8

Dimensional regularization

We shall discuss here the procedure how these divergences can be removed in QCD. Our discussion will be based on the previous QSSR1 book and review in [2,3].

8.1 On some other types of regularization

8.1.1 Pauli–Villars regularization

In QED, one regulates an UV divergent integral using a Pauli–Villars [107] regularization (PVR), by replacing the propagator as:

$$\frac{1}{q^2 - m^2} \to \frac{1}{q^2 - m^2} - \frac{1}{q^2 - \Lambda^2_{\rm UV}} , \tag{8.1}$$

where $\Lambda_{\rm UV}$ is a UV cut-off. PVR respects translational and Lorentz invariance, and, in QED, the gauge invariance. However, the renormalization programme of QED [106] cannot be extended trivially to QCD. PVR, which is successful in QED, is not often convenient. For instance, using PVR, the proof of unitarity for massless Yang–Mills theory is quite cumbersome. For massive Yang–Mills such as the Electroweak Standard Model [61], PVR does not maintain gauge invariance [113].

8.1.2 Analytic regularization

Like the case of PVR, the *analytic regularization* proposed in the literature [114], does not also maintain gauge invariance. It consists by replacing the propagator as:

$$\frac{1}{q^2 - m^2} \to \frac{1}{(q^2 - m^2)^\alpha} , \tag{8.2}$$

where α is a complex number with Re α >1, which ensures the convergence of the integral. The original propagator is recovered for $\alpha \to 1$.

8.1.3 Lattice regularization

Another type of regularization is *the lattice regularization* [115] dedicated to lattice calculations of hadron parameters but not suitable for analytic gauge theories as it breaks translation and Lorentz invariance. It is based on the fact that the space–time is discretized and made of small cells of size a (lattice spacing). Due to the lattice structure of space–time, the short-distance contribution to the space–time is eliminated and then leads to a convergent integral.

8.2 Dimensional regularization

In QCD continuum theory or/and in the Standard Model, one uses instead the method of dimensional regularization and renormalization (so-called *MS* scheme [108–112,123]) which is proven to preserve gauge invariance to all orders of perturbation theory. Its most important feature is the concept of analytical continuation of the dimension of space–time to complex $n(n = 4$ for low-energy space–time). In practice, this means that Dirac algebra, Fierz rearrangments and the momentum integration are done in n dimensions, and then analytically continued to four dimensions.[1] As mentioned in the introduction, in this approach, the IR and UV divergences are transformed into poles in $\epsilon \equiv n - 4$, as we shall see in the following explicit example of the two-point correlator of the pseudoscalar current. However, there are different variants of dimensional regularization, where the difference is due to the definitions of the Dirac matrices used in n dimensions, and in particular, on the one of γ_5 which is more delicate when one works in $n > 4$ space–time dimensions. Among possible others, there are the so-called *naïve dimensional regularization (NDR)* and *'t Hooft-Veltman (HV)* [108,117] schemes, which we shall briefly sketch below.

8.2.1 Naïve dimensional regularization

In this case, only the n-dimension metric tensor satisfying the properties is introduced:

$$g_{\mu\nu} = g_{\nu\mu} , \qquad g_{\mu\rho} g^{\rho}_{\nu} = g_{\mu\nu} , \qquad g^{\mu}_{\mu} = n , \tag{8.3}$$

while the γ matrices obey the same rules as in four dimensions (see Appendix D.5):

$$Tr\, \mathbf{1} = 4 , \qquad \{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} , \qquad \{\gamma_\mu, \gamma_5\} = 0 . \tag{8.4}$$

where γ_5 anti-commutes with the other Dirac matrices. NDR is very convenient and widely used in the literature because of its easy implementation in a software program. The definition of γ_5 in four dimensions given in Eqs. (D.10) and (D.11) has been proven to maintain chiral symmetry to all orders of QCD perturbation series [118]. However, care must be taken when odd parity fermion loops appear in the calculation due to the presence of the parity-violating term $Tr\,(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma)$ [117,119], as in fact, one does not know how to deal with such a term in n dimensions.

[1] Useful packages for doing these n-dimension calculations are given in the Appendices D and F.

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8.2.2 Dimensional reduction for supersymmetry

Dimensional reduction [120] is a variant of dimensional regularization, and is convenient for supersymmetric theories,[2] because the conventional dimensional regularization does not preserve supersymmetry. Indeed, in n dimensions, the numbers of bosonic and fermionic degrees of freedom increases, the Fierz rearrangements need more covariants, while one also has to worry about the supersymmetric anomalies and Ward identities. This is obvious in the superfield language since the integral:

$$\int d^n x (D^\alpha D_\alpha)(\bar{D}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}})\mathcal{L}(x,\ \theta,\ \bar{\theta})\ , \tag{8.5}$$

is θ independent only for $n = 4$ [122]. Here, θ is a four-component anti-commuting variable, D^α is a covariant derivative and x is the space-time variable. The *dimensional reduction* technique is based on analytically continuing the number of co-ordinates and momenta, *but not* the number of components of the fields. In other words, the Dirac algebra should be done in four dimensions but the momentum integration has to be done in $n = 4 - \epsilon$ in order to regulate UV divergences. In particular, the average of the momentum integral should be done in n dimensions:

$$\int \frac{d^n k}{(2\pi)^n} k_\mu k_\nu f(k^2, m^2) = \frac{1}{n} g_{\mu\nu} \int \frac{d^n k}{(2\pi)^n} k^2 f(k^2, m^2)\ . \tag{8.6}$$

More specifically, the tensor metric $\tilde{g}_{\mu\nu}$ is defined in the same way as in Eq. (8.3), except that:

$$\tilde{g}^\mu_\mu = 4\ , \qquad \tilde{g}^\rho_\mu g^\rho_\nu = g_{\mu\nu}\ , \tag{8.7}$$

where the last equality is needed for preserving gauge invariance for $n < 4$.

8.2.3 ’t Hooft-Veltman regularization

The HV rule can be satisfied by introducing a new metric $\hat{g}$ in addition to the previous g n-dimensional and $\tilde{g}$ four-dimensional metrics. In 4–ϵ space–time, one has the same properties as in Eq. (8.3), except that:

$$\hat{g}^\mu_\mu = -\epsilon\ . \tag{8.8}$$

The difference with dimensional reduction is that, instead of the rule in Eq. (8.7), one has:

$$\tilde{g}^\rho_\mu g^\rho_\nu = \tilde{g}_{\mu\nu}\ , \tag{8.9}$$

which does not lead to inconsistencies, while, one also has:

$$\hat{g}^\rho_\mu g^\rho_\nu = \hat{g}_{\mu\nu}\ , \qquad \hat{g}_{\mu\rho}\tilde{g}^\rho_\nu = 0\ . \tag{8.10}$$

[2] For a review on supersymmetry, see e.g. [121].

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The n-dimensional Dirac matrices are now split into 4- and $-\epsilon$-dimensional parts:

$$\gamma_\mu = \tilde{\gamma}_\mu + \hat{\gamma}_\mu \ , \tag{8.11}$$

where γ, $\tilde{\gamma}$ and $\hat{\gamma}$ satisfy the usual commutation rule analogue to the one in Eq. (8.4), but, in addition, one has the novel properties:

$$\{\hat{\gamma}_\mu, \tilde{\gamma}_\nu\} = 0 \ , \qquad \hat{\gamma}_\mu \tilde{\gamma}^\mu = 0 \ , \qquad \tilde{g}^\mu_\nu \hat{\gamma}_\mu = 0 \ , \qquad \hat{g}^\mu_\nu \tilde{\gamma}_\mu = 0 \ . \tag{8.12}$$

The γ_5 matrix can be be introduced [117] which anti-commutes with $\tilde{\gamma}$ but commutes with $\hat{\gamma}$:

$$\gamma_5^2 = 1 \ , \qquad \{\gamma_5, \tilde{\gamma}_\mu\} = 0 \ , \qquad [\gamma_5, \hat{\gamma}_\mu] = 0 \ . \tag{8.13}$$

As γ_5 does not have simple commutation rules, it is important to check that chiral Ward identities are respected at each step of the calculation, where anomalous genuine terms have to be cancelled by the counterterms of the Lagrangian [119,117]. For instance, in practice, one has:

$$\frac{1}{2}(1+\gamma_5)\gamma_\mu(1-\gamma_5) = \tilde{\gamma}_\mu(1-\gamma_5) \ . \tag{8.14}$$

Equivalently, one can represent the γ_5 matrix as:

$$\gamma_5 = \frac{i}{4!}\epsilon_n^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma \ , \tag{8.15}$$

where for $n \geq 4$:

$$\epsilon_n^{\mu\nu\rho\sigma} = \epsilon^{\mu\nu\rho\sigma} \quad \text{for} \quad \mu\nu\rho\sigma = 0,\ldots,3 \ , \qquad \epsilon_n^{\mu\nu\rho\sigma} = 0 \quad \text{for} \quad \mu\nu\rho\sigma > 3 \ . \tag{8.16}$$

It is clear that contrary to the NDR scheme, the HV scheme is more cumbersome, in particular, when one tries to implement it in the computer. Neverthless, it is the only dimensional regularization scheme which has been demonstrated to be consistent [119,117].

8.2.4 Momentum integrals in n dimensions

Let us consider the typical one-loop integral:

$$I(m,r) = \int \frac{d^n k}{(2\pi)^n} \frac{(k^2)^r}{[k^2 - \mathbf{R}^2]^m} \ . \tag{8.17}$$

It is convenient to rotate (Wick's rotation) the path of integration in the complex k_0 plane $[k \equiv (k_0, \vec{k})]$ by $+\pi/2$ without crossing the two poles:

$$k_0 = \pm\sqrt{|\vec{k}|^2 + \mathbf{R}^2} \ . \tag{8.18}$$

Therefore, the k_0 integration has the limits $-i\infty$ to $+i\infty$. Going to the Euclidian space, one can define:

$$k_0 \equiv i\tilde{k}_0 \ , \quad \vec{k} \equiv \vec{\tilde{k}} \quad \text{and} \quad \tilde{k} \equiv (\tilde{k}_0, \vec{\tilde{k}}) \ , \tag{8.19}$$

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such that the $\tilde{k}_0$ integral goes from $-\infty$ to $+\infty$. It is easy to find:

$$I(m,r) = (-1)^{r-m} i \int \frac{d^n\tilde{k}}{(2\pi)^n} \frac{(\tilde{k}^2)^r}{[\tilde{k}^2 - \mathbf{R}^2]^m} . \tag{8.20}$$

Going over polar co-ordinates, one has:

$$\int d^n\tilde{k} = \int_0^\infty \rho^{n-1} d_\rho \int_0^\pi d\theta_{n-1}(\sin\theta_{n-1})^{n-2} \cdots \int_0^\pi d\theta_2(\sin\theta_2) \int_0^{2\pi} d\theta_1 , \tag{8.21}$$

where ρ is the length of the vector $\tilde{k}$. In this way, the integrand of $I(m,r)$ only depends on ρ, and one can perform the angular integration using the formula:

$$\int_0^\pi d\theta\ (\sin\theta)^m = \sqrt{\pi}\frac{\Gamma\left((m+1)/2\right)}{\Gamma\left((m+2)/2\right)} , \tag{8.22}$$

where Γ is the gamma function defined and having the properties in Appendix F. Then, one obtains:

$$I(m,r) = (-1)^{r-m}\, i \frac{2(\pi)^{n/2}}{\Gamma(n/2)} \int_0^\infty d\rho\ \rho^{n-1} \frac{(\rho^2)^r}{[\rho^2 - \mathbf{R}^2]^m} , \tag{8.23}$$

which leads to the basic formula:

$$\begin{aligned} I(m,r) &\equiv \int \frac{d^nk}{(2\pi)^n} \frac{(k^2)^r}{[k^2 - R^2]^m} \\ &= \frac{i}{(16\pi^2)^{n/4}} (-1)^{r-m} (\mathbf{R}^2)^{r-m+n/2} \frac{\Gamma(r+n/2)\Gamma(m-r-n/2)}{\Gamma(n/2)\Gamma(m)} . \end{aligned} \tag{8.24}$$

Using the symmetry of the integration, it is easy to show that:

$$\int \frac{d^nk}{(2\pi)^n} \frac{k_\mu k_\nu}{[k^2 - \mathbf{R}^2]^m} = \frac{1}{n} g_{\mu\nu} \int \frac{d^nk}{(2\pi)^n} \frac{k^2}{[k^2 - \mathbf{R}^2]^m} . \tag{8.25}$$

In the same way:

$$k_\mu k_\nu k_\rho k_\sigma \to \frac{1}{n(n+2)} (k^2)^2 (g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}) . \tag{8.26}$$

In the case where r is odd:

$$\int \frac{d^nk}{(2\pi)^n} \frac{k_{\mu_1} \cdots k_{\mu_r}}{[k^2 - \mathbf{R}^2]^m} = 0 . \tag{8.27}$$

Finally, it is important to notice that tadpole type integral vanishes identically in dimensional regularization:

$$\int \frac{d^nk}{(2\pi)^n} (k^2)^{\beta-1} = 0 \quad \text{for} \quad \beta = 0, 1, 2, \ldots \tag{8.28}$$

We shall also see that the divergent part of $I(m,r)$ can be tranformed into $\epsilon \equiv 4 - n$ poles thanks to the properties of the Γ function.

8.2.5 Example of the pseudoscalar two-point correlator

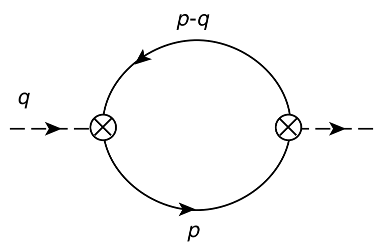
Let us consider the pseudoscalar correlator:

$$\Psi_5(q^2) \equiv i \int d^4x \, e^{iqx} \langle 0|\mathcal{T} J_P(x) \, (J_P(0))^\dagger \, |0\rangle \, , \tag{8.29}$$

where:

$$J_P = (m_i + m_j)\bar{\psi}_i(i\gamma_5)\psi_j \, , \tag{8.30}$$

is the light quark pseudoscalar current; m_i is the mass of the quark ψ_i. In order to simplify the discussion, we shall work to lowest order of perturbative QCD and work with massless quarks in the fermion loop given in the following diagram (Fig. 8.1):[3]



(8.31)

Using Feynman rules given in Appendix E, it reads:

$$i\Psi_5(q^2) = (m_i + m_j)^2(-1)N \int \frac{d^4p}{(2\pi)^4} \mathbf{Tr} \left\{ (i\gamma_5)\frac{i}{\hat{p} + i\epsilon'}(i\gamma_5)\frac{i}{\hat{p} - \hat{q} + i\epsilon'} \right\} \, , \tag{8.32}$$

where one can notice that for large k^2, one has a divergent integral:

$$I = \int \frac{d^4k}{k^2} = \infty \, . \tag{8.33}$$

One can use either PVR, but it is more convenient to use dimensional regularization. In so doing, one works in $n \equiv 4 - \epsilon$ space–time dimensions, such that the previous expression becomes:

$$\nu^\epsilon \Psi_5(q^2) = (m_i + m_j)^2(-i)N \int \frac{d^np}{(2\pi)^n} Tr \left\{ (i\gamma_5)\frac{1}{\hat{p} + i\epsilon'}(i\gamma_5)\frac{1}{\hat{p} - \hat{q} + i\epsilon'} \right\} \, . \tag{8.34}$$

The arbitrary scale ν has been introduced for dealing with dimensionless quantities in $4 - \epsilon$ dimensions.

One can parametrize the quark propagators à la Feynman (see Appendix E):

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[(a-b)x + b]^2} \equiv \int_0^1 \frac{dx}{[(p-l)^2 - \mathbf{R}^2]^2} \, , \tag{8.35}$$

[3] The case of massive quarks will be discussed later on in Section 11.14.

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where:

$$\begin{aligned} a &= (p-q)^2 + i\epsilon' , \\ b &= p^2 + i\epsilon' , \\ l &= qx , \\ \mathbf{R}^2 &= -q^2 x(1-x) - i\epsilon' . \end{aligned} \tag{8.36}$$

One uses the properties of the Dirac matrices in n dimensions given previously:

$$Tr\ \hat{p}\gamma_5(\hat{p}-\hat{q})\gamma_5 = -4p(p-q) , \tag{8.37}$$

and does the shift:

$$p \to \tilde{p} + l . \tag{8.38}$$

Therefore, one arrives at the momentum integration of the type:

$$\int \frac{d^n\ \tilde{p}}{(2\pi)^n} \frac{\tilde{p}^k}{[\tilde{p}^2 - \mathbf{R}^2]^2} , \tag{8.39}$$

which one can evaluate using the formula given in the previous section. It is easy to obtain:

$$\begin{aligned} \nu^\epsilon \Psi_5^B(q^2) = (m_i + m_j)^2 \frac{N}{4\pi^2} \int_0^1 dx \Gamma\left(\frac{\epsilon}{2}\right) \left(\frac{\mathbf{R}^2 - i\epsilon'}{\nu^2}\right)^{-\epsilon/2} \\ \times \left(3 + \frac{\epsilon}{2}\right) q^2 x(1-x) . \end{aligned} \tag{8.40}$$

where $\gamma_E = 0.5772\ldots$ is the Euler constant. The loop UV divergence appears as a pole at $\epsilon = 0$ of the Γ-function:

$$\lim_{\epsilon\to 0} \Gamma\left(\frac{\epsilon}{2}\right) \simeq \frac{2}{\epsilon} + \ln 4\pi - \gamma_E + \mathcal{O}(\epsilon) , \tag{8.41}$$

which, as you may have noticed simplify the calculation, which is remarkable when one does a higher order calculation. For large value of q^2, one then obtains to leading order:

$$\nu^\epsilon \Psi_5^B\left(q^2\right) = (m_i + m_j)^2 q^2 \frac{N}{8\pi^2} \left\{\frac{2}{\epsilon} + \ln 4\pi - \gamma_E - \ln\left(\frac{-q^2}{\nu^2}\right)\right\} \tag{8.42}$$

As we have discussed in the introduction, the UV (and IR) divergences originated from the Γ-function, are transformed into poles in $\epsilon \equiv n - 4$, and are, more generally, of the form:

$$\sum_{p=1} \frac{Z^{(p)}}{\epsilon^p} , \tag{8.43}$$

in the so-called [123] *Minimal Subtraction (MS) scheme*.

Later on, it has been remarked [124] that the combination in Eq. (8.41) appears always in the stage of the calculation. Therefore, the authors in [124] find that it is natural to also

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subtract the constant terms $\ln 4\pi - \gamma_E$ together with the ϵ-pole:

$$\frac{2}{\epsilon} \to \frac{2}{\hat{\epsilon}} \equiv \frac{2}{\epsilon} + \ln 4\pi - \gamma_E \tag{8.44}$$

This is the modified version of the *MS* scheme, and called: $\overline{MS}$ *scheme*, which will be used in the forthcoming discussions of this book. These divergences will appear as counterterms in the initial Langragian constrained by the Slavnov–Taylor identities [104]. One should notice that for renormalizable theories the $Z^{(p)}$ are local, i.e. constants or polynomials in the inverse of the square of some momentum. These features will be discussed in the forthcoming section.