J. Appl. Probab. **61,** 214–229 (2024)
*doi:*10.1017/jpr.2023.45

THE PROPORTION OF TRIANGLES IN A CLASS OF ANISOTROPIC POISSON LINE TESSELLATIONS

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Abstract

Stationary Poisson processes of lines in the plane are studied, whose directional distributions are concentrated on $k \geq 3$ equally spread directions. The random lines of such processes decompose the plane into a collection of random polygons, which form a so-called Poisson line tessellation. The focus of this paper is to determine the proportion of triangles in such tessellations, or equivalently, the probability that the typical cell is a triangle. As a by-product, a new deviation of Miles's classical result for the isotropic case is obtained by an approximation argument.

Keywords: Poisson line tessellation; random triangle; stochastic geometry; triangle probability; typical cell.

2020 Mathematics Subject Classification: 60D05

1. Introduction and results

The study of random polygons induced by a Poisson process of random lines in the plane is among the most classical topics in stochastic geometry. The distribution of a stationary Poisson line process $X = X(\gamma, G)$ in the plane is completely determined by its intensity $\gamma > 0$ and its directional distribution G. For us, the latter is a probability measure on the interval $[0, \pi)$ satisfying $G(\theta) < 1$ for each $\theta \in [0, \pi)$. We refer to the monographs [9] and [15] for further background material and detailed descriptions and explanations. The typical cell $Z = Z(\gamma, G)$ of a stationary Poisson line tessellation with intensity γ and directional distribution G can intuitively be thought of as a random polygon selected 'uniformly at random' among the collection of all polygons (in a very large observation window) induced by X, regardless of size and shape. It is a classical descriptor of the statistical properties of the random polygons generated by this Poisson line process. Formally, its distribution can be defined using Palm calculus as explained in detail in [15]; see also (2.4) below. The geometry of the typical cell of a Poisson line tessellation and its analogue in higher dimensions has been investigated intensively over the past few decades, and numerous articles are dedicated to the study of its size or its combinatorial structure. As examples we mention the articles [3], [5], [11], [12], and [16], which

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deal with first- and second-order moments as well as integral expressions in the planar case, and the works [2], [4], [6], [10], [13], and [14], which mainly discuss first-order properties in higher-dimensional situations.

However, despite the many results just mentioned, even in the planar case, for most of the geometric and combinatorial quantities the precise distribution is unknown and even good approximation results are rarely available. In particular, this is the case for the number of vertices of the typical cell Z, which is the principal object we study in this paper. More precisely, we are interested in the exact probability that Z takes the simplest possible shape: a triangle. Since the intensity γ only acts as a scaling parameter, this probability cannot depend on γ and we can take $\gamma = 1$ for simplicity and write $Z(G)$ instead of $Z(1, G)$. Further, we define the triangle probability

$$p_3(G) := \mathbb{P}[Z(G) \text{ is a triangle}],$$

which can equivalently be described as the proportion of triangles among the polygons of the Poisson line tessellation:

$$p_3(G) = \lim_{R\to\infty} \frac{1}{\sum_{c\subset B_R} 1} \sum_{c\subset B_R} \mathbf{1}\{c \text{ is a triangle}\},$$

where B_R stands for a disk of radius $R > 0$ centred at the origin and the sum runs over all tessellation cells c contained in B_R. If the directional distribution $G = G_{\text{unif}}$ is the uniform distribution on $[0, \pi)$ and the Poisson line tessellation is isotropic, it has been known since [11] (see Theorem 6 therein) that

$$p_3(G_{\text{unif}}) = 2 - \frac{\pi^2}{6} \approx 0.35507; \tag{1.1}$$

compare also with [12] and with the computations indicated in Section 3. A realization of an isotropic Poisson line process is shown in Figure 1(c). In Section 5 we will provide an alternative proof for (1.1) using new results from the present paper. We further remark that in the isotropic case too the probability

$$\mathbb{P}[Z(G_{\text{unif}}) \text{ is a quadrangle}] = \pi^2 \log 2 - \frac{1}{3} - \frac{7\pi^2}{36} - \frac{7}{2}\sum_{i=1}^{\infty}\frac{1}{i^3} \approx 0.381466$$

is known from [16]. However, for $k \geq 5$ the probabilities $\mathbb{P}[Z(G_{\text{unif}})$ has exactly k vertices] can be expressed only as rather involved multiple integrals, which can be evaluated numerically; see [3]. On the other hand, it is well known that the expected number of vertices of the typical cell is 4, independently of the choice of the directional distribution G; see [15, Section 10.5.1].

At the other extreme, if G is concentrated on only two different values, all cells are almost surely parallelograms. So in this case we have $p_3(G) = 0$. Thus the next non-trivial case arises if the directional distribution G is concentrated on three different values. For simplicity and concreteness we start with the case where G is given by

$$G_3(p, q) := p\delta_0 + q\delta_{\pi/3} + (1 - p - q)\delta_{(2\pi)/3}, \tag{1.2}$$

where we write $\delta_{(\cdot)}$ for the Dirac measure and where $p, q \in (0, 1)$ are weights satisfying $0 < p + q < 1$. In other words, $G_3(p, q)$ is concentrated on the angles $0, \pi/3$, and $2\pi/3$ with weights p, q, and $1 - p - q$, respectively. A simulation of a Poisson line tessellation with directional

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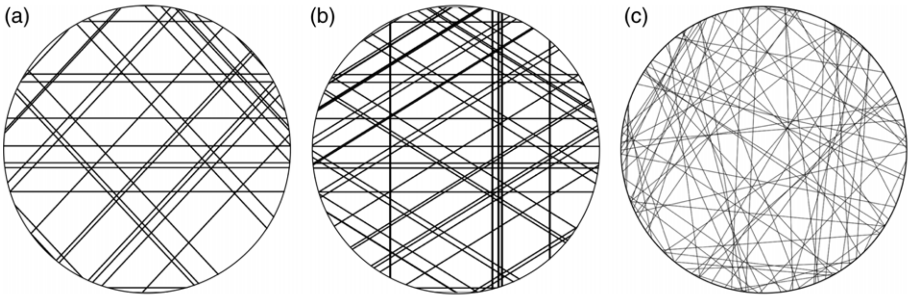

FIGURE 1. Simulation of a Poisson line tessellation with directional distribution G_3 (a), G_4 (b), and G_{unif} (c).

distribution $G_3(1/3, 1/3)$ is shown in Figure 1(a). We remark that a stationary Poisson line process with directional distribution $G_3(1/3, 1/3)$ is of course not invariant under *all* rotations in the plane. However, it is invariant under rotations whose angle is an integer multiple of $\pi/3$. The corresponding Poisson line tessellation can thus be called $G_3(1/3, 1/3)$-pseudo-isotropic.

Our first result is a formula for $p_3(G_3(p, q))$ in terms of the weights p and q. Also, we determine those weights for which $p_3(G_3(p, q))$ attains its maximal value; see Figure 2.

Theorem 1.1. *For all* $0 < p, q < 1$ *with* $0 < p + q < 1$*, we have*

$$p_3(G_3(p, q)) = \frac{2pq(1-p-q)}{p+q-p^2-q^2-pq}.$$

The maximal value for $p_3(G_3(p, q))$ *is attained precisely if* $p = q = 1/3$ *and is given by*

$$\max_{0<p+q<1} p_3(G_3(p, q)) = p_3(G_3(1/3, 1/3)) = \frac{2}{9}.$$

It is a special feature of the case of three directions that the formula for the triangle probability carries over to general orientation angles.

Corollary 1.1. *Fix* $0 \le \varphi_1 < \varphi_2 < \varphi_3 < \pi$*, weights* $p, q \in (0, 1)$ *with* $0 < p + q < 1$*, and consider the directional distribution* $G := p\delta_{\varphi_1} + q\delta_{\varphi_2} + (1-p-q)\delta_{\varphi_3}$*. Then* $p_3(G) = p_3(G_3(p, q))$ *with* $p_3(G_3(p, q))$ *as in Theorem 1.1.*

In analogy with the case of three directions just studied, one can consider a Poisson line tessellation with directional distribution $G_4(p, q, r) := p\delta_0 + q\delta_{\pi/4} + r\delta_{\pi/2} + (1-p-q-r)\delta_{(3\pi)/4}$ with weights $0 < p, q, r < 1$ satisfying $0 < p + q + r < 1$, as shown in Figure 1(b). The corresponding triangle probability is in this case given by

$$\begin{aligned} &p_3(G_4(p, q, r)) \\ &\quad = \frac{2p}{\sqrt{2}p + 2q + \sqrt{2}r - \sqrt{2}p^2 - 2q^2 - \sqrt{2}r^2 - 2pq + (2-2\sqrt{2})pr - 2qr} \\ &\qquad \times \left(\frac{3qr}{2 + p(-2+\sqrt{2}) - q + r(-2+\sqrt{2})} + \frac{3\sqrt{2}q(1-p-q-r)}{2 + (-2+\sqrt{2})p + r(-2+2\sqrt{2})} \right. \\ &\qquad\quad \left. + \frac{2r\sqrt{2}(1-p-q-r)}{\sqrt{2} + p(2-\sqrt{2}) + \sqrt{2}q + r(2-\sqrt{2})} \right), \end{aligned}$$

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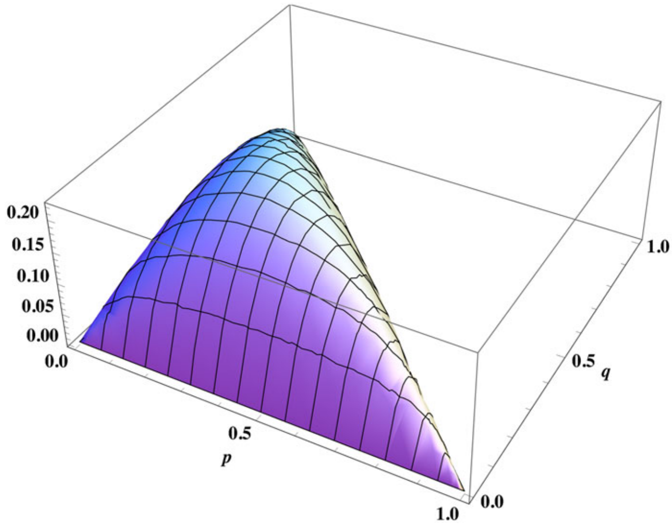

FIGURE 2. A plot of $p_3(G_3(p, q))$.

as demonstrated in [8]. Since the triangle probabilities for five or more directions with arbitrary weights become increasingly more involved, from now on we concentrate on the special case where all weights are equal. Namely, for integers $k \geq 3$ we take as directional distribution the probability measure

$$G_k := \frac{1}{k} \sum_{\ell=0}^{k-1} \delta_{(\ell\pi)/k},$$

which for $k = 3$ and $k = 4$ reduces to $G_3(1/3, 1/3)$ and $G_4(1/4, 1/4, 1/4)$, respectively. In other words, G_k puts weight $1/k$ onto k equally spread directions. The Poisson line tessellation induced by such a directional distribution is G_k-pseudo-isotropic in the sense that it is invariant under rotations in the plane whose angle is an integer multiple of π/k. In our second result we determine the triangle probabilities $p_3(G_k)$.

Theorem 1.2. *For $k \geq 3$ we have that*

$$p_3(G_k) = \frac{4}{k} \tan^2 \frac{\pi}{2k} \sum_{i=1}^{k-2} \left[(k-i) \sum_{j=1}^{k-i-1} \frac{\sin \frac{i\pi}{k} \, \sin \frac{j\pi}{k} \, \sin \frac{(i+j)\pi}{k}}{\sin \frac{i\pi}{k} + \sin \frac{j\pi}{k} + \sin \frac{(i+j)\pi}{k}} \right].$$

The exact and approximate values for $p_3(G_k)$ for $k \in \{3, 4, 5, 6\}$ are summarized in the table in Figure 3(a), some further values are visualized in Figure 3(b). The latter also shows that, as $k \to \infty$, the value $p_3(G_k)$ tends to $2 - \pi^2/6 = p_3(G_{\text{unif}})$, the triangle probability appearing in the isotropic case. This observation is confirmed in the following corollary.

Corollary 1.2. *For $k \geq 3$, let $p_3(G_k)$ be as in Theorem 1.2. Then $\lim_{k\to\infty} p_3(G_k) = p_3(G_{\text{unif}})$.*

The proof of both Theorem 1.1 and Theorem 1.2 is based on the sampling procedure for the typical cell of stationary Poisson line tessellation developed in [5]. It generalizes one of the stochastic constructions described in [12] to general directional distributions. To keep the paper reasonably self-contained we recall the relevant elements of this construction in the next

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(a)

$k = 3$	$k = 4$	$k = 5$	$k = 6$
$\frac{2}{9}$	$4(5\sqrt{2} - 7)$	$\frac{32}{\sqrt{5}} - 14$	$\frac{4}{3}(70\sqrt{3} - 121)$
0.2222	0.2843	0.3108	0.3274

(b)

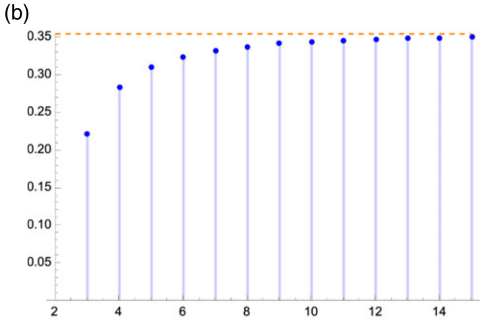

FIGURE 3. (a) Concrete values of $p_3(G_k)$ for $k = 3, 4, 5, 6$. (b) Visualization of values of $p_3(G_k)$ for $k = 3, 4, \ldots, 15$ and the limiting value $p_3(G_{\mathrm{unif}})$ (dashed line).

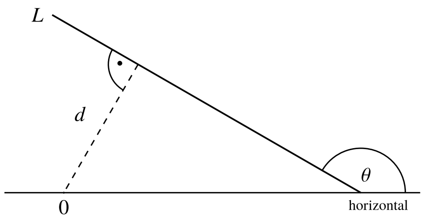

FIGURE 4. A line L parametrized by (d, θ).

section. Then in Section 3 we show how the probability $p_3(G_{\mathrm{unif}})$ can be determined using this sampling procedure. Using the same approach, the proofs of Theorem 1.1, Theorem 1.2, and Corollary 1.2 are the subject of Section 4. The final section of this paper provides an alternative proof of Miles's result (1.1) regarding the proportion of triangles in an isotropic Poisson line tessellation.

2. Sampling random triangles

In this paper a line is parametrized by a pair (θ, d), where $d \in \mathbb{R}$ is the signed distance of the line to the origin and $\theta \in [0, \pi)$ is the north-east angle this line makes with the horizontal; see Figure 4. We refer to θ as the orientation angle of the line.

Following [5], it will turn out to be convenient to extend the range of the possible orientation angles to the larger interval $[-\pi, \pi)$, where negative angles should be thought of modulo π. For example, we identify the orientation angles $-\pi/3$ and $2\pi/3$.

Throughout the remainder of this work, we denote random variables by a capital letter and their realizations by small ones; for example, Φ denotes a random angle and ϕ a given realization.

2.1. General facts about Poisson line processes

We consider a stationary Poisson line process $X = X(\gamma, G)$ with intensity $\gamma > 0$ and directional distribution G. We assume G to be non-degenerate, meaning that $G(\theta) < 1$ for each $\theta \in [0, \pi)$. The following facts are taken from [5], but see also [15].

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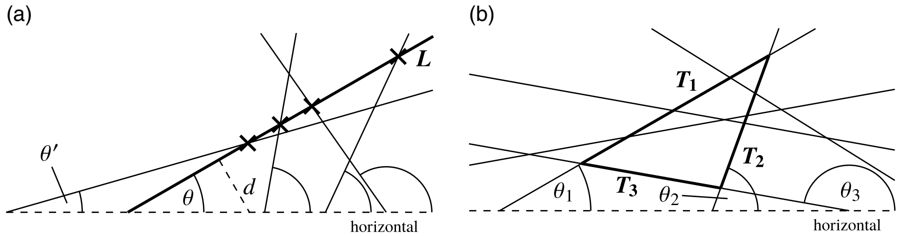

FIGURE 5. (a) Intersection of a Poisson line process X (thin lines) with a fixed line $L = (\theta, d)$ (thick). (b) Lines of X (thin lines) intersecting triangle sides T_1 and T_2 but not T_3.

Intersection with a fixed line. Let L be a fixed line with orientation angle $\theta \in [0, \pi)$. Its intersection with X is a stationary Poisson point process on L with intensity $\gamma\lambda(\theta)$, where

$$\lambda(\theta) := \int_{[0,\pi)} |\sin(\theta - \theta')|\, G(\mathrm{d}\theta'), \tag{2.1}$$

see Figure 5(a). Furthermore, the random orientation angles of the lines associated with these points of intersection are independent and identically distributed with common conditional density

$$\theta' \mapsto \frac{1}{\lambda(\theta)} |\sin(\theta - \theta')|, \quad 0 \le \theta' < \pi,$$

with respect to G.

Intersection of two random lines. Let L and L' be two different lines from X, and let (Θ, Θ') be the two orientation angles at the intersection point $L \cap L'$. Then the pair (Θ, Θ') has joint density

$$(\theta, \theta') \mapsto \frac{1}{\lambda} |\sin(\theta - \theta')|, \quad 0 \le \theta, \theta' < \pi, \tag{2.2}$$

with respect to the product measure $G \otimes G$ on $[0, \pi) \times [0, \pi)$, where

$$\lambda := \int_0^{\pi} \lambda(\theta)\, G(\mathrm{d}\theta). \tag{2.3}$$

Intersection with a triangle. Consider an arbitrary triangle T in the plane with sides T_1, T_2, and T_3 having lengths t_1, t_2, t_3 and whose supporting lines have orientation angles θ_1, θ_2, θ_3, respectively. Then the number of lines of X intersecting T but do not intersect T_3 has a Poisson distribution with mean

$$\frac{\gamma}{2}(t_1\lambda(\theta_1) + t_2\lambda(\theta_2) - t_3\lambda(\theta_3));$$

see Figure 5(b).

2.2. Stochastic construction of a typical triangle

A stochastic construction of the typical cell of a stationary Poisson line tessellation induced by a Poisson line process X with intensity $\gamma > 0$ and a general directional distribution G was introduced in [5], adopting previously developed methods of [12] for the isotropic case. We

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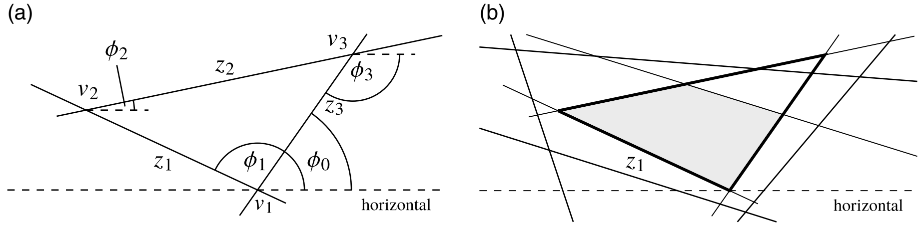

FIGURE 6. (a) Construction of the random triangle Δ. (b) The random triangle Δ (thick lines) and the random line process X'' (thin lines). The intersection (2.5) is the shaded polygon.

rephrase it here in the special case of a triangle, that is, we describe the distribution of the typical cell given that it is a triangle; for brevity we refer to it as the typical triangle. Formally, the distribution P_Z of the typical cell Z of the Poisson line tessellation induced by X is given as follows. Namely, if for a polygon $c \subset \mathbb{R}^2$, $m(c)$ is the lexicographically smallest vertex, the distribution P_Z of the random polygon Z is given by

$$P_Z(\,\cdot\,) := \frac{1}{\mathbb{E}\sum_{c\,:\,m(c)\in[0,1]^2} 1}\,\mathbb{E}\sum_{c\,:\,m(c)\in[0,1]^2} \mathbf{1}\{c - m(c) \in \,\cdot\,\}, \tag{2.4}$$

where each sum runs over all cells c of the Poisson line tessellation with $m(c) \in [0, 1]^2$ (or any other Borel set with unit area). The distribution of the typical triangle is then the conditional distribution $P_Z(\cdot|Z \text{ is a triangle})$.

Starting with the lexicographically smallest vertex of the typical triangle, we label the vertices consecutively in clockwise direction by v_1, v_2, v_3. For $i \in \{1, 2, 3\}$, let z_i be the length of the segment $\overline{v_i v_{i+1}}$, where we formally put $v_4 := v_1$. Moreover, we denote the angle between $\overline{v_i v_{i+1}}$ and the eastern horizontal at v_i by ϕ_i; see Figure 6(a). Hence ϕ_0 denotes the initial angle. The typical triangle is completely determined by the 4-tuple $(\Phi_0, \Phi_1, Z_1, \Phi_2)$; all other angles and edge lengths (especially Φ_3, Z_2, and Z_3) can be computed from these data.

We shall now describe the (conditional) distribution of the random variables Φ_0, Φ_1, Z_1, and Φ_2, which are clearly dependent.

- The joint density with respect to $G \otimes G$ of (Φ_0, Φ_1) is given by (2.2).
- Given $\Phi_1 = \phi_1$, the intersection of X with the line having orientation angle Φ_1 is a stationary Poisson point process with intensity $\lambda(\phi_1)$ according to (2.1). The distance from v_1 to the first point of this process above the horizontal line is exponentially distributed with mean $\lambda(\phi_1)$. As a result, the conditional Lebesgue density of Z_1 given $\Phi_1 = \phi_1$ equals
$$z_1 \mapsto \lambda(\phi_1)\, \mathrm{e}^{-\lambda(\phi_1) z_1}, \quad z_1 > 0.$$
- Given $\Phi_1 = \phi_1$ and $Z_1 = z_1$, the random variable Φ_2 has density
$$\phi_2 \mapsto \frac{\sin(\phi_1 - \phi_2)}{\int_{a(z_1)}^{\phi_1} \sin(\phi_1 - \theta)\, G(\mathrm{d}\theta)}, \quad a(z_1) \le \phi_2 < \phi_1,$$
with respect to G. Here $a(z_1)$ is given by
$$a(z_1) := \arctan\left(\frac{y_1}{x_1}\right) - \pi,$$
where (x_1, y_1) are the coordinates of the first vertex v_1.

The construction just described leads to a random triangle Δ in the plane, which is determined by the four random variables Φ_0, Φ_1, Φ_2, and Z_1. It has the conditional distribution of the typical cell $Z = Z(\gamma, G)$, given that Z is a triangle. To obtain from Δ the (unconditional) typical cell $Z = Z(\gamma, G)$, let X' be an independent stationary Poisson line process with intensity γ and directional distribution G. From X' we remove all lines hitting the first edge of Δ with length Z_1 and call X'' the resulting collection of random lines; see Figure 6(b). Then the typical cell Z has the same distribution as

$$\Delta \cap \bigcap_{L \in X''} L^+, \tag{2.5}$$

where for each line L, L^+ denotes the closed half-space bounded by L and containing the origin; see [5].

3. Triangle probability in the isotropic case

In this section we consider the isotropic case and indicate how to compute $p_3 := p_3(G_{\text{unif}})$ using the stochastic construction outlined in the previous section. So, let $G := G_{\text{unif}}$ be the uniform distribution on $[0, \pi)$ with constant density $\theta \mapsto \theta/\pi$. We also recall our choice $\gamma = 1$. It follows from (2.1) and (2.3) that

$$\lambda = \lambda(\phi) = 1/\pi \int_0^\pi |\sin(\phi - \theta)| \, \mathrm{d}\theta = 2/\pi \quad \text{for any } \phi \in [0, \pi).$$

Due to the rotation invariance of the Poisson line tessellation in the isotropic case, the distribution of the initial angle Φ_0 is irrelevant and we can just choose $\Phi_0 \equiv 0$ in the construction of the typical triangle for simplicity. Then the following hold.

- The random variable Φ_1 has density $\phi_1 \mapsto (\pi - \phi_1)\sin(\phi_1)/\pi$, for $0 \le \phi_1 < \pi$, which is the marginal density of the pair (Φ_0, Φ_1) with respect to the second coordinate.
- The random variable Z_1 is independent of Φ_1 and has density $z_1 \mapsto 2\mathrm{e}^{-2z_1/\pi}/\pi$, for $z_1 > 0$.
- The random variable Φ_2 only depends on Φ_1 and has conditional density $\phi_2 \mapsto \sin(\phi_1 - \phi_2)/2$, given $\Phi_1 = \phi_1$. Here $\phi_1 - \pi \le \phi_2 < 0$, since $x_1 = z_1 \cos\phi_1$, $y_1 = z_1 \sin\phi_1$, which in turn implies $a(z_1) = \phi_1 - \pi$, independently of z_1.

Given these distributions, the probability p_3 that the typical cell is a triangle can now be written as follows:

$$p_3 = \int_0^\pi \int_0^\infty \int_0^{\phi_1 - \pi} \mathrm{e}^{-(\lambda(\phi_2)z_2 + \lambda(\phi_3)z_3 - \lambda(\phi_1)z_1)/2} \\ \times \frac{\pi - \phi_1}{\pi} \sin\phi_1 \times \frac{2}{\pi} \mathrm{e}^{-2z_1/\pi} \times \frac{1}{2} \sin(\phi_1 - \phi_2) \, \mathrm{d}\phi_2 \, \mathrm{d}z_1 \, \mathrm{d}\phi_1.$$

In fact, in order to ensure that the typical cell is a triangle, we need to ensure that after the stochastic construction of the typical triangle, given $\Phi_1 = \phi_1$, $Z_1 = z_1$, and $\Phi_2 = \phi_2$, the two edges with length z_2 and z_3 are not intersected by lines of the random line process X''; recall (2.5). Thus, by the intersection-with-a-triangle property, the above event has probability $\exp(-(\lambda(\phi_2)z_2 + \lambda(\phi_3)z_3 - \lambda(\phi_1)z_1)/2)$, which is the probability that a Poisson random variable with mean $(\lambda(\phi_2)z_2 + \lambda(\phi_3)z_3 - \lambda(\phi_1)z_1)/2$ takes the value zero. The other terms in the above integral representation are just the densities of the random variables Φ_1, Z_1, and Φ_2.

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It is not difficult to verify that

$$z_2 = -z_1 \frac{\sin\phi_1}{\sin\phi_2}, \quad z_3 = z_2 \cos\phi_1 - z_1 \frac{\sin\phi_1}{\sin\phi_2} \cos\phi_2 \quad \text{and} \quad \phi_3 = \phi_0 - \pi, \tag{3.1}$$

which yields

$$z_1 + z_2 + z_3 = z_1 \frac{\sin\phi_2 - \sin\phi_1 - \sin(\phi_1 - \phi_2)}{\sin\phi_2}.$$

Inserting this together with the values of $\lambda(\phi_1) = \lambda(\phi_2) = \lambda(\phi_3) = 2/\pi$, we arrive at

$$p_3 = \int_0^{\pi} \frac{\pi - \phi_1}{\pi} \int_{\phi_1 - \pi}^{0} \frac{\sin\phi_1 \sin\phi_2 \sin(\phi_1 - \phi_2)}{\sin\phi_2 - \sin\phi_1 - \sin(\phi_1 - \phi_2)} \,\mathrm{d}\phi_2 \,\mathrm{d}\phi_1, \tag{3.2}$$

where we have already carried out the integration with respect to z_1. Rewriting the integrand by means of trigonometric identities eventually leads to (1.1); all details of the computation are contained in [8].

4. Proofs

4.1. Triangle probability in the $G_3(p, q)$ case: Proof of Theorem 1.1

In this section we compute the triangle probability $p_3 := p_3(G_3(p, q))$ if the underlying directional distribution is given by (1.2), again using the stochastic construction of the typical cell. We recall that we choose $\gamma = 1$ as our intensity.

Before we actually compute p_3, we deal with the possible constructions for triangles with only three edge directions corresponding to the orientation angles 0, $\pi/3$, and $2\pi/3$. In fact we only have two ways to construct a triangle with these orientation angles, as demonstrated in Figure 7. Further, writing G for $G_3(p, q)$ for brevity, we can now compute

$$\begin{aligned}
\lambda(0) &= \int_0^{\pi} |\sin\theta| \, G(\mathrm{d}\theta) = q \sin\frac{\pi}{3} + (1 - p - q) \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}(1 - p), \\
\lambda\left(\frac{\pi}{3}\right) &= \int_0^{\pi} \left|\sin\left(\theta - \frac{\pi}{3}\right)\right| G(\mathrm{d}\theta) \\
&= p\left|\sin\left(-\frac{\pi}{3}\right)\right| + (1 - p - q)\left|\sin\left(\frac{2\pi}{3} - \frac{\pi}{3}\right)\right| = \frac{\sqrt{3}}{2}(1 - q), \\
\lambda\left(\frac{2\pi}{3}\right) &= \int_0^{\pi} \left|\sin\left(\theta - \frac{2\pi}{3}\right)\right| G(\mathrm{d}\theta) \\
&= p\left|\sin\left(-\frac{2\pi}{3}\right)\right| + q\left|\sin\left(\frac{\pi}{3} - \frac{2\pi}{3}\right)\right| = \frac{\sqrt{3}}{2}(p + q),
\end{aligned}$$

according to (2.1), which implies that

$$\begin{aligned}
\lambda &= \int_0^{\pi} \lambda(\theta) \, G(\mathrm{d}\theta) \\
&= p\lambda(0) + q\lambda\left(\frac{\pi}{3}\right) + (1 - p - q)\lambda\left(\frac{2\pi}{3}\right) \\
&= \sqrt{3}(p + q - p^2 - q^2 - pq).
\end{aligned}$$

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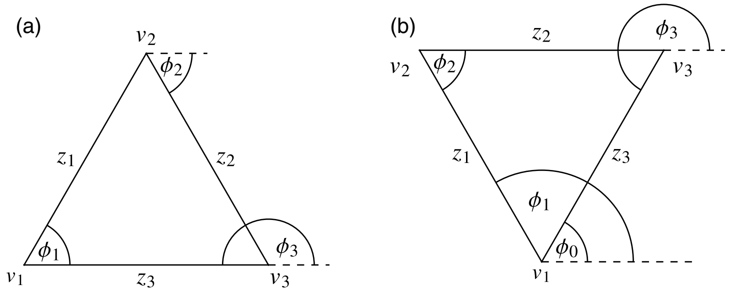

FIGURE 7. The two possible triangles in a Poisson line tessellation with directional distribution $G_3(p, q)$: (a) with angles $\phi_0 = 0$, $\phi_1 = \frac{1}{3}\pi$, and $\phi_2 = -\frac{1}{3}\pi$, (b) with angles $\phi_0 = \frac{1}{3}\pi$, $\phi_1 = \frac{2}{3}\pi$, and $\phi_2 = 0$.

Moreover, from (2.2) it follows that the pair (Φ_0, Φ_1) has joint density

$$(\phi_0, \phi_1) \mapsto \frac{2}{\sqrt{3}(p+q-p^2-q^2-pq)} \sin(\phi_0 - \phi_1), \quad 0 \le \phi_0, \phi_1 < \pi,$$

with respect to $G \otimes G$. Given $\Phi_1 = \phi_1$, the random variable Z_1 is exponentially distributed with mean $\lambda(\phi_1)$. Finally, as in the isotropic case, we have $a(z_1) = \phi_1 - \pi$ and so the random variable Φ_2 has conditional density

$$\phi_2 \mapsto \frac{1}{\lambda(\phi_1)} \sin(\phi_1 - \phi_2), \quad \phi_1 - \pi \le \phi_2 < 0,$$

with respect to G, given $\Phi_1 = \phi_1$.

With the same argument as in the isotropic case, we can now represent the triangle probability as follows:

$$\begin{aligned} p_3 = &\int_0^\pi \int_0^\pi \int_{\phi_1-\pi}^0 \int_0^\infty \mathrm{e}^{-(\lambda(\phi_2)z_2+\lambda(\phi_3)z_3-\lambda(\phi_1)z_1)/2} \\ &\times \frac{2}{\sqrt{3}(p+q-p^2-q^2-pq)} \sin(\phi_0-\phi_1) \\ &\times \frac{1}{\lambda(\phi_1)} \sin(\phi_1-\phi_2) \times \lambda(\phi_1)\,\mathrm{e}^{-\lambda(\phi_1)z_1}\,\mathrm{d}z_1 G(\mathrm{d}\phi_2)(G\otimes G)(\mathrm{d}(\phi_0,\phi_1)) \\ = &\int_0^\pi \int_0^\pi \int_{\phi_1-\pi}^0 \int_0^\infty \mathrm{e}^{-(\lambda(\phi_1)z_1+\lambda(\phi_2)z_2+\lambda(\phi_3)z_3)/2} \times \frac{2}{\sqrt{3}(p+q-p^2-q^2-pq)} \\ &\times \sin(\phi_0-\phi_1)\sin(\phi_1-\phi_2)\,\mathrm{d}z_1 G(\mathrm{d}\phi_2)(G\otimes G)(\mathrm{d}(\phi_0,\phi_1)); \end{aligned}$$

the term $\mathrm{e}^{-(\lambda(\phi_2)z_2+\lambda(\phi_3)z_3-\lambda(\phi_1)z_1)/2}$ represents the probability that after the stochastic construction of the typical triangle the random line process X'' does not intersect the two edges with lengths z_2 and z_2, whereas the other terms are the (conditional) densities of (Φ_0, Φ_1), Φ_2, and Z_1. From the discussion at the beginning of this section we know the three outer integrals are

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just a sum of two terms corresponding to the following angles:

$$\begin{aligned}
&\text{case 1} \quad \phi_0 = 0, \quad \phi_1 = \frac{\pi}{3}, \quad \phi_2 = \frac{2\pi}{3}, \quad \phi_3 = -\pi, \\
&\text{case 2} \quad \phi_0 = \frac{\pi}{3}, \quad \phi_1 = \frac{2\pi}{3}, \quad \phi_2 = 0, \quad \phi_3 = -\frac{2\pi}{3}.
\end{aligned}$$

In both cases, using (3.1), we conclude that, given $\Phi_1 = \phi_1$, $Z_1 = z_1$, and $\Phi_2 = \phi_2$, we have $z_1 = z_2 = z_3$, formally confirming that we are dealing with regular triangles. Moreover, in both cases we have $\lambda(\phi_1) + \lambda(\phi_2) + \lambda(\phi_3) = \sqrt{3}$, implying that

$$\int_0^\infty \mathrm{e}^{-(\lambda(\phi_1)z_1+\lambda(\phi_2)z_2+\lambda(\phi_3)z_3)/2}\,\mathrm{d}z_1 = \int_0^\infty \mathrm{e}^{-\sqrt{3}z_1/2}\,\mathrm{d}z_1 = \frac{2}{\sqrt{3}}.$$

Hence

$$\begin{aligned}
p_3 = \frac{2}{\sqrt{3}} \times \frac{2}{\sqrt{3}(p+q-p^2-q^2-pq)} \\
\times \int_0^\pi \int_0^\pi \int_{\phi_1-\pi}^0 \sin(\phi_0-\phi_1)\sin(\phi_1-\phi_2)\,G(\mathrm{d}\phi_2)(G\otimes G)(\mathrm{d}(\phi_0,\phi_1)).
\end{aligned}$$

Finally, in case 1, which has weight $pq(1-p-q)$, the integrand equals $3/4$, and in case 2, which has weight $p(1-p-q)p$, the integrand equals $3/4$ as well. This eventually leads to

$$p_3 = \frac{2}{\sqrt{3}} \times \frac{2}{\sqrt{3}(p+q-p^2-q^2-pq)} \times 2 \times pq(1-p-q) \times \frac{3}{4} = \frac{2pq(1-p-q)}{p+q-p^2-q^2-pq}$$

and concludes the proof of the first part of Theorem 1.1.

For the second part, define the function

$$F(p,q) := \frac{2pq(1-p-q)}{p+q-p^2-q^2-pq}$$

on the domain $D := \{(p,q) \in (0,1)^2 : 0 < p+q < 1\}$ whose gradient is

$$\mathrm{grad}F(p,q) = \frac{2}{(p+q-p^2-q^2-pq)^2}(q(1-p-q)-pq,\, p(1-p-q)-pq).$$

Solving $\mathrm{grad}F(p,q) = (0,0)$ leads to the only solution $(p,q) = (1/3, 1/3)$ on D. One can easily check that this is indeed the global maximum of $F(p,q)$ on D. Since $p_3(G_3(1/3, 1/3)) = 2/9$, the proof of Theorem 1.1 is complete. □

4.2. Triangle probability for three general directions: Proof of Corollary 1.1

Recall the definition of the directional distribution G from the statement of Corollary 1.1 and define the unit vectors

$$\begin{aligned}
v_1 &:= (0,1), & v_2 &:= \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), & v_3 &:= \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \\
w_1 &:= (\cos\varphi_1, \sin\varphi_1), & w_2 &:= (\cos\varphi_2, \sin\varphi_2), & w_3 &:= (\cos\varphi_3, \sin\varphi_3).
\end{aligned}$$

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Then we can find a non-degenerate affine map $A:\mathbb{R}^2\to\mathbb{R}^2$ which satisfies $A(v_i)=w_i$ for $i\in\{1, 2, 3\}$. Applying A to a Poisson line tessellation X with intensity one and directional distribution $G_3(p, q)$ leads again to a Poisson line tessellation AX by the well-known mapping property of general Poisson processes. By definition of A, its directional distribution equals $G=AG_3(p, q)$ and the intensity is given by the determinant of A. Moreover, the application of A leaves invariant the number of vertices of each of the cells of X. As a consequence, the two tessellations X and AX have the same proportion of triangles. As this quantity only depends on the directional distribution and not on the intensity parameter, it follows that $p_3(G_3(p, q))=p_3(G)$. □

4.3. Triangle probability in the case of k directions: Proof of Theorem 1.2

Recall the construction of a typical triangle based on the random angles Φ_0, Φ_1, Φ_2 and the random edge length Z_1. Since the Poisson line tessellation with directional distribution G_k is G_k-pseudo-isotropic, the initial angle Φ_0 is irrelevant and we can just take $\Phi_0=0$. Moreover, recall that $\Phi_3=\Phi_0-\pi=-\pi$. It is now a crucial observation that the stochastic construction described above leads to a triangle if and only if

$$(\phi_1, \phi_2)\in\left\{\left(\frac{i\pi}{k}, -\frac{j\pi}{k}\right): \begin{array}{c}1\le i\le k-2\\ 1\le j\le k-i-1\end{array}\right\},$$

since the angle sum of a triangle is equal to π and since we require the vertex v_1 to be the lexicographically smallest vertex of the triangle. Moreover, for fixed $1\le i\le k-2$ each such triangle can be rotated by the angles $0, \pi/k, \dots, (k-i-1)\pi/k$ to yield another admissible triangle.

We now determine the distribution of the relevant random variables and start with Φ_1. According to (2.1) and using the identity for sums of sines in arithmetic progressions from [7] (with $a=0$ and $d=\pi/k$ there), we have

$$\lambda(0)=\int_0^\pi |\sin(\theta)|\, G_k(\mathrm{d}\theta)=\frac{1}{k}\sum_{\ell=0}^{k-1}\sin\frac{\ell\pi}{k}=\frac{1}{k}\frac{\sin\frac{(k-1)\pi}{2k}}{\sin\frac{\pi}{2k}}=\frac{1}{k}\cot\frac{\pi}{2k},$$

and because of G_k-pseudo-isotropy we also have $\lambda(\pi/k)=\cdots=\lambda((k-1)\pi/k)=\lambda(0)$. Thus it follows from (2.3) that

$$\lambda=\int_0^\pi \lambda(\theta)\, G_k(\mathrm{d}\theta)=\frac{1}{k}\cot\frac{\pi}{2k}.$$

We can now conclude from (2.2) that the pair (Φ_0, Φ_1) has joint density

$$(\phi_0, \phi_1)\mapsto\frac{2k}{\cot\frac{\pi}{2k}}\sin(\phi_1-\phi_0), \quad 0\le\phi_0,\ \phi_1<\pi,$$

with respect to $G_k\otimes G_k$. Integration with respect to ϕ_0 now yields the marginal density

$$\phi_1\mapsto\frac{2k}{\cot\frac{\pi}{2k}}\int_0^{\pi-\phi_1}\sin(\phi_1-\phi_0)\, G_k(\mathrm{d}\phi_0)=\frac{2\Sigma_k(\phi_1)}{\cot\frac{\pi}{2k}}\sin(\phi_1), \quad 0\le\phi_1<\pi,$$

of Φ_1 with respect to G_k, where

$$\Sigma_k(\phi_1):=\sum_{\substack{\phi_0\in\{0,\pi/k,\dots,((k-1)\pi)/k\}\\ \phi_0<\pi-\phi_1}} 1.$$

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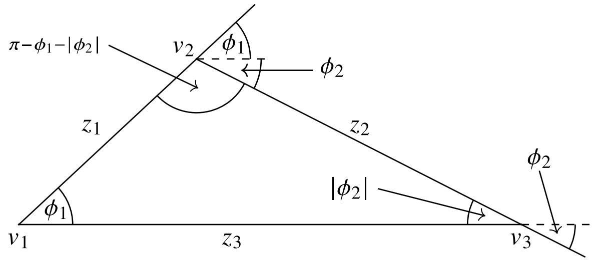

FIGURE 8. Determination of z_2 and z_3.

Note that $\Sigma_k(\phi_1) = k - \ell$ if $\phi_1 = \ell\pi/k$ for some $\ell \in \{0, \ldots, k-1\}$. The distribution of Z_1 is an exponential distribution with mean $(\cot \pi/2k)/k$ and so Z_1 has density

$$z_1 \mapsto \frac{1}{k} \cot \frac{\pi}{2k} \exp\left(-\frac{1}{k} \cot \frac{\pi}{2k} z_1\right), \quad z_1 > 0,$$

with respect to the Lebesgue measure. Finally, we deal with the conditional distribution of Φ_2 given Φ_1. As above, we have that the conditional density with respect to G_k of Φ_2 given $\Phi_1 = \phi_1$ equals

$$\phi_2 \mapsto \frac{\sin(\phi_1 - \phi_2)}{\int_{\phi_1-\pi}^{\phi_1} \sin(\phi_1 - \phi)\, G_k(\mathrm{d}\phi)}, \quad \phi_1 - \pi \le \phi_2 < \phi_1.$$

Since the integral in the denominator is just $\lambda(\phi_1)$, we arrive at

$$\phi_2 \mapsto \frac{k}{\cot \frac{\pi}{2k}} \sin(\phi_1 - \phi_2), \quad \phi_1 - \pi \le \phi_2 < \phi_1,$$

for the conditional density of Φ_2.

As in the two previous sections, we can now express $p_3 := p_3(G_k)$ as follows:

$$\begin{aligned} p_3 &= \int_0^\pi \int_{\phi_1-\pi}^0 \int_0^\infty \frac{2\Sigma_k(\phi_1)}{\cot \frac{\pi}{2k}} \sin(\phi_1) \times \frac{1}{k} \cot \frac{\pi}{2k} \exp\left(-\frac{1}{k} \cot \frac{\pi}{2k} z_1\right) \\ &\quad \times \frac{k}{\cot \frac{\pi}{2k}} \sin(\phi_1 - \phi_2) \times \mathrm{e}^{-(\lambda(\phi_2)z_2 + \lambda(\phi_3)z_3 - \lambda(\phi_1)z_1)/2}\, \mathrm{d}z_1 G_k(\mathrm{d}\phi_2) G_k(\mathrm{d}\phi_1) \\ &= \frac{2}{\cot \frac{\pi}{2k}} \int_0^\pi \int_{\phi_1-\pi}^0 \Sigma_k(\phi_1) \sin(\phi_1) \sin(\phi_1 - \phi_2) \\ &\quad \times \int_0^\infty \exp\left(-\frac{1}{2k} \cot \frac{\pi}{2k}(z_1 + z_2 + z_3)\right) \mathrm{d}z_1 G_k(\mathrm{d}\phi_2) G_k(\mathrm{d}\phi_1), \end{aligned}$$

where in the last step we used that $\lambda(\phi) = \lambda(0)$ for all angles ϕ in the support of G_k.

To determine z_2 and z_3 we can use the law of sines as illustrated in Figure 8.

If $\phi_1 = i\pi/k$, $1 \le i \le n-2$, and $\phi_2 = -j\pi/k$, $1 \le j \le n-i-1$, this yields

$$z_2 = z_1 \frac{\sin \frac{i\pi}{k}}{\sin \frac{j\pi}{k}} \quad \text{and} \quad z_3 = z_1 \frac{\sin \frac{(i+j)\pi}{k}}{\sin \frac{j\pi}{k}}.$$

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Thus

$$\frac{1}{2k}\cot\frac{\pi}{2k}(z_1+z_2+z_3)=\frac{1}{2k}\cot\frac{\pi}{2k}\frac{\sin\frac{i\pi}{k}+\sin\frac{j\pi}{k}+\sin\frac{(i+j)\pi}{k}}{\sin\frac{j\pi}{k}}z_1,$$

and the integral with respect to z_1 evaluates to

$$\int_0^\infty \exp\biggl(-\frac{1}{2k}\cot\frac{\pi}{2k}(z_1+z_2+z_3)\biggr)\,\mathrm{d}z_1=\frac{2k\sin\frac{j\pi}{k}}{\cot\frac{\pi}{2k}\bigl(\sin\frac{i\pi}{k}+\sin\frac{j\pi}{k}+\sin\frac{(i+j)\pi}{k}\bigr)}.$$

Plugging this back into the expression for p_3, we see that

$$p_3=\frac{4k}{\cot^2\frac{\pi}{2k}}\frac{1}{k^2}\sum_{i=1}^{k-2}\Sigma_k\Bigl(\frac{i\pi}{k}\Bigr)\sum_{j=1}^{k-i-1}\frac{\sin\frac{i\pi}{k}\,\sin\frac{j\pi}{k}\,\sin\frac{(i+j)\pi}{k}}{\sin\frac{i\pi}{k}+\sin\frac{j\pi}{k}+\sin\frac{(i+j)\pi}{k}}. \tag{4.1}$$

Using $\Sigma_k(i\pi/k)=k-i$, we can complete the proof of Theorem 1.2. □

4.4. The convergence to the isotropic case: Proof of Corollary 1.2

We start with the observation that

$$\frac{4}{k}\tan^2\frac{\pi}{2k}=\frac{\pi^2}{k^3}+\mathrm{O}(k^{-5}), \tag{4.2}$$

as $k\to\infty$. Combining this with the representation for $p_3(G_k)$ in Theorem 1.2 implies

$$\begin{aligned}\lim_{k\to\infty}p_3(G_k)&=\lim_{k\to\infty}\frac{4}{k}\tan^2\frac{\pi}{2k}\sum_{i=1}^{k-2}\Biggl[(k-i)\sum_{j=1}^{k-i-1}\frac{\sin\frac{i\pi}{k}\,\sin\frac{j\pi}{k}\,\sin\frac{(i+j)\pi}{k}}{\sin\frac{i\pi}{k}+\sin\frac{j\pi}{k}+\sin\frac{(i+j)\pi}{k}}\Biggr]\\&=\lim_{k\to\infty}\frac{\pi^2}{k^3}\sum_{i=1}^{k-2}\Biggl[(k-i)\sum_{j=1}^{k-i-1}\frac{\sin\frac{i\pi}{k}\,\sin\frac{j\pi}{k}\,\sin\frac{(i+j)\pi}{k}}{\sin\frac{i\pi}{k}+\sin\frac{j\pi}{k}+\sin\frac{(i+j)\pi}{k}}\Biggr]\\&=\lim_{k\to\infty}\pi^2\frac{1}{k}\sum_{i=1}^{k-2}\Biggl[\Bigl(1-\frac{i}{k}\Bigr)\frac{1}{k}\sum_{j=1}^{k-i-1}\frac{\sin\frac{i\pi}{k}\,\sin\frac{j\pi}{k}\,\sin\frac{(i+j)\pi}{k}}{\sin\frac{i\pi}{k}+\sin\frac{j\pi}{k}+\sin\frac{(i+j)\pi}{k}}\Biggr].\end{aligned}$$

Interpreting the two sums as Riemann sums with $i/k\to\mathrm{d}t$ and $j/k\to\mathrm{d}s$, as $k\to\infty$, and noting that the condition $j\le k-i-1$ asymptotically translates to $s<1-t$, we conclude that

$$\lim_{k\to\infty}p_3(G_k)=\pi^2\int_0^1(1-t)\int_0^{1-t}\frac{\sin(\pi t)\,\sin(\pi s)\,\sin((t+s)\pi)}{\sin(\pi t)+\sin(\pi s)+\sin((t+s)\pi)}\,\mathrm{d}s\,\mathrm{d}t. \tag{4.3}$$

This, up to the substitutions $u=\pi t$ and $v=-\pi s$, is exactly the integral expression for $p_3(G_{\mathrm{unif}})$ we encountered already in (3.2). This completes the argument. □

5. Alternative proof of Miles's result (1.1)

As mentioned in Section 1, it is known from [11] that $p_3(G_{\mathrm{unif}})=2-\pi^2/6$. In this section, we use our Theorem 1.2 to give a 'continuous-mapping-type' argument leading to the same result. Our strategy is to prove that the weak convergence of G_k to G_{unif} implies the convergence of $p_3(G_k)$ to $p_3(G_{\mathrm{unif}})$, as $k\to\infty$. To conclude, we can then use Corollary 1.2, which shows that $p_3(G_{\text{unif}}) = \lim_{k\to\infty} p_3(G_k)$. The value of this limit is given by the integral (4.3), which has the value $2 - \pi^2/6$. The approach can be summarized in the following chain of equalities, in which $\lim^w$ stands for the weak limit of probability measures:

$$p_3\Big(\lim_{k\to\infty}^{w} G_k\Big) \overset{\text{shown below}}{=} \lim_{k\to\infty} p_3(G_k) \overset{\text{Corollary 1.2}}{=} p_3(G_{\text{unif}}) \overset{(1.1)}{=} 2 - \frac{\pi^2}{6}.$$

To prove the first equality, we recall that the weak convergence of G_k to G_{unif} implies the weak convergence of the product measures $G_k \otimes G_k \otimes G_k$ to $G_{\text{unif}} \otimes G_{\text{unif}} \otimes G_{\text{unif}}$; see [1, Proposition 2.7.7]. For each $k \geq 3$, the triangle probability $p_3(G_k)$ can be represented as the integral

$$p_3(G_k) = \int_{[0,\pi)\times[0,\pi)\times[0,\pi)} f_k(\phi_0, \phi_1, \phi_2)\,(G_k \otimes G_k \otimes G_k)(\mathrm{d}(\phi_0, \phi_1, \phi_2))$$

with the function $f_k : [0, \pi) \times [0, \pi) \times [0, \pi) \to \mathbb{R}$ given by

$$f_k(\phi_0, \phi_1, \phi_2) := 4k^2 \tan^2 \frac{\pi}{2k} T(\phi_0, \phi_1, \phi_2)\mathbf{1}\{\phi_0 < \pi - \phi_1, \phi_1 < \phi_2\},$$

where

$$T(\phi_0, \phi_1, \phi_2) := \frac{\sin(\phi_0 - \phi_1) \sin \phi_1 \ \sin \phi_2 \ \sin(\phi_1 - \phi_2)}{\sin \phi_1 + \sin \phi_2 + \sin(\phi_1 - \phi_2)}.$$

Note that if the integration with respect to ϕ_0 is carried out, we arrive precisely at (4.1). It remains to observe that $4k^2 \tan^2 (\pi/2k) \to \pi^2$ as $k \to \infty$ by (4.2) and that the function $(\phi_0, \phi_1, \phi_2) \mapsto T(\phi_0, \phi_1, \phi_2)\mathbf{1}\{\phi_0 < \pi - \phi_1, \phi_1 < \phi_2\}$ is bounded and $G_{\text{unif}} \otimes G_{\text{unif}} \otimes G_{\text{unif}}$-a.e. continuous on $[0, \pi) \times [0, \pi) \times [0, \pi)$. The result thus follows from [1, Corollary 2.2.10]. □

Acknowledgements

We are grateful to Tom Kaufmann and Daniel Rosen for inspiring ideas and constructive discussions on the subject of this paper. We also thank an Associate Editor and the referee for useful comments and remarks.

Funding information

CT was supported by the DFG priority programme SPP 2265 *Random Geometric Systems*.

Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

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