An extension of the curious binomial identity of Simons

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In 2001 Simons [1] discovered a curious binomial identity which can be written as

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} x^k = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k}\binom{n+k}{k} (x+1)^k. \tag{1}$$

Identity (1) has received the attention of many researchers and proofs by using a variety of techniques has been given by different authors. Chapman [2] gave an elegant and short proof using a generating function method. In [3] Prodinger presented another nice proof of it using the Cauchy integral formula. Wang and Sun [4] gave a very short proof using a special linear transformation on the space of polynomials. Munarini [5], by using the Cauchy integral formula, offered the following generalisation of (1)

$$\sum_{k=0}^{n} \binom{\alpha}{n-k}\binom{\beta+k}{k} x^k y^{n-k} = \sum_{k=0}^{n} (-1)^{n+k} \binom{\beta-\alpha+n}{n-k}\binom{\beta+k}{k} (x+y)^k y^{n-k}. \tag{2}$$

Note that (2) reduces to (1) when we put $\alpha = \beta = n$ and $y = 1$. See also the related papers [6], [7], [8] and [9]. The aim of this short Article is to provide the following extension of identity (1) by means of basic calculus. For any non-negative integers m, n, and any complex number x we have

$$\sum_{k=0}^{n} \binom{n}{k}\binom{m+k}{k} x^k = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k}\binom{m+k}{k} (1+x)^k, \tag{3}$$

where $\binom{m+k}{n}$ is counted as zero when $n > m + k$. We can now give a proof of (3). Our proof is a modification of the method used in [1] and uses the Leibniz rule for the product of two differentiable functions and Taylor's theorem. We believe that this unusual application of calculus will be interesting and useful for undergraduate students. In order to prove this identity we need to define

$$g(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{m+k}{k} x^k, \tag{4}$$

and $k = 0, 1, 2, \dots, n$

$$f_k(x) = x^{m+k} \; (k = 0, 1, 2, \dots, n). \tag{5}$$

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Differentiating (5) m times with respect to x we find

$$f_k^{(m)}(x) = (m + k)(m + k - 1)\dots(k + 1)x^k = \frac{(m + k)!}{k!}x^k,$$

so that

$$\frac{f_k^{(m)}(x)}{m!} = \binom{m + k}{k}x^k. \tag{6}$$

Thus, substituting (6) into (4) we have

$$g(x) = \sum_{j=0}^{n}\binom{n}{j}\frac{f_k^{(j)}(x)}{j!} = \frac{1}{m!}\frac{d^m}{dx^m}\left(x^m\sum_{k=0}^{n}\binom{n}{k}x^k\right)$$

$$= \frac{1}{m!}\frac{d^m}{dx^m}\left[x^m(1 + x)^n\right]. \tag{7}$$

Since g is a polynomial of degree n, we have $g^{(k)}(x) = 0$ for $k \geqslant n + 1$. Thus, its Taylor polynomial at $x = -1$ is

$$g(x) = \sum_{k=0}^{n}\frac{g^{(k)}(-1)}{k!}(1 + x)^k. \tag{8}$$

By (7) we have

$$g^{(k)}(x) = \frac{1}{m!}\frac{d^{m+k}}{dx^{m+k}}\left[x^m(1 + x)^n\right]. \tag{9}$$

For n times differentiable functions f and g the Leibniz's rule implies that their product is also n times differentiable and

$$(f(x)g(x))^{(n)} = \sum_{k=0}^{n}\binom{n}{k}f^{(n-k)}(x)\ g^{(k)}(x).$$

Here we adopt the notation $h(x)^{(n)}$ for $h^{(n)}(x)$. Applying Leibniz rule to (9) we have

$$g^{(k)}(x) = \frac{1}{m!}\sum_{p=0}^{m+k}\binom{m + k}{p}\left(x^m\right)^{(m+k-p)}\left((1 + x)^n\right)^{(p)}. \tag{10}$$

Note that $\left(x^m\right)^{(m+k-p)} = 0$ for $p < k$. Then we can easily show that for $p \geqslant k$

$$\left(x^m\right)^{(m+k-p)} = \frac{m!\,x^{p-k}}{(p - k)!} \text{ and } \left((1 + x)^n\right)^{(p)} = \frac{n!\,(1 + x)^{n-p}}{(n - p)!}.$$

Substituting these two identities in (10) and simplifying the resulting identity one gets that

$$\frac{g^{(k)}(x)}{k!} = \sum_{p=k}^{m+k}\binom{m + k}{p}\binom{p}{k}\binom{n}{p}x^{p-k}(1 + x)^{n-p}.$$

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Putting $x = -1$ gives

$$\frac{g^{(k)}(-1)}{k!} = (-1)^{n+k}\binom{m+k}{n}\binom{n}{k}. \tag{11}$$

Substituting (11) into (8), the conclusion follows. It is worth noting that $\binom{m+k}{n}$ appears on the right-hand side of (3), not $\binom{m+k}{k}$ as in (1). Clearly (3) reduces to (1) when we set $m = n$.

Remark

Formula (2) enables us to derive many binomial coefficient identities by specialising the parameters α and β. If we set $\alpha = n$ and $y = 1$, and note that

$$\binom{\beta}{n-k}\binom{\beta+k}{k} = \binom{n}{k}\binom{\beta+k}{n},$$

we can rewrite (2) as follows: For real numbers β, other than negative integers, integers $n \geqslant 0$, and $x \in \mathbb{R}$,

$$\sum_{k=0}^{n}\binom{n}{k}\binom{\beta+k}{k}x^k = \sum_{k=0}^{n}(-1)^{n+k}\binom{n}{k}\binom{\beta+k}{n}(1+x)^k. \tag{12}$$

(a) If we set $\beta = -\frac{1}{2}$ in (12), we get

$$\sum_{k=0}^{n}\binom{n}{k}\binom{-\frac{1}{2}+k}{k}x^k = \sum_{k=0}^{n}(-1)^{n+k}\binom{n}{k}\binom{-\frac{1}{2}+k}{n}(1+x)^k. \tag{13}$$

We have

$$\binom{-\frac{1}{2}+k}{k} = \frac{\Gamma\left(k+\frac{1}{2}\right)}{k!\,\Gamma\left(\frac{1}{2}\right)} = \frac{1}{4^k}\binom{2k}{k}$$

and

$$\binom{-\frac{1}{2}+k}{k} = \frac{\Gamma\left(k+\frac{1}{2}\right)}{n!\,\Gamma\left(\frac{1}{2}+k-n\right)} = \frac{(-1)^{n+k}\binom{2k}{k}\binom{2n-2k}{n-k}}{4^n\binom{n}{k}},$$

both of which follow from Legendre's duplication and reflection formulas for the classical gamma function Γ. See, for example, [10]. Replacing the last two identities in (13) it follows that

$$\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\frac{x^k}{2^{2k}} = \frac{1}{2^{2n}}\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}(1+x)^k. \tag{14}$$

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(b) For brevity we write $B_k = \binom{2k}{k}$. For the values $x = 0$ and $x = -1$ in (14) we obtain the following interesting identities, respectively:

$$\sum_{k=0}^{n} B_k B_{n-k} = 4^n, \tag{15}$$

and

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{B_k}{4^k} = \frac{B_n}{4^n}.$$

(c) If we set $\beta = -\frac{1}{2} + n$ and $x = 0$ in (12) we obtain

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{-\frac{1}{2} + n + k}{n} = (-1)^n.$$

Using the Legendre duplication formula for the gamma function once more we find immediately that

$$\begin{aligned}\binom{n}{k}\binom{-\frac{1}{2} + n + k}{n} &= \frac{n!}{k!\,(n-k)!} \frac{\Gamma\left(n + k + \frac{1}{2}\right)}{n!\,\Gamma\left(k + \frac{1}{2}\right)} \\ &= \frac{1}{4^n}\binom{2n + 2k}{2k}\binom{2n}{n + k},\end{aligned}$$

which leads to

$$\sum_{k=0}^{n} (-1)^k \binom{2n + 2k}{2k}\binom{2n}{n + k} = (-4)^n.$$

Identity (15) is not new and was posed as a problem in Sved [11]. In the literature, many different proofs of it appeared. For a brief account of this nice identity we refer to [12] and [13].

(d) Putting $\beta = -\frac{1}{2} - n$ in (12) and proceeding as above we get

$$\sum_{k=0}^{n} \frac{(-1)^k}{4^k} \binom{2n}{2k}\binom{2k}{k} x^k = \frac{1}{4^n} \sum_{k=0}^{n} (-1)^k \binom{2n}{k}\binom{4n - 2k}{2n} (1 + x)^k.$$

For $x = -1$ this leads to

$$\sum_{k=0}^{n} \frac{1}{4^k}\binom{2n}{2k}\binom{2k}{k} = \frac{1}{4^n}\binom{4n}{2n}$$

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