bar{u}_{L_1}) = (\bar{a}_1, \ldots, \bar{a}_{L_1}),$$

then we apply N_1 to generators in A_{L_1} given in (8) to define $N(x_1, u_2, \ldots, u_{L_1}) = (a_1, \ldots, a_{L_1})$.

Suppose now that a free base $\{a_1, \ldots, a_{L_{k-1}}\}$ for $A_{L_{k-1}}$ has been chosen as required. Now from (8), (6) for ℓ_k, and the inductive hypothesis for $A_{L_{k-1}}$

$$A_{L_k} = gp(a_1, \ldots, a_{L_{k-1}}, v_{L_{k-1}+1}, \ldots, v_{\ell_k}, u_{\ell_k+1}, \ldots, u_{L_k}). \tag{10}$$

Consider $\bar{A}_{L_k}$, then it follows from (10) and (2) that there exists a Nielsen transformation N_k, identical for elements with indices $i \le L_{k-1}$ such that

$$N_k(\bar{a}_1, \ldots, \bar{a}_{L_{k-1}}, \bar{v}_{L_{k-1}+1}, \ldots, \bar{v}_{\ell_k}, \bar{u}_{\ell_k+1}, \ldots, \bar{u}_{L_k}) = (\bar{a}_1, \ldots, \bar{a}_{L_k}).$$

Using (10) we then define the required abelian base in A_{L_k} of representatives $a_i \in \bar{a}_i$, $i \le L_k$ by

$$N_k(a_1, \ldots, a_{L_{k-1}}, v_{L_{k-1}+1}, \ldots, v_{\ell_k}, u_{\ell_k+1}, \ldots, u_{L_k}) = (a_1, \ldots, a_{L_k}).$$

Thus we have defined the set $\{a_i, i = 1, 2, \ldots\}$ of representatives in $\bar{a}_i$, $i = 1, 2, \ldots$ which by (9) generate F freely. Hence the mapping $\alpha : x_i \to a_i$, $i = 1, 2, \ldots$ defines the required automorphism on F. The Theorem is proved.

References

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