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HOMOTOPY THEORY OF MODULES AND ADAMS COCOMPLETION

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Abstract. In this note, we have obtained a Whitehead-like tower of a module by considering a suitable set of morphisms and shown that the different stages of the tower are the Adams cocompletions of the module with respect to the suitably chosen set of morphisms.

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1. Introduction. The notion of Adams completion was suggested by Adams [**1–3**]. Initially, this was considered for admissible categories and generalized homology or cohomology theories. Later on, Deleanu, Frei and Hilton [**8**] developed the concept in a more general framework, where an arbitrary category and an arbitrary set of morphisms of the category are considered. The dual notion, that is, the Adams cocompletion was also suggested by them.

Let $\mathscr{C}$ be a category and S be a set of morphisms of $\mathscr{C}$. Let $\mathscr{C}[S^{-1}]$ denote the category of fractions [**13**] of $\mathscr{C}$ with respect to S and $F : \mathscr{C} \to \mathscr{C}[S^{-1}]$ be the canonical functor. Let $\mathscr{S}$ denote the category of sets and functions. Then, for a given object Y of $\mathscr{C}$, $\mathscr{C}[S^{-1}](Y, -) : \mathscr{C} \to \mathscr{S}$ defines a covariant functor. If this functor is representable by an object Y_S of $\mathscr{C}$, i.e., $\mathscr{C}[S^{-1}](Y, -) \cong \mathscr{C}(Y_S, -)$, then Y_S is called the *generalized Adams cocompletion* of Y with respect to the set of morphisms S or simply the *S-cocompletion* of Y. Y_S is also referred as the *cocompletion* of Y [**4**].

Given a set S of morphisms of $\mathscr{C}$, its *saturation* $\bar{S}$, is defined as the set of all morphisms u in $\mathscr{C}$ such that $F(u)$ is an isomorphism in $\mathscr{C}[S^{-1}]$. Further, S is said to be *saturated* [**4**], if $S = \bar{S}$.

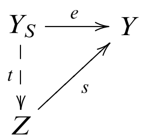
The following theorem shows that under certain condition the Adams cocompletion of an object in a category $\mathscr{C}$ always exists.

THEOREM 1.1 ([**9**], p. 32, dual of Theorem 1). *Let $\mathscr{C}$ be a complete small $\mathscr{U}$-category, where $\mathscr{U}$ is a fixed Grothendieck universe and S be a set of morphisms of $\mathscr{C}$ admitting a calculus of right fractions. Suppose that the following compatibility condition with product is satisfied:*

If each $s_i : X_i \to Y_i, i \in I$ lies in S, where the index set I is an element of $\mathscr{U}$, then $\underset{i\in I}{\wedge} s_i : \underset{i\in I}{\wedge} X_i \to \underset{i\in I}{\wedge} Y_i$ lies in S.
Then, every object X of $\mathscr{C}$ has an Adams cocompletion X_S with respect to the set of morphisms S.

The theorem given below characterizes Adams cocompletion in terms of a couniversal property.

THEOREM 1.2 ([**4**], p. 224, Proposition 1.1). *Let S be a set of morphisms of $\mathscr{C}$ admitting a calculus of right fractions. Then, an object Y_S of $\mathscr{C}$ is the S-cocompletion of the object Y with respect to S if and only if there exists a morphism $e : Y_S \to Y$ in $\bar{S}$ which is couniversal with respect to morphisms in S : given a morphism $s : Z \to Y$ in S there exists a unique morphism $t : Y_S \to Z$ in $\bar{S}$ such that $st = e$. In other words, the following diagram is commutative:*



In order to show the morphism $e : Y_S \to Y$, as constructed in the above theorem, belongs to S, the following result will be used.

THEOREM 1.3 ([**5**], p. 533, dual of Theorem 1.3). *Let S be a set of morphisms in a category $\mathscr{C}$ admitting a calculus of right fractions. Let $e : Y_S \to Y$ be the canonical morphism as defined in Theorem 1.2, where Y_S is the S-cocompletion of Y. Furthermore, let S_1 and S_2 be sets of morphisms in the category $\mathscr{C}$ which have the following properties:*

(*a*) *S_1 and S_2 are closed under composition;*
(*b*) *$fg \in S_1$ implies that $g \in S_1$;*
(*c*) *$fg \in S_2$ implies that $f \in S_2$;*
(*d*) *$S = S_1 \cap S_2$. Then, $e \in S$.*

2. The category $\tilde{\mathscr{M}}$. Behera and Nanda [**4**] have obtained the Cartan–Whitehead decomposition of a 0-connected-based CW-complex with the help of a suitable set of morphisms. This note contains a Cartan–Whitehead-like decomposition of a module over a ring with unity.

The relative homotopy theory of modules, including the (module) homotopy exact sequence was introduced by Peter Hilton ([**11**], Chapter 13). Infact, he has developed homotopy theory in module theory, parallel to the existing homotopy theory in topology. Later, C. J. Su [**15–17**] has extensively studied homotopy theory of modules. Unlike homotopy theory in topology, there are two types of homotopy theory in module theory, namely, the injective theory and the projective theory. They are dual but not isomorphic [**17**]. Using injective theory, we have obtained the Cartan–Whitehead-like decomposition of a module. We do this in a general framework by considering a Serre class $\mathcal{C}$ of modules [**7, 14**]. The narrative may be recalled from ([**11**], Chapter 13). We briefly describe some of the concepts towards notational view-points.

Let Λ be a Dedekind domain. Let $\mathscr{U}$ be a fixed Grothendieck universe [**13**]. Let $\mathscr{M}$ denote the category of right Λ-modules and Λ-module homomorphisms and let $\tilde{\mathscr{M}}$ be the corresponding i-homotopy category, that is, the objects of $\tilde{\mathscr{M}}$ are all right Λ-modules and the morphisms are i-homotopic classes of Λ-homomorphisms. We assume that the underlying sets of elements of $\mathscr{M}$ are elements of $\mathscr{U}$.

Let M and N be right Λ-modules and $f : M \to N$ be a Λ-homomorphism. Then, f is *i-nullhomotopic*, denoted $f \simeq_i 0$, if f can be extended to some injective module

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$\overline{M}$ containing M. Also, if $g : M \to N$, then $f \simeq_i g$, if $f - g \simeq_i 0$ [**11**]. We denote the i-homotopy class of f by $[f]_i$.

We now choose a suitable set of morphisms S_n for the category $\tilde{\mathscr{M}}$. Let A be any right Λ-module. A morphism $\alpha : X \to Y$ in $\tilde{\mathscr{M}}$ is in S_n if and only if $\alpha_* : \bar{\pi}_m(A, X) \to \bar{\pi}_m(A, Y)$ is a $\mathcal{C}$-isomorphism for $m > n$ and a $\mathcal{C}$-monomorphism for $m = n$.

We show that this set of morphisms S_n of the category $\tilde{\mathscr{M}}$ admits a calculus of right fractions [**10, 13**].

Proposition 2.1. *S_n admits a calculus of right fractions.*

Proof. Clearly, S_n is a closed family of morphisms of the category $\tilde{\mathscr{M}}$. We shall verify conditions (i) and (ii) of Theorem 1.3* ([**8**], p.70).

Let $\alpha : X \to Y$ and $\beta : Y \to Z$ be two morphisms in $\tilde{\mathscr{M}}$. We show if $\beta\alpha \in S_n$ and $\beta \in S_n$, then $\alpha \in S_n$. Since $\beta\alpha \in S_n$ and $\beta \in S_n$, $(\beta\alpha)_* = \beta_*\alpha_* : \bar{\pi}_m(A, X) \to \bar{\pi}_m(A, Z)$ and $\beta_* : \bar{\pi}_m(A, Y) \to \bar{\pi}_m(A, Z)$ are $\mathcal{C}$-isomorphisms for $m > n$ and $\mathcal{C}$-monomorphisms for $m = n$. Therefore, α_* is a $\mathcal{C}$-monomorphism for $m \geq n$. In order to show α_* to be a $\mathcal{C}$-isomorphism for $m > n$, we need to show α_* is a $\mathcal{C}$-epimorphism for $m > n$. We have $\beta_*\alpha_*(\bar{\pi}_m(A, X)) = \bar{\pi}_m(A, Z)$ for $m > n$, that is, $\beta_*(\alpha_*(\bar{\pi}_m(A, X))) = \beta_*(\bar{\pi}_m(A, Y))$ for $m > n$. From this, we conclude that $\alpha_*(\bar{\pi}_m(A, X)) = \bar{\pi}_m(A, Y)$ for $m > n$, that is, α_* is a $\mathcal{C}$-epimorphism for $m > n$. Therefore, α_* is a $\mathcal{C}$-isomorphism for $m > n$ and a $\mathcal{C}$-monomorphism for $m = n$. Hence, condition (i) of Theorem 1.3* ([**8**], p.70) holds.

In order to prove the condition (ii) of Theorem 1.3* ([**8**], p.70), consider the diagram

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle\alpha} \\ Y & \xrightarrow[\gamma]{} & Z \end{array}$$

with $\gamma \in S_n$ in $\tilde{\mathscr{M}}$. We assert that the above diagram can be completed to a weak pull-back diagram in $\tilde{\mathscr{M}}$

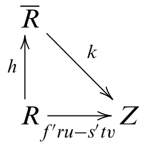
$$\begin{array}{ccc} W & \overset{\delta}{\dashrightarrow} & X \\ {\scriptstyle\beta}\,\vdots & & \downarrow{\scriptstyle\alpha} \\ Y & \xrightarrow[\gamma]{} & Z \end{array}$$

with $\delta \in S_n$. Let $\alpha = [f]_i$ and $\gamma = [s]_i$. We replace f and s by fibrations [**18**], that is, $f = f'r$ and $s = s't$, where f' and s' are fibrations and r, t are i-homotopy equivalences. Let $\bar{r}$ and $\bar{t}$ be i-homotopy inverses of r and t, respectively. Let $P_f = X \oplus D$ and $P_s = Y \oplus D$, where D is the maximal divisible submodule of Z. Let W be the usual pull-back of f' and s'; hence there exist $p : W \to P_f$ and $q : W \to P_s$ such that $f'p = s'q$. Let $\delta = [\bar{r}p]_i$ and $\beta = [\bar{t}q]_i$. Hence, $\alpha\delta = [f]_i[\bar{r}p]_i = [f\bar{r}p]_i = [f'r\bar{r}p]_i = [f'p]_i = [s'q]_i = [s't\bar{t}q]_i = [s\bar{t}q]_i = [s]_i[\bar{t}q]_i = \gamma\beta$. Thus, we have a commutative diagram

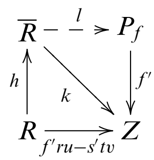
$$\begin{array}{ccc} W & \xrightarrow{\delta} & X \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\alpha} \\ Y & \xrightarrow[\gamma]{} & Z \end{array}$$

in $\tilde{\mathscr{M}}$.

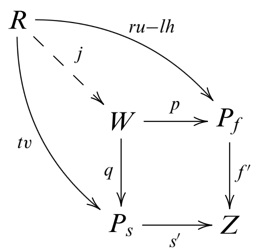
Moreover, let $\varphi : R \to X$ and $\psi : R \to Y$ in $\tilde{\mathscr{M}}$ be such that $\alpha\varphi = \gamma\psi$. Let $\varphi = [u]_i$, $\psi = [v]_i$. Thus, we have $fu \simeq_i sv$. This implies $f'ru \simeq_i s'tv$, that is, $f'ru - s'tv \simeq_i 0$, that is, $f'ru - s'tv$ can be extended to some injective module $\overline{R}$ containing R.



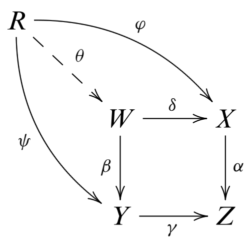
Thus, $kh = f'ru - s'tv$. Consider the following diagram:



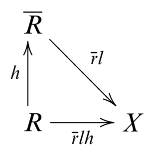
Since f' is a fibration, there exists $l : \overline{R} \to P_f$ such that $f'l = k$. Thus, $f'lh = kh$ and $f'ru - s'tv = kh = f'lh$, that is, $f'(ru - lh) = s'(tv)$. In the following diagram:



since W is the pull-back of f' and s' in $\mathscr{M}$, there exists $j : R \to W$ such that $pj = ru - lh$ and $qj = tv$. Let $\theta = [j]_i$. In the following diagram, in $\tilde{\mathscr{M}}$,



we have $\delta\theta = [\bar{r}p]_i[j]_i = [\bar{r}pj]_i = [\bar{r}(ru - lh)]_i = [\bar{r}ru - \bar{r}lh]_i = [u - \bar{r}lh]_i$. We claim that $[u - \bar{r}lh]_i = [u]_i$, that is, $u - \bar{r}lh \simeq_i u$; hence we need to show that $\bar{r}lh \simeq_i 0$, which is evident from the following commutative diagram.



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Also, $\beta\theta = [\bar{\imath}q]_i[j]_i = [\bar{\imath}qj]_i = [\bar{\imath}tv]_i = [v]_i = \psi$. Thus, we have the required pull-back diagram in $\tilde{\mathscr{M}}$.

It remains to show that $\delta \in S_n$. Let $F = \ker \beta$ and from the commutative diagram

$$\begin{CD}
F @= F \\
@VVV @VVV \\
W @>{\delta}>> X \\
@V{\beta}VV @VV{\alpha}V \\
Y @>>{\gamma}> Z
\end{CD}$$

in $\tilde{\mathscr{M}}$, we have the following commutative diagram:

$$\begin{CD}
\cdots @>>> \bar{\pi}_{m+1}(A, Y) @>>> \bar{\pi}_m(A, F) @>>> \bar{\pi}_m(A, W) @>>> \\
@. @V{\gamma_*}VV @| @VV{\delta_*}V @. \\
\cdots @>>> \bar{\pi}_{m+1}(A, Z) @>>> \bar{\pi}_m(A, F) @>>> \bar{\pi}_m(A, X) @>>>
\end{CD}$$

$$\begin{CD}
\bar{\pi}_m(A, Y) @>>> \bar{\pi}_{m-1}(A, F) @>>> \cdots \\
@V{\gamma_*}VV @| @. \\
\bar{\pi}_m(A, Z) @>>> \bar{\pi}_{m-1}(A, F) @>>> \cdots
\end{CD}$$

By five Lemma [**6**], δ_* is a $\mathcal{C}$-isomorphism for $m > n$ and a $\mathcal{C}$-monomorphism for $m = n$, that is, $\delta \in S_n$. This completes the proof. □

We show the following result always holds in the category $\tilde{\mathscr{M}}$ together with the chosen set of morphisms S_n.

PROPOSITION 2.2. *Let $s_j : X_j \to Y_j$ lie in S_n for each $j \in J$, where the index set J is an element of $\mathscr{U}$. Then, $\underset{j\in J}{\wedge} s_j : \underset{j\in J}{\wedge} X_j \to \underset{j\in J}{\wedge} Y_j$ lies in S_n.*

Proof. Let $s = \prod\limits_{j\in J} s_j$, $X = \prod\limits_{j\in J} X_j$ and $Y = \prod\limits_{j\in J} Y_j$. Define a map $s : X \to Y$ by the rule $s(x) = (s_j(x_j))_{j\in J}$, where $x = (x_j)_{j\in J}$. Clearly, s is well defined and is also a morphism in $\tilde{\mathscr{M}}$. Consider the commutative diagram

$$\begin{CD}
X @>{s}>> Y \\
@V{p_j}VV @VV{q_j}V \\
X_j @>>{s_j}> Y_j
\end{CD}$$

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where p_j and q_j are the projections. Let $F = \ker p_j$ and from the commutative diagram

$$\begin{array}{ccc} F & = & F \\ \downarrow & & \downarrow \\ X & \xrightarrow{s} & Y \\ {\scriptstyle p_j}\downarrow & & \downarrow{\scriptstyle q_j} \\ X_j & \xrightarrow[s_j]{} & Y_j \end{array}$$

we have the following commutative diagram:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \bar{\pi}_{m+1}(A, X_j) & \longrightarrow & \bar{\pi}_m(A, F) & \longrightarrow & \bar{\pi}_m(A, X) & \longrightarrow & \\ & & {\scriptstyle s_{j*}}\downarrow & & \| & & \downarrow{\scriptstyle s_*} & & \\ \cdots & \longrightarrow & \bar{\pi}_{m+1}(A, Y_j) & \longrightarrow & \bar{\pi}_m(A, F) & \longrightarrow & \bar{\pi}_m(A, Y) & \longrightarrow & \end{array}$$

$$\begin{array}{ccccc} \bar{\pi}_m(A, X_j) & \longrightarrow & \bar{\pi}_{m-1}(A, F) & \longrightarrow & \cdots \\ {\scriptstyle s_{j*}}\downarrow & & \| & & \\ \bar{\pi}_m(A, Y_j) & \longrightarrow & \bar{\pi}_{m-1}(A, F) & \longrightarrow & \cdots \end{array}$$

By five Lemma [**6**], s_* is a $\mathcal{C}$-isomorphism for $m > n$ and a $\mathcal{C}$-monomorphism for $m = n$, that is, $s \in S_n$. This completes the proof. □

The following result is well known.

PROPOSITION 2.3. *The category $\tilde{\mathscr{M}}$ is complete.*

3. Existence of Adams cocompletion in $\tilde{\mathscr{M}}$. Using Propositions 2.1–2.3, from Theorem 1.1, we draw the following result.

THEOREM 3.1. *Every object X in the category $\tilde{\mathscr{M}}$ has an Adams cocompletion X_{S_n} with respect to the set of morphisms S_n.*

Since every object in the category $\tilde{\mathscr{M}}$ has Adams cocompletion with respect to the set of morphisms S_n, from Theorem 1.2, we will have the following result.

THEOREM 3.2. *Every object X of the category $\tilde{\mathscr{M}}$ has an S_n cocompletion with respect to the set of morphisms S_n if and only if there exists a morphism $e_n : X_{S_n} \to X$ in $\bar{S}_n$ which is couniversal with respect to the morphisms in S_n : given a morphism $s : Y \to X$ in S_n there exists a unique morphism $t_n : X_{S_n} \to Y$ in $\bar{S}_n$ such that $st_n = e_n$. In other words, the following diagram is commutative:*

$$\begin{array}{ccc} X_{S_n} & \xrightarrow{e_n} & X \\ {\scriptstyle t_n}\downarrow & \nearrow{\scriptstyle s} & \\ Y & & \end{array}$$

The morphism $e_n : X_{S_n} \to X$ as constructed in Theorem 3.2 is in S_n.

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THEOREM 3.3. $e_n \in S_n$.

Proof. Let $S_n^1 = \{\alpha : X \to Y$ in $\tilde{\mathcal{M}} \mid \alpha_* : \bar{\pi}_m(A, X) \to \bar{\pi}_m(A, Y)$ is a $\mathcal{C}$-monomorphism for $m \geq n\}$ and $S_n^2 = \{\alpha : X \to Y$ in $\tilde{\mathcal{M}} \mid \alpha_* : \bar{\pi}_m(A, X) \to \bar{\pi}_m(A, Y)$ is a $\mathcal{C}$-epimorphism for $m > n\}$. Clearly, $S_n = S_n^1 \cap S_n^2$ and S_n^1 and S_n^2 satisfy all the conditions of Theorem 1.3. Hence, $e_n \in S_n$. This completes the proof. □

Now, we will obtain a Whitehead-like tower for a module with the help of chosen set of morphisms S_n whose different stages are the Adams cocompletion with respect to the set of morphisms S_n.

THEOREM 3.4. *Let X be a right Λ-module. Then, for $n \geq 0$, there exist right Λ-modules X_{S_n}, maps $e_n : X_{S_n} \to X$ and maps $\theta_{n+1} : X_{S_{n+1}} \to X_{S_n}$ such that*

(*i*) $e_{n_*} : \bar{\pi}_m(A, X_{S_n}) \to \bar{\pi}_m(A, X)$ *is a $\mathcal{C}$-isomorphism for $m > n$ and $\bar{\pi}_m(A, X_{S_n}) = 0$ for $m \leq n$.*

(*ii*) $e_{n+1} = e_n \circ \theta_{n+1}$.

Proof. For every $n \geq 0$, let X_{S_n} be the S_n-cocompletion of X and $e_n : X_{S_n} \to X$ be the canonical map. We have already shown $e_n \in S_n$. So $e_{n_*} : \bar{\pi}_m(A, X_{S_n}) \to \bar{\pi}_m(A, X)$ is a $\mathcal{C}$-isomorphism for $m > n$. Every module has an injective resolution [**6**]. Consider an injective resolution of A as $A \to \overline{A} \to \overline{SA} \to \cdots \to \overline{S^m A} \to \cdots$ with successive cokernels $SA, S^2A, \ldots, S^{m+1}A, \ldots$. We claim that $\overline{S^m A}$ is injective. We can decompose the above sequence by the following short exact sequences.

$$\begin{array}{l} 0 \to A \to \overline{A} \to SA \to 0 \\ 0 \to SA \to \overline{SA} \to S^2A \to 0 \\ \qquad \vdots \qquad\quad \vdots \qquad\quad \vdots \\ 0 \to S^{m-1}A \to \overline{S^{m-1}A} \to S^mA \to 0 \\ \qquad \vdots \qquad\quad \vdots \qquad\quad \vdots \end{array}$$

Applying $\mathrm{Ext}^j_\Lambda(A, -)$ for every integer $j \geq 1$ to the short exact sequence $0 \to S^{m-1}A \to \overline{S^{m-1}A} \to S^mA \to 0$, we get the following short exact sequence [**6**].

$$0 \to \mathrm{Ext}^j_\Lambda(A, S^{m-1}A) \to \mathrm{Ext}^j_\Lambda(A, \overline{S^{m-1}A}) \to \mathrm{Ext}^j_\Lambda(A, S^mA) \to 0$$

Since $\overline{S^{m-1}A}$ is injective, $\mathrm{Ext}^j_\Lambda(A, \overline{S^{m-1}A}) = 0$ for every integer $j \geq 1$ [**6**]. So, $\mathrm{Ext}^j_\Lambda(A, S^mA) = 0$ for every integer $j \geq 1$. This concludes S^mA is injective [**6**]. Therefore, $\bar{\pi}_m(A, X_{S_n}) = 0$ for $m \leq n$ [**11**]. Next, we have $e_n \in S_n \subset S_{n+1}$. By the couniversal property of e_{n+1}, there exists a unique morphism $\theta_{n+1} : X_{S_{n+1}} \to X_{S_n}$ such that the following diagram commutes, that is, $e_{n+1} = e_n \circ \theta_{n+1}$.

$$\begin{array}{ccc} X_{S_{n+1}} & \xrightarrow{e_{n+1}} & X \\ {\scriptstyle \theta_{n+1}} \downarrow & \nearrow {\scriptstyle e_n} & \\ X_{S_n} & & \end{array}$$

Thus, we get a Whitehead-like tower of a module in $\tilde{\mathcal{M}}$. This completes the proof. □

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