COLLINEATIONS OF AFFINE MOULTON PLANES

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1. Introduction. In a recent article on Moulton planes **(8)**, I have generalized the non-Desarguesian planes introduced by F. R. Moulton **(6)** and Pickert **(7**, pp. 93–94**)**. Let $F = \{0, 1; a, b, \ldots, x, y, \ldots\}$ denote a given field, having P as a multiplicative subgroup of index 2. Define $x > 0$ or $x < 0$ according as x lies in P or in the other coset of non-zero elements. The function ϕ is order-preserving, and can be assumed to satisfy $\phi(0) = 0$, $\phi(1) = 1$. For any $n < 0$, the maps $x \to \phi(x)$ and $x \to \phi(x) - nx$ both carry F *onto* itself. The elements of F form a Cartesian group, $\{+, \circ\}$ being defined so that

$$a \circ b = \begin{cases} ab & \text{if} \quad b \geqslant 0, \\ \phi(a)\cdot b & \phantom{\text{if}} \quad b < 0. \end{cases}$$

A Moulton plane $M_\phi(F)$ consists of points which are the ordered pairs (x, y), and lines, given by $\{x = c\}$ and $\{y = b + m \circ x\}$, for all choices of x, y, c, b, m in F. The basic geometry of Moulton planes has been discussed in **(8)**.

The purpose of the present paper is to determine the collineations from one Moulton plane onto a second $M_\psi(K)$, with F as above and $K = \{0, 1; a', b', \ldots, x', y', \ldots\}$. The functions ϕ and ψ will be assumed non-trivial, so that $M_\phi(F)$ and $M_\psi(K)$ must be non-Desarguesian (**8**, Theorem 4**)**. The collineations of $M_\phi(F)$ onto $M_\psi(K)$ are classical when $M_\phi(F)$ is Desarguesian. If $\phi = \psi$ and $F = K$, these collineations form a group on $M_\phi(F)$. Since Moulton planes are affine in their natural representation, the present discussion will be confined to affine collineations, i.e. to those which map the ideal line of $M_\phi(F)$ onto the ideal line of $M_\psi(K)$.

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2. Preliminary Lemmas. The x-axis, y-axis, ideal line, ideal point on the x-axis, and ideal point on the y-axis of M_ϕ(F) are denoted respectively by $\{y = 0\}$, $\{x = 0\}$, l_∞, X_∞, and Y_∞. The corresponding elements of $M_\psi(K)$ are denoted respectively by $\{y' = 0\}$, $\{x' = 0\}$, l_∞', X_∞', and Y_∞'. A succession of lemmas will lead to the main theorems.

LEMMA 1. *If ϕ is non-trivial, every affine collineation of $M_\phi(F)$ onto $M_\psi(K)$ maps Y_∞ onto Y_∞'.*

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Proof. Since $\{+, \circ\}$ on either $M_\phi(F)$ or $M_\psi(K)$ defines a Cartesian group, $M_\phi(F)$ is (Y_∞, l_∞)-transitive and $M_\psi(K)$ is (Y_∞', l_∞')-transitive. A given affine collineation maps l_∞ onto l_∞'. If it mapped Y_∞ onto $Z_\infty' \neq Y_\infty'$, $M_\psi(K)$ would be (Q', l_∞')-Desarguesian for distinct choices of Q', implying the Little Theorem from l_∞'. By **(8**, Theorem 4**)**, the full theorem of Desargues would follow, and ψ would be trivial, a contradiction.

Remark. This lemma is false if collineations of *projective* Moulton planes are allowed. J. C. D. Spencer has shown **(9)** that both the original Moulton plane **(6)** and Pickert's generalizations of it **(7**, pp. 93–94**)** support collineations displacing Y_∞. In a forthcoming paper I shall prove that, up to isomorphism, these are the *only* such planes among the Moulton type here considered.

COROLLARY 1. *If ϕ is non-trivial, every affine collineation on $M_\phi(F)$ maps Y_∞ onto itself.*

LEMMA 2. *Every Moulton plane $M_\phi(F)$ supports non-trivial* $(0, 0)^- l_\infty$ *elations:* $(x, y) \rightarrow (px, py)$ *for arbitrary* $p > 0$; *and non-trivial translations:* $(x, y) \rightarrow (x, y + a)$ *for arbitrary* a, l_∞ *being pointwise fixed under both transformations.*

If $n < 0$, $(x, y) \rightarrow (nx, ny)$ *defines a collineation on* $M_\phi(F)$ *if and only if* ϕ^2 *is the identity. Such a collineation—if it exists—interchanges the ideal points of slopes* m *and* $\phi(m)$, *for all* m.

Proof. Let $p > 0$. Substitution of $x' = px$, $y' = py$ in $y' = b + m \circ x'$ gives $\{y = (b/p) + m \circ x\}$ as the inverse image of $\{y' = b + m \circ x'\}$, each ideal point of slope m being fixed. The point Y_∞ is fixed, by Lemma 1. Since x' depends only on x, the lines through Y_∞ are permuted among themselves.

Let $n < 0$. Substitution of $x' = nx$, $y' = ny$ in $y' = b + m \circ x'$ gives

$$y = \begin{cases} (b/n) + \phi(m)\cdot x & \text{if} \quad x > 0, \\ (b/n) + mx & \phantom{\text{if}} \quad x < 0. \end{cases}$$

The necessary and sufficient condition that $y' = b + m \circ x'$ be equivalent to $y = (b/n) + m' \circ x$, for appropriate m' depending only on m, is that $\phi(m) = m'$ and $m = \phi(m')$. Thus $\phi^2(m) = m$ for all m.

The existence of the translation $x' = x$, $y' = y + a$ for any $a \in F$ (l_∞ being pointwise fixed) follows at once from **(3**. Theorem 3**)**.

LEMMA 3. *The map* $\lambda : (x, y) \rightarrow (x, y + a \circ x + d)$, *with given* a, d, *is a collineation on* $M_\phi(F)$ *if and only if*

$$\phi(m) - \phi(a) = \phi(m - a) \textit{ for all } m.$$

This collineation can then be extended to l_∞ *by sending* $P_\infty(r)$ (the ideal point of slope r) *to* $P_\infty(r + a)$.

Proof. Since the first co-ordinate of every point is kept unchanged, each ordinary line through Y_∞ is mapped onto itself.

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Substitution of $x' = x$, $y' = y + a \circ x + d$ in an equation $y' = b + m \circ x'$ yields

$$y = (b - d) + (m \circ x - a \circ x)$$
$$= \begin{cases} (b - d) + (m - a)x & \text{if } x \geqslant 0, \\ (b - d) + (\phi(m) - \phi(a))x & \text{if } x < 0. \end{cases}$$

Hence $y = (b - d) + r \circ x$ for some r (viz., $r = m - a$) and all x if and only if $\phi(m) - \phi(a) = \phi(m - a)$ for all m. In that case $m = r + a$. This completes the proof.

COROLLARY 2. *If ϕ is additive, the map $(x, y) \to (x, y + a \circ x + d)$ is a collineation on $M_\phi(F)$ for arbitrary a, d. It is extended to the ideal line by sending $P_\infty(r)$ to $P_\infty(r + a)$.*

LEMMA 4. *Let γ be an affine collineation from $M_\phi(F)$ onto $M_\psi(K)$ mapping each ordinary point (x, y) of $M_\phi(F)$ onto (x', y') of $M_\psi(K)$. Assume that γ sends $(0, 0)$ onto $(x_0', 0)$. Then $x' = \mu(x)$, and $y' = e' \circ \mu(x) - e' \circ x_0' + \sigma(y)$, for all x, y. Here σ is additive and $\{y' = e' \circ x' - e' \circ x_0'\}$ is the image of $\{y = 0\}$ under γ.*

Proof. By Lemma 1, $\gamma(Y_\infty) = Y_\infty'$. Hence x' is independent of y, say $x' = \mu(x)$. Put $y' = \eta(x, y)$. For any $b \in F$, the line $\{y = b\}$ is mapped by γ onto a line $\{y' = e' \circ x' + b'\}$ for some e', b' in K, b' depending only on b. Substitution in the equation $y' = e' \circ x' + b'$ gives $\eta(x, b) = e' \circ x' + b'$ for all x; in particular $\eta(0, b) = e' \circ x_0' + b'$, for all $b \in F$. Putting $\eta(0, b) = \sigma(b)$, we obtain

$$\eta(x, b) = e' \circ \mu(x) - e' \circ x_0' + \sigma(b). \tag{1}$$

Since $\eta(0, 0) = 0$, (1) implies that $\sigma(0) = 0$, $\eta(x, 0) = e' \circ x' - e' \circ x_0'$, and

$$\eta(x, b) = \eta(x, 0) + \eta(0, b), \qquad \text{for all } x, b \in F. \tag{2}$$

This shows, incidentally, that γ maps $\{y = 0\}$ onto $\{y' = e' \circ x' - e' \circ x_0'\}$.

It remains to prove that σ is additive. For any b, r, and non-zero $x \in F$, the line $(0, b) \cup (x, r \circ x + b)$ is mapped by γ onto

$$(x_0', \eta(0, b)) \cup (\mu(x), \eta(x, r \circ x + b)) = \{y' = (m' \circ x' - m' \circ x_0') + \eta(0, b)\}.$$

Thus $\eta(x, r \circ x + b) = (m' \circ \mu(x) - m' \circ x_0') + \eta(0, b)$, and if $b = 0$ $\eta(x, r \circ x) = m' \circ \mu(x) - m' \circ x_0'$. Hence $\eta(x, r \circ x + b) = \eta(x, r \circ x) + \eta(0, b)$, for any $x \neq 0$.

Any non-zero $a \in F$ is equal to $r \circ x$ for some $r \neq 0$ and some $x \neq 0$. Thus

$$\eta(x, a + b) = \eta(x, a) + \eta(0, b) \tag{3}$$

for all $x \neq 0$, $a, b \in F$ (even $a = 0$ by (2)).

Finally, by (2), (3), and substitution,

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$$\eta(x, 0) + \eta(0, a + b) = \eta(x, a + b) = \eta(x, a) + \eta(0, b)$$
$$= \eta(x, 0) + \eta(0, a) + \eta(0, b);$$
$$\text{for all } a, b, \text{ non-zero } x \in F.$$

Hence $\eta(0, a + b) = \eta(0, a) + \eta(0, b)$ or, by the definition of σ,

$$\sigma(a + b) = \sigma(a) + \sigma(b)$$

for all $a, b \in F$. Thus σ is additive and our proof is complete.

The above proof is a simplified version of my original proof, as suggested by Professor Pickert. He comments that it can be interpreted so that it applies to *any* cartesian group.

LEMMA 5. *Every affine collineation of $M_\phi(F)$ onto $M_\psi(K)$ maps $\{x = 0\}$ onto $\{x' = 0\}$.*

Proof. Suppose there exists an affine collineation α from $M_\phi(F)$ onto $M_\psi(K)$ mapping $\{x = r \neq 0\}$ onto $\{x' = 0\}$. Applying first a translation of $M_\phi(F)$, we can assume that $\alpha[(r, 0)] = (0, 0)$; cf. Lemma 2. Again using Lemma 2, let ρ denote a non-trivial elation $(x', y') \to (p'x', p'y')$ on $M_\psi(K)$, where $p' \in K$, $p' > 0$, $p' \neq 1$. Then the collineation $\gamma = \alpha\rho\alpha^{-1}$ on $M_\phi(F)$ leaves l_∞ pointwise fixed and maps each line through $(r, 0)$ onto itself. This applies in particular to the line $\{y = 0\}$. (If β, δ are transformations and if "β followed by δ" is meaningful, then $\beta\delta$ denotes the map $u \to (\beta\delta)(u) = \delta(\beta(u))$.)

Let γ map (x, y) onto $(x', y') = (\mu(x), \eta(y))$ in $M_\phi(F)$, for all $x, y \in F$. (This is possible since both Y_∞ and X_∞ are fixed points of γ.) Unless $x = r$, $\mu(x) \neq x$ for any x (γ being an affine conjugate of ρ). In particular, $c = \mu(0) \neq 0$ and $d = \mu^{-1}(0) \neq 0$. With $e = 0$ and $\sigma = \eta$ in Lemma 4, it follows that η is additive. Since γ preserves slopes and since $\gamma[(0, 0)] = (c, 0)$, each line $\{y = m \circ x\}$ is mapped onto $\{y' = m \circ x' - m \circ c\}$. Putting

$$y' = \eta(y) = \eta(m \circ x),$$

and $x' = \mu(x)$, we conclude that

$$\eta(m \circ x) = m \circ \mu(x) - m \circ c, \qquad \text{for all } m, x \in F; \tag{4}$$

in particular $\eta(m \circ d) = -m \circ c$. If $cd < 0$, this yields

$$\eta(z) = (-c)\cdot\lambda(z/d), \quad \text{where} \quad \lambda = \begin{cases} \phi & \text{if} \quad d > 0, \\ \phi^{-1} & \phantom{\text{if}} \quad d < 0, \end{cases} \tag{5}$$

but, if $cd > 0$,

$$\eta(z) = kz, \qquad \text{where} \qquad k = -c/d \neq 0. \tag{6}$$

Substituting $\eta(m \circ d) = -m \circ c$ in (4) and using the additivity of η we obtain

$$\eta(m \circ x - m \circ d) = m \circ \mu(x). \tag{7}$$

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We shall now show in all possible cases that there exist constants $a \neq 0$, b such that

$$\phi(am) = bm \qquad \text{for all } m \in F. \tag{8}$$

This condition is impossible because, putting $m = 1/a$ (a being non-zero), we would obtain $b = a$ and ϕ would be the identity.

Case (i), $cd < 0$. Choosing $x = r$, so that $\mu(x) = r$, and substituting (5) into (4) and (7) we obtain

$$\left.\begin{aligned} &(9) \qquad (-c)\lambda[(m \circ r)/d] = m \circ r - m \circ c \\ &(10) \qquad (-c)\lambda[(m \circ r - m \circ d)/d] = m \circ r \end{aligned}\right\} \text{for all } m \in F.$$

If rc is positive, rd is negative and we apply (9). If $rc < 0$, then $rd > 0$ and we apply (10). In either case we obtain an identity (8).

Case (ii), $cd > 0$. Choosing x so that $cx < 0$, and substituting (6) into (4), we obtain

$$k(m \circ x) = m \circ \mu(x) - m \circ c.$$

This yields an identity (8), unless $\mu(x) = c$ or $\mu(x) = kx$, in which case $bm = 0 \cdot \phi(m) = 0$, for all $m \in F$. Hence a contradiction is reached and the proof of Lemma 5 is complete.

COROLLARY 3. *Let γ be any affine collineation from $M_\phi(F)$ onto $M_\psi(K)$, mapping each ordinary point (x, y) of $M_\phi(F)$ onto (x', y') in $M_\psi(K)$. Then $x' = \mu(x)$, and*

$$y' = e' \circ \mu(x) + \sigma(y) + a', \qquad \textit{for all} \quad x, y \in F.$$

Here a' and e' are constants in K; μ and σ are one-to-one functions from F onto K with $\mu(0) = 0$; σ is additive.

Proof. By Lemma 5, $\gamma(\{x = 0\}) = \{x' = 0\}$. Hence we may take $x_0' = 0$ in Lemma 4. If the collineation of Lemma 4 is succeeded by the arbitrary translation $(x', y') \to (x', y' + a')$ on $M_\psi(K)$, the general collineation of the present Corollary is obtained.

LEMMA 6. *The most general affine collineation of $M_\phi(F)$ onto $M_\psi(K)$ sending $(0, 0)$ to $(0, 0)$, $(1, 0)$ to $(1, 0)$, and $(0, 1)$ to $(0, 1)$ (provided such a collineation exists) is given by $(x, y) \to (\sigma(x), \sigma(y))$, where σ is an isomorphism of the Cartesian group $\{+, \circ_{(\phi)}\}$ onto the Cartesian group $\{+, \circ_{(\psi)}\}$. Each ideal point of slope r is mapped onto the ideal point of slope $\sigma(r)$.*

Proof. Since $\gamma[(0, 0)] = (0, 0)$ and $\gamma[(1, 0)] = (1, 0)$, Corollary 3 implies that

$$\gamma[(x, y)] = (\mu(x), \sigma(y)) \qquad \text{for all } x, y \in F,$$

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where σ is additive. Since $\gamma[(0, 0)] = (0, 0)$ and $\gamma[(1, r)] = (1, \sigma(r))$, each ideal point of slope r is mapped onto the one of slope $r' = \sigma(r)$.

It remains to be shown that $\mu = \sigma$ and that

$$\sigma(r \circ x) = \sigma(r) \circ \sigma(x), \qquad \text{for all } r, x \in F.$$

The collineation γ will map each line $\{y = r \circ x\}$ onto some line $\{y' = r' \circ x'\}$. Thus $\sigma(r \circ x) = r' \circ \mu(x) = \sigma(r) \circ \mu(x)$. Specializing $x = 1$, we obtain $\sigma(x) = \mu(x)$. This completes our proof.

Remark. In a sense, Lemma 6 is a special case of a result of Pickert (**7**, p. 36). While that result is stated only for collineations of one plane onto itself, the proof given by Pickert is easily extended to the case of distinct planes.

3. The nine case. The affine collineations from $M_\phi(F)$ onto $M_\psi(K)$ are exceptional if F has order 9. This case is treated in Theorem A below, leaving the general case of order >9 for Theorem 1. This case is also exceptional in its extension from the affine to the projective collineation group. The non-Desarguesian Moulton plane over a field of order 9 is the only such plane admitting a collineation that leaves the point Y_∞ fixed without fixing or interchanging the lines l_∞ and $\{x = 0\}$. This will be proved in a forthcoming paper.

If F has order 9, it is isomorphic to K and $M_\phi(F)$ is isomorphic to $M_\psi(K)$. Thus no real generality is sacrificed in taking $F = K$ and $\phi = \psi$.

Let $F_3 = \{0, +1, -1\}$ denote the prime field of characteristic three. The field F_9 of order 9 consists of the elements $x + jy$, where $x, y \in F_3$ and $j^2 = -1$. Its four positive elements are ± 1, $\pm j$.

THEOREM A. *Let ϕ denote the automorphism $x + jy \to x - jy$ on F_9. The general affine collineation ω on $M_\phi(F_9)$ is a product $\omega = \beta\lambda$ of an affine collineation $\beta : (x, y) \to (x', y')$ that maps the x-axis onto itself, by a transformation*

$$\lambda : (x', y') \to (x'', y'') = (x', y' + d \circ x' + c),$$

with d, c arbitrary in F_9. The functions y' and x' are given by

$$y' = lk\alpha(y), \qquad x' = \begin{cases} l\alpha(x) & \text{if } \alpha(x) \geqslant 0, \\ lk\alpha(x)/\phi(k) & \text{if } \alpha(x) < 0. \end{cases}$$

Here l, k are arbitrary non-zero elements of F_9, and α is any one-to-one linear transformation on F_9 mapping each element of F_3 onto itself. The ideal point $P_\infty(r)$ (of slope r) is moved by ω to $P_\infty(k\alpha(r) + d)$ or to $P_\infty(\phi[k\alpha(r)] + d)$ according as $l > 0$ or $l < 0$.

Proof. Let ω be an affine collineation on $M_\phi(F_9)$, and suppose that

$$\omega(\{y = 0\}) = \{y = d \circ x + c\}.$$

Define $\lambda : (x, y) \to (x, y + d \circ x + c)$ and $\lambda[P_\infty(r)] = P_\infty(r + d)$. By Corollary 2, λ is a collineation, ϕ being additive. The collineation $\beta = \omega\lambda^{-1}$ maps the x-axis onto itself.

By Corollary 3, β has the form

$$\beta : x' = \mu(x), \quad y' = e \circ \mu(x) + \sigma(y) + a,$$

where μ and σ are one-to-one mappings of F_9 onto itself, σ is additive, and $\mu(0) = 0$.

Since β must map the line $\{y = 0\}$ onto itself, we have $e \circ \mu(x) + a = 0$ for all x. Hence $e = a = 0$. Thus β is given by

$$\beta : x' = \mu(x), \qquad y' = \sigma(y).$$

By Lemma 2,

$$\gamma : x \to \mu(1)x, \qquad y \to \mu(1)y$$

is an affine collineation. It maps the ideal point with slope m onto the ideal point with slope m or $\phi(m)$ respectively as $\mu(1) > 0$ or $\mu(1) < 0$. We have $\beta = \beta_0\gamma$, where

$$\beta_0 : (x, y) \to (\mu_0(x), \sigma_0(y)), \quad \mu_0(x) = \mu(x)/\mu(1), \quad \sigma_0(x) = \sigma(x)/\mu(1).$$

Thus μ_0 and σ_0 have the same properties as μ and σ respectively; in addition $\mu_0(1) = 1$.

The collineation β_0 maps each straight line $\{y = a \circ x\}$ through the origin onto a straight line $\{y' = a'(a) \circ x'\}$ through the origin. Hence

$$y' = \sigma_0(y) = \sigma_0(a \circ x) = a'(a) \circ \mu_0(x).$$

Specializing $x = 1$, we obtain

$$\sigma_0(a) = \sigma_0(a \circ 1) = a'(a) \circ \mu_0(1) = a'(a) \circ 1 = a'(a).$$

Hence

$$\sigma_0(a \circ x) = \sigma_0(a) \circ \mu_0(x) \qquad \text{for all } a, x \in F_9. \tag{11}$$

Conversely, (11) is readily seen to imply that β_0, and hence β, is an affine collineation.

Specializing $a = 1$ in (11) we obtain

$$\sigma_0(x) = \sigma_0(1 \circ x) = \sigma_0(1) \circ \mu_0(x) = \begin{cases} \sigma_0(1)\mu_0(x) & \text{if} \quad \mu_0(x) \geqslant 0, \\ \phi(\sigma_0(1))\cdot\mu_0(x) & \phantom{\text{if}} \quad \mu_0(x) < 0. \end{cases} \tag{12}$$

Put

$$\alpha(x) = \sigma_0(x)/\sigma_0(1); \qquad \text{so that } \sigma_0(x) = \sigma_0(1)\cdot\alpha(x). \tag{13}$$

Then α is an additive mapping of F_9 onto itself. We have $\alpha(1) = 1$; hence $\alpha(-1) = -1$. Thus α fixes each element of F_3. Since $\sigma_0(1)/\phi\sigma_0(1)$ is positive, (12) and (13) imply that $\mu_0(x)$ and $\alpha(x)$ have the same sign for every $x \in F_9$; and

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$$(14) \qquad \mu_0(x) = \begin{cases} \alpha(x) & \text{if } \alpha(x) \geqslant 0, \\ \dfrac{\sigma_0(1)}{\phi(\sigma_0(1))}\cdot\alpha(x) & \phantom{\text{if }} \alpha(x) < 0. \end{cases}$$

Suppose conversely that $\alpha(x)$ is an additive mapping of F_9 onto itself which keeps each element of F_3 fixed. Choose $\sigma_0(1)$ arbitrarily and define σ_0 and μ_0 through (12) and (13). We wish to show that (11) is satisfied. Formula (11) is equivalent to

$$(15) \qquad \alpha(a \circ x) = \alpha(a) \circ \alpha(x) \qquad \text{for all } a, x \in F_9.$$

Since α is additive and linear over F_3, it suffices to prove (15) for $a = 1$ and $a = j$. (This uses the left-distributive law for $\circ$; cf. (**8**, Theorem 5).) On account of $\alpha(1) = 1$, the case $a = 1$ is trivial. Thus it is sufficient to prove that

$$(16) \qquad \alpha(j \circ x) = \alpha(j) \circ \alpha(x) \qquad \text{for all } x \in F_9.$$

For any $z \in F_9$, $-z$ has the same sign as z. Hence if (16) holds true for a particular x, it will also hold true for $-x$. Formula (16) being trivial for $x \in F_3$, it suffices to prove (16) for $x = j$ and $x = 1 \pm j$.

Put $\alpha(j) = \delta$. Then

$$(17) \qquad \delta \notin F_3, \qquad 1 \pm \delta \notin F_3.$$

Since

$$\alpha(r + sj) = \alpha(r) + \alpha(sj) = r\cdot\alpha(1) + s\cdot\alpha(j) = r + s\delta$$

for every $r \in F_3$, $s \in F_3$, (16) follows from the formulas

$$(18) \qquad \delta \circ \delta = -1,$$

$$(19) \qquad \delta \circ (1 \pm \delta) = \pm 1 - \delta.$$

It is readily verified that $z \circ z = -1$ for any $z \in F_9 - F_3$. Using (17), we obtain (18) and also $(1 \pm \delta) \circ (1 \pm \delta) = -1$. This yields

$$\begin{aligned} -1 = (1 \pm \delta) \circ (1 \pm \delta) &= [1 \circ (1 \pm \delta)] \pm [\delta \circ (1 \pm \delta)] \\ &= (1 \pm \delta) \pm [\delta \circ (1 \pm \delta)], \end{aligned}$$

which reduces to (19).

Remark. The plane $M_\phi(F_9)$ and its co-ordinate system were introduced by Veblen and Wedderburn (**10**). It is discussed by M. Hall, Jr. (**4**). It is dual to the plane considered by J. André in (**1**, section 6). This plane is given by Pickert (**7**, pp. 95–96) as an example of a cartesian group which forms a non-distributive, associative left quasi-field; cf. also Zassenhaus (**11**) and Pierce (**8**). (Corollary 7 of the latter should read: "... is associative if and only if $\phi(ab) = \phi(a)\cdot\phi(b)$ when a or b is non-negative, and $\phi(ab) = \phi^{-1}(a)\cdot\phi(b)$ for all $a \in F$ when $b < 0$." This correction will be made in the Appendix to the present paper.)

4. The case of order greater than nine.

THEOREM 1. *Let $M_\phi(F)$ be non-Desarguesian with order exceeding* 9. *The most general affine collineation γ of $M_\phi(F)$ onto a plane $M_\psi(K)$ maps each ordinary point (x, y) of $M_\phi(F)$ onto a point $(\mu(x), e' \circ \mu(x) + g' \cdot \alpha(y) + a')$. Here α is a sign-preserving isomorphism of F onto K; and*

$$\mu(x) = \begin{cases} g' \cdot \alpha(x)/(c' - e') & \text{if} \quad \mu(x) \geqslant 0, \\ g' \cdot \alpha(x)/[\psi(c') - \psi(e')] & \phantom{\text{if}} \quad \mu(x) < 0. \end{cases}$$

The constants c' and e' are distinct and satisfy

$$[\psi(c') - \psi(e')] \cdot \alpha\phi(r) + \psi(e') = \psi[(c' - e')\alpha(r) + e'] \qquad \textit{for all } r \in F \textit{ if } \mu(1) > 0,$$

$$[\psi(c') - \psi(e')] \cdot \alpha(r) + \psi(e') = \psi[(c' - e') \cdot \alpha\phi(r) + e'] \qquad \textit{for all } r \in F \textit{ if } \mu(1) < 0.$$

Each ideal point of slope r in $M_\phi(F)$ is sent to the ideal point of slope r' in $M_\psi(K)$, where

$$\alpha(r) = \begin{cases} (r' - e')/(c' - e') & \text{if} \quad \mu(1) > 0, \\ [\psi(r') - \psi(e')]/[\psi(c') - \psi(e')] & \phantom{\text{if}} \quad \mu(1) < 0. \end{cases}$$

The image of Y_∞ under γ is Y_∞'.

Proof. By Lemma 1, $\gamma(Y_\infty) = Y_\infty'$.

By Corollary 3, γ maps (x, y) onto

$$(\mu(x), e' \circ \mu(x) + \sigma(y) + a'),$$

where μ and σ are one-to-one functions from F onto K, e' and a' are constants, $\mu(0) = 0$, and σ is additive. Any such map is the product of a collineation sending origin to origin, followed by a translation on $M_\psi(K)$; cf. Lemma 2. Since the translations on $M_\psi(K)$ are transitive on the ordinary points of $\{x' = 0\}$, we need consider only maps in which $a' = 0$.

The collineation γ maps each straight line $\{y = b + r \circ x\}$ onto a straight line $\{y' = b' + r' \circ x'\}$. Hence

$$y' = e' \circ \mu(x) + \sigma(y) = e' \circ \mu(x) + \sigma(b) + \sigma(r \circ x) = b' + r' \circ \mu(x).$$

The special case $x = 0$ yields $b' = \sigma(b)$. Thus γ is a collineation if and only if μ, σ, and $r' = r'(e', r)$ satisfy the *Collinearity Condition*

$$\sigma(r \circ x) = r' \circ \mu(x) - e' \circ \mu(x). \tag{20}$$

We determine r' by specializing $x = 1$:

$$\sigma(r)/\mu(1) = \begin{cases} r' - e' & \text{if} \quad \mu(1) > 0, \\ \psi(r') - \psi(e') & \phantom{\text{if}} \quad \mu(1) < 0. \end{cases} \tag{21}$$

Put $c' = r'(e', 1)$. Thus

$$\sigma(1)/\mu(1) = \begin{cases} c' - e' & \text{if } \mu(1) > 0, \\ \psi(c') - \psi(e') & \phantom{\text{if }} \mu(1) < 0. \end{cases} \tag{22}$$

Specializing $r = 1$ and thus $r' = c'$ in (20), we determine the function $\mu(x)$:

$$\mu(x) = \begin{cases} \sigma(x)/(c' - e') & \text{if } \mu(x) \geqslant 0, \\ \sigma(x)/(\psi(c') - \psi(e')) & \phantom{\text{if }} \mu(x) < 0. \end{cases} \tag{23}$$

Condition (20) now reduces to

$$\sigma(r \circ x) = \begin{cases} \dfrac{r' - e'}{c' - e'}\,\sigma(x) & \text{if } \mu(x) = \sigma(x)/(c' - e') \geqslant 0, \\ \dfrac{\psi(r') - \psi(e')}{\psi(c') - \psi(e')}\,\sigma(x) & \phantom{\text{if }} \mu(x) = \sigma(x)/[\psi(c') - \psi(e')] < 0, \end{cases} \tag{24}$$

where

$$\frac{\sigma(r)}{\sigma(1)} = \begin{cases} \dfrac{r' - e'}{c' - e'} & \text{if } \mu(1) = \sigma(1)/(c' - e') > 0, \\ \dfrac{\psi(r') - \psi(e')}{\psi(c') - \psi(e')} & \phantom{\text{if }} \mu(1) = \sigma(1)/[\psi(c') - \psi(e')] < 0. \end{cases}$$

Since r' is determined as a function of r by c' and e', the *Collinearity Condition* (20) is satisfied by a choice of constants c', e' and a one-to-one additive function σ from F onto K if and only if c, e, and σ satisfy (24). Hence (24) is the exact condition that a collineation be obtained in the prescribed way.

Define the mapping α of F onto K through

$$\alpha(x) = \sigma(x)/\sigma(1).$$

Thus $\alpha(1) = 1$ and α is additive and one-to-one. By discussing the four possible cases we readily deduce from (21) to (23) that

$$\operatorname{sign} \mu(x) = \operatorname{sign} \alpha(x) \cdot \operatorname{sign} \mu(1). \tag{25}$$

Hence (24) can be rewritten

$$\alpha(r \circ x) = \begin{cases} \dfrac{r' - e'}{c' - e'}\,\alpha(x) & \text{if } \alpha(x)\mu(1) > 0, \\ \dfrac{\psi(r') - \psi(e')}{\psi(c') - \psi(e')}\,\alpha(x) & \phantom{\text{if }} \alpha(x)\mu(1) < 0. \end{cases}$$

In particular (as also follows from division of (21) by (22))

$$\alpha(r) = \begin{cases} \dfrac{r' - e'}{c' - e'} & \text{if } \mu(1) > 0, \\ \dfrac{\psi(r') - \psi(e')}{\psi(c') - \psi(e')} & \phantom{\text{if }} \mu(1) < 0. \end{cases} \tag{26}$$

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Thus

$$\alpha(r \circ x) = \begin{cases} \alpha(r)\alpha(x) & \text{if} \quad \alpha(x) > 0, \\ \beta(r)\alpha(x) & \phantom{\text{if}} \quad \alpha(x) < 0, \end{cases} \tag{27}$$

where we define

$$\beta(r) = \begin{cases} \dfrac{\psi(r') - \psi(e')}{\psi(c') - \psi(e')} & \text{if} \quad \mu(1) > 0, \\ \dfrac{r' - e'}{c' - e'} & \phantom{\text{if}} \quad \mu(1) < 0. \end{cases} \tag{28}$$

Thus β maps F into K. By (26), β satisfies $\beta(0) = 0$, $\beta(1) = 1$, and

$$\operatorname{sign} \beta(r) = \operatorname{sign} \alpha(r) \qquad \text{for all } r \in F. \tag{29}$$

Let $P = \{x/x > 0\}$, the set of all positive $x \in F$; $S = \{x/x > 0$ and $\alpha(x) > 0\}$; $N = \{x/x < 0\}$, the set of all negative $x \in F$. We have $1 \in S \subseteq P$. By (27)

$$\alpha(rx) = \alpha(r)\alpha(x) \qquad \text{for all } r \in F,\ x \in S. \tag{30}$$

It follows at once that S is a subgroup of P.

If there is an $x \in P - S$, (27) implies that $\alpha(rx) = \beta(r)\alpha(x)$ for any such x and for all $r \in F$, also that $\beta(r)$ (and hence $\alpha(r)$) has sign opposite that of $\alpha(rx)$, unless $r = 0$. In particular $r \in P - S$ and $x \in P - S$ imply $rx \in S$; whence either $S = P$ or S is a subgroup of P with index 2.

We next show that if $S = P$, α is multiplicative, i.e. that

$$\alpha(rx) = \alpha(r)\alpha(x) \qquad \text{for all } r \in F,\ x \in F. \tag{31}$$

By (30) it suffices to assume $x \in N$.

Since

$$x(1 + x)\left(1 + \frac{1}{x}\right) = (1 + x)^2 \in P,$$

either $1 + x \in P$ or $1 + 1/x \in P$. In the first case we have

$$\begin{aligned} \alpha(rx) &= (\alpha(rx) + \alpha(r)) - \alpha(r) = \alpha(rx + r) - \alpha(r) = \alpha(r(x + 1)) - \alpha(r) \\ &= \alpha(r)\alpha(x + 1) - \alpha(r) = \alpha(r)(\alpha(x) + 1) - \alpha(r) = \alpha(r)\alpha(x). \end{aligned}$$

In the second case we obtain similarly

$$\alpha\left(r \cdot \frac{1}{x}\right) = \alpha(r) \cdot \alpha\left(\frac{1}{x}\right), \qquad \text{whence } \alpha(r) = \alpha(rx) \cdot \alpha\left(\frac{1}{x}\right)$$

for all $r \in F$. With $r = 1$, the latter formula yields $\alpha(1/x) = 1/\alpha(x)$; and substitution of this in the same formula gives (31).

We now consider the case $S \neq P$. Here there exists an element $q \in P - S$ such that $P - S = qS$.

Let $s_1 \in S$, $s_2 \in S$. By (30)

$$\alpha(r(s_1 - s_2)) = \alpha(rs_1 - rs_2) = \alpha(rs_1) - \alpha(rs_2) = \alpha(r)\alpha(s_1) - \alpha(r)\alpha(s_2)$$
$$= \alpha(r)(\alpha(s_1) - \alpha(s_2)) = \alpha(r)\alpha(s_1 - s_2).$$

If $s_1 - s_2 \in qS$, we would have

$$\alpha(r(s_1 - s_2)) = \beta(r)\alpha(s_1 - s_2).$$

Hence $\beta(r) = \alpha(r)$ for all $r \in F$; and (from (26) and (28)) $\psi(r') = A'r' + B'$ for all $r' \in K$ with A', B' suitable constants $\in K$. Since ψ keeps 0 and 1 fixed, $\psi(r') = r'$ for all $r' \in K$, making $M_\psi(K)$ Desarguesian (**8**, Theorem 4). Thus the case $s_1 \in S$, $s_2 \in S$, $s_1 - s_2 \in qS$ is impossible and we have

(32) $$S - S \subseteq S \cup N \cup \{0\}.$$

Suppose first that

(33) $$S - S \subseteq S \cup \{0\};$$

whence also $qS - qS \subseteq qS \cup \{0\}$.

Since F has more than nine elements, S and qS have more than two each. Choose $s \in S$, $s \neq 1$. By (33), $s - 1 \in S$ and $(-1)(s-1) = 1 - s \in S$. Hence $-1 \in S$, $S + S \subseteq S \cup \{0\}$, and $qS + qS \subseteq qS \cup \{0\}$.

Since qS has more than two elements, there exists an $r \in qS$ such that $r^2 + 1 \neq 0$. From $r^2 \in S$, we have

$$r^2 + 1 \in S + S \subseteq S \cup \{0\}; \qquad \text{hence } r^2 + 1 \in S.$$

Put $u = r + 1$, obtaining $(r^2 + 1) + 2r = (r+1)^2 = u^2 \in P$. If $u^2 \in S$, we conclude that

$$2r = u^2 - (r^2 + 1) \in qS \cap (S - S) \subseteq qS \cap (S \cup \{0\}) = \emptyset,$$

where $\emptyset$ is the null set. If $u^2 \in qS$, we have

$$r^2 + 1 = u^2 - 2r \in S \cap (qS - qS) \subseteq S \cap (qS \cup \{0\}) = \emptyset.$$

Thus $u^2 \notin S$, $u^2 \notin qS$, and (33) has led to a contradiction.

Formula (32) therefore (and in fact the assumption $S \neq P$) implies that

$$(S - S) \cap N \neq \emptyset.$$

Let $s_1 \in S$, $s_2 \in S$, $s_1 - s_2 \in N$. Then

(34) $$\frac{s_1}{s_2} - 1 = n \in N \quad \text{and} \quad n + 1 \in S.$$

Using (30) we obtain

$$\alpha(rn) = (\alpha(rn) + \alpha(r)) - \alpha(r) = \alpha(rn + r) - \alpha(r)$$
$$= \alpha(r(n+1)) - \alpha(r) = \alpha(r)\alpha(n+1) - \alpha(r)$$
$$= \alpha(r)(\alpha(n) + 1) - \alpha(r)$$

or

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$$\alpha(rn) = \alpha(r)\alpha(n) \qquad \text{for all } r \in F. \tag{35}$$

In particular

$$\alpha(n^2) = (\alpha(n))^2 > 0,$$

and thus $n^2 \in S$.

Let $x \in N$. Then $x = np$, $p \in P$, and $x^2 = n^2p^2 \in S \cdot S = S$. This yields

$$x^2 \in S \qquad \text{for all } x \neq 0 \text{ in } F. \tag{36}$$

Let $q \in P - S$; $q \neq -1$. By (36), the elements 2^2, $(q+1)^2$, and $(q-1)^2$ lie in S. Hence, using (32),

$$2^2 \cdot q = (q+1)^2 - (q-1)^2 \in qS \cap (S - S) = \emptyset.$$

We now know that $S = P$; whence $\alpha(x) > 0$ if $x > 0$. Let $n < 0$, $x < 0$. Then $x = np$, where $p \in P$ and hence $\alpha(p) > 0$. By (30), $\alpha(x) = \alpha(n) \cdot \alpha(p)$. Thus, $\operatorname{sign} \alpha(x) = \operatorname{sign} \alpha(n)$, and $\operatorname{sign} \alpha(x)$ is constant for all $x \in N$. Since α is a mapping onto K, this yields $\alpha(x) < 0$ for all $x \in N$. Thus $\operatorname{sign} \alpha(x) = \operatorname{sign} x$ for all $x \in F$, $x \neq 0$; and by (31), α is multiplicative.

Choose any $x < 0$. By (27) and (31)

$$\beta(r)\alpha(x) = \alpha(r \circ x) = \alpha(\phi(r)x) = \alpha\phi(r)\alpha(x).$$

Hence

$$\beta(r) = \alpha\phi(r) \qquad \text{for all } r \in F. \tag{37}$$

Equations (37), (26), and (28) readily yield the additional conditions of Theorem 1 for c', e', ϕ, and ψ. If they are satisfied, (37) will hold and the functions $\sigma(x)$, $\mu(x)$, $r'(r)$ will satisfy the Collinearity Condition (20). This completes the proof of Theorem 1.

COROLLARY 4. *If the order of F is greater than 9, and if $M_\phi(F)$ is non-Desarguesian, any affine collineation, $(x, y) \to (x', y')$, from $M_\phi(F)$ onto a plane $M_\psi(K)$, preserves all the signs of x ($\neq 0$), or else reverses all of them.*

Since $x' = \mu(x)$, the proof follows immediately from (25) and the fact that α is sign-preserving.

COROLLARY 5. *Let $M_\phi(F)$ be non-Desarguesian. There cannot exist an affine collineation from $M_\phi(F)$ onto a plane $M_\psi(K)$ unless there is a sign-preserving isomorphism α of F onto K.*

Proof. The case of order 9 is consistent with this corollary. If the order of F exceeds 9, such an α is required in Theorem 1.

For convenience, the next corollary gives an explicit determination of γ^{-1}, the inverse of the transformation γ in Theorem 1.

COROLLARY 6. *Under the assumptions of Theorem 1, the inverse γ^{-1} of γ sends (x', y') to (x, y), where*

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$$x = \begin{cases} g\alpha^{-1}(x')/(c-e) & \text{if} \quad x \geqslant 0, \\ g\alpha^{-1}(x')/(\phi(c) - \phi(e)) & \phantom{\text{if}} \quad x < 0, \end{cases}$$

and

$$y = e \circ x + g\alpha^{-1}(y') + a.$$

The constants c, e, g, a are related to c', e', g', a' as follows: $g = 1/\alpha^{-1}(g')$, $a = -\alpha^{-1}(a'/g')$; *and* (i) *if μ is sign-preserving*

$$\alpha(e) = \frac{e'}{e' - c'}, \qquad \alpha\phi(e) = \frac{\psi(e')}{\psi(e') - \psi(c')},$$

$$\alpha(c) = \frac{e' - 1}{e' - c'}, \qquad \alpha\phi(c) = \frac{\psi(e') - 1}{\psi(e') - \psi(c')},$$

$$[\phi(c) - \phi(e)]\cdot\alpha^{-1}\psi(x') + \phi(e) = \phi[(c-e)\alpha^{-1}(x') + e], \qquad \textit{for all } x' \in K;$$

while (ii) *if μ is sign-reversing*

$$\alpha(e) = \frac{\psi(e')}{\psi(e') - \psi(c')}, \qquad \alpha\phi(e) = \frac{e'}{e' - c'},$$

$$\alpha(c) = \frac{\psi(e') - 1}{\psi(e') - \psi(c')}, \qquad \alpha\phi(c) = \frac{e' - 1}{e' - c'},$$

$$[\phi(c) - \phi(e)]\cdot\alpha^{-1}(x') + \phi(e) = \phi[(c-e)\alpha^{-1}\psi(x') + e], \quad \textit{for all } x' \in K.$$

Proof. Theorem 1, with $M_\phi(F)$ and $M_\psi(K)$ interchanged, yields

$$x = \mu'(x'), \qquad y = e \circ \mu'(x') + g\cdot\alpha'(y') + a, \tag{38}$$

where α' is a sign-preserving isomorphism of K onto F.

$$\mu'(x') = \begin{cases} g\alpha'(x')/(c-e) & \text{if} \quad x \geqslant 0, \\ g\alpha'(x')/(\phi(c) - \phi(e)) & \phantom{\text{if}} \quad x < 0, \end{cases}$$

and c, e, g are constants, $c \neq e$.

By (25), we have $\operatorname{sign} x' = \operatorname{sign} x \cdot \operatorname{sign} \mu(1)$. By (38)

$$\begin{aligned} y &= e \circ x + g\cdot\alpha'[e' \circ \mu(x) + g'\alpha(y) + a'] + a \\ &= e \circ x + g\cdot\alpha'[e' \circ \mu(x)] + g\cdot\alpha'(g')\cdot\alpha'[\alpha(y)] + g\alpha'(a') + a \end{aligned}$$

identically in x and y. Putting first $x = y = 0$, then $x = 0$, we obtain

$$g\alpha'(a') + a = 0; \qquad y = g\cdot\alpha'(g')\cdot\alpha'(\alpha(y)) \qquad \text{for all } y \in F.$$

The case $y = 1$ yields $g\cdot\alpha'(g') = 1$ and hence $y = \alpha'(\alpha(y))$ for all $y \in F$; that is

$$\alpha' = \alpha^{-1}.$$

The equations for g and a are thus established.

The relations for e and c as functions of e' and c' are easily computed from the equations

$$e \circ x + g \cdot \alpha^{-1}[e' \circ x'] = 0; \qquad \mu(\mu'(x')) = x'.$$

The other conditions are direct reformulations from Theorem 1.

Remark. If $\gamma(X_\infty) = X_\infty'$, the expressions for x' and y' may be simplified. These formulae are useful in treating the projective case, and so they will be given in the form of a corollary.

COROLLARY 7. *Let $M_\phi(F)$ be non-Desarguesian of order >9. Then $\gamma : (x, y) \to (x', y')$ is an affine collineation from $M_\phi(F)$ onto $M_\psi(K)$ with $\gamma(X_\infty) = X_\infty'$, if and only if*

$$x' = g'\alpha(d \circ x), \qquad y' = g'\alpha(y) + a',$$

for appropriate non-zero $d \in F$, and constants $g' \neq 0$ and a' in K. Here α is a sign-preserving isomorphism of F onto K and

$$\psi\alpha(r/d) = \alpha\phi(r) \cdot \psi[1/\alpha(d)] \qquad \text{if } g'\alpha(d) > 0,$$
$$\psi\alpha[\phi(r)/\phi(d)] = \alpha(r) \cdot \psi[1/\alpha\phi(d)] \qquad \text{if } g'\alpha(d) < 0.$$

Each ideal point of slope r in $M_\phi(F)$ is sent to the ideal point of slope r' in $M_\psi(K)$, where

$$\alpha(r) = \begin{cases} r'\alpha(d) & \text{if } g'\alpha(d) > 0, \\ \psi(r') \cdot \alpha(d) & g'\alpha(d) < 0. \end{cases}$$

The image of Y_∞ under γ is Y_∞'.

Proof. By Theorem 1, $e' = 0$. Thus $\mu(1) = g'/c'$ or $g'/\psi(c')$ as $\mu(1) > 0$ or $\mu(1) < 0$. Define

$$d = \begin{cases} 1/\alpha^{-1}(c') & \text{if } \mu(1) > 0, \\ \phi^{-1}[1/\alpha^{-1}(c')] & \mu(1) < 0, \end{cases}$$

and apply the formulae of Theorem 1.

The expressions for x' in the case $\mu(x) < 0$ are obtained by using the other formulae of this Corollary to obtain

$$\begin{aligned} \alpha\phi(d) &= 1/\psi[1/\alpha(d)] & \text{if } x < 0, \\ \alpha(d) &= 1/\psi[1/\alpha\phi(d)] & x > 0. \end{aligned}$$

5. Concluding remarks. A forthcoming paper, "Collineations of Projective Moulton Planes," will complete the solution of the collineation problem for Moulton planes.

Further research, as suggested by the present series, may take several directions. For example, the sets P, N of the Moulton construction can be replaced by a collection $\{S_i\}$ of sets; and the functions $\mathfrak{I}$ (on P), ϕ (on N), by ϕ_i on S_i. This "multi-bending" arrangement supplies a large new class of Baer planes. In particular, the subgroup P may have index >2 and the field F may or may not have characteristic 2. (A simple non-Desarguesian "generalized Moulton Plane" can be constructed, for example, from a field F of order 16, relative to the multiplicative subgroup P, of order 5, whose elements are the non-zero cubes, by using $\mathfrak{I}$ on P, and $\phi : x \to x^4$ elsewhere.)

A further extension of the Carlitz theorem (**3**; **8**, pp. 428–429) has been communicated to me by Robert McConnel. He hopes that his theorem will have applications to the geometry of higher dimensions.

McConnel's Theorem (*unpublished*). Given a Galois field GF(p^n) [with p odd or even], let d_s be a divisor of $(p^n - 1)$ for each $s = 1, \ldots, m$. Put $\psi_s(x) = x^{([p^n-1)/d_s]}$ for all $x \in$ GF(p^n) and $s = 1, \ldots, m$. If

$$\psi_s[f(x_1, \ldots, x_{(s-1)}, x_s, x_{(s+1)}, \ldots, x_m) - f(x_1, \ldots, x_{(s-1)}, y_s, x_{(s+1)}, \ldots, x_m)] = \lambda_s \psi_s(x_s - y_s) \qquad [s = 1, \ldots, m]$$

for all $x_i, y_i \in$ GF(p^n), where each λ_i is a fixed element of GF(p^n) such that $(\lambda_i)^{d_i} = 1$, then

$$f(x_1, \ldots, x_m) = \sum_{i=1}^{m} a_i \cdot x^{(p^{r_i})} + b \qquad (0 \leqslant r_i < m),$$

where $\psi_i(a_i) = \lambda_i$ for $i = 1, \ldots, m$.

Appendix. Corrections and remarks on "Moulton planes" (**8**).

1. On page 435, Theorem 7 (ii), replace "ϕ is multiplicative on F" by "$\phi(ab) = \phi(a)\phi(b)$ for all $a \in F$."

The necessary and sufficient condition in Corollaries 7 and 8 should read "$\phi(ab) = \phi(a)\phi(b)$ when a or b is non-negative, and $\phi(ab) = \phi^{-1}(a)\phi(b)$ for all $a \in F$ when $b < 0$." (The latter equality is obtained from the formula given in Theorem 7 (iii) if we replace $\phi(a)$ by a.)

The correct necessary and sufficient condition of the Corollaries implies that $\circ$ cannot be associative unless $\phi(x) = \phi^{-1}(x)$ for all non-negative $x \in F$. In the case of additive ϕ, a necessary and sufficient condition for associativity of $\circ$ is that ϕ be an automorphism of order 2.

2. On page 434, the second part of Theorem 6 states that associativity of $\circ$ and the right-distributive law $c \circ (a + b) = c \circ a + c \circ b$ together are equivalent to the $(Y_\infty, \{y = 0\})$-Desargues' condition. This is correct; in fact, it is a direct application of Pickert's *Satz 45*. However, the right-distributive law is already equivalent to Desargues' Theorem (by Theorem 4), and so the second part of Theorem 6 is extraneous. It would be more appropriate to include the $(Y_\infty, \{y = 0\})$-Desargues' Theorem as one of the conditions in Theorem 4 equivalent to the Theorem of Desargues. (The equivalence follows from Pickert's *Satz 45*.)

3. On page 429, line 3, each r_i $(i = 1, \ldots, k)$ should be raised above p.

4. On page 435, reference 3, replace "2" by "11".

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