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KASTELEYN'S THEOREM AND ARBITRARY GRAPHS

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1. Introduction and basic definitions. Familiarity with the basic notions of graph theory is assumed. Loops and multiple edges are not permitted.

An *orientation of an edge e* of a graph G is a designation of one of the ends of e as the positive end and the other as the negative end. We say that e is oriented from the positive end to the negative end. If e joins vertex v to vertex w and v is the positive end of e, we write $e = (v, w)$. An *orientation of G* is a set of orientations, one for each edge of G; a graph with an orientation is called a *directed* graph.

Let G be a planar graph, and M a representation of G on the plane. Then according to a theorem of Kasteleyn [**1**], the edges of M may be oriented so that for every circuit C, the number of edges of C that are oriented in the clockwise sense is of opposite parity to the number of vertices enclosed by C. In section 2 of this paper, we give an easy proof of an extension of Kasteleyn's theorem to non-planar graphs, and we present a few simple related results. In section 3, a much more general theorem is proved and we show how Kasteleyn's theorem can also be derived from it.

Let G be a finite graph with vertex set VG and edge set EG. If $X, Y \subseteq VG$ and $X \cap Y = \emptyset$, we define $\delta_G(X, Y)$ to be the set of edges of G each with one end in X and the other in Y. We define $\delta_G X = \delta_G(X, VG - X)$. Whenever there is no ambiguity, the symbol δ_G will be replaced by δ. A *coboundary* is defined as a set of edges equal to δX for some vertex set X. A *cutset* is defined to be a minimal nonnull coboundary. A graph G is called *connected* if δX is nonnull for each nonnull proper subset X of VG. A *component* of G is a maximal nonnull connected subgraph.

If $X \subseteq VG$, we define $G[X]$ to be the subgraph of G whose vertex set is X and whose edge set is the set of edges of G having both ends in X. Define $r(X) = |EG[X]| - |X| + p_0(X)$ where $p_0(X)$ is the number of components of $G[X]$. Thus $r(X)$ is the number of chords of a spanning forest of $G[X]$. If G is a directed graph, let δ^+X be the set of edges of δX with positive end in X. Then if $v \in VG$, an orientation of G is defined to be *odd relative to v* if for every $X \subseteq VG - \{v\}$ we have

$$|\delta^+X| \equiv r(X) + p_0(X) \bmod 2$$

The *outvalency* of v is defined as the number of edges of G whose positive end is v.

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2. Kasteleyn's theorem for non-planar graphs. In this section we show how to generalize Kasteleyn's theorem to non-planar graphs. Since Kasteleyn's theorem in the form in which it has been stated is meaningless for non-planar graphs, the method used is to find a dual form of the theorem that does have meaning in the non-planar case. Theorem 1 contains such a form as a special case. Kasteleyn's theorem then follows from this special case and the principle of duality for planar graphs. Theorems 2 and 4 present some further related results.

THEOREM 1. *If G is a finite connected graph and $v \in VG$, then G has an odd orientation relative to v.*

We need a simple lemma.

LEMMA 1. *Let G be a finite connected graph and let $v \in VG$. Then there is an orientation of G such that every vertex of $VG - \{v\}$ has odd outvalency.*

Proof. Let R be an orientation of G such that the number of vertices of $VG - \{v\}$ of even outvalency is minimal. Let $u \neq v$ be a vertex of even outvalency in R. Since G is connected, there is a simple path P joining u and v. Let R' be the orientation obtained from R by interchanging the positive and negative ends of each edge of P. Then u is of odd outvalency in R', but all other vertices of $VG - \{v\}$ have the same outvalency in R' as in R. Thus fewer vertices of $VG - \{v\}$ have even outvalency in R' than in R and the minimality property of R is contradicted. Therefore no vertex of $VG - \{v\}$ has even outvalency in R.

Proof of Theorem 1. Let G be oriented so that every vertex of $VG - \{v\}$ has odd outvalency. Let $X \subseteq VG - \{v\}$. Thus

$$r(X) = |EG[X]| - |X| + p_0(X). \tag{1}$$

The total outvalency summed over all vertices in X is $|EG[X]| + |\delta^+X|$. Since every vertex of X has odd outvalency,

$$|EG[X]| + |\delta^+X| \equiv |X| \bmod 2,$$

so that $|EG[X]| - |X| \equiv |\delta^+X| \bmod 2$. Therefore, from (1),

$$|\delta^+X| \equiv r(X) + p_0(X) \bmod 2,$$

so that the given orientation is odd. The theorem is proved.

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Remark. If G is planar, the dual of Kasteleyn's theorem is easily seen to be the case where δX is a cutset, so that $p_0(X) = 1$.

It is now a simple matter to characterize odd orientations.

THEOREM 2. *An orientation of the edges of a finite connected graph is odd relative to a vertex v if and only if every vertex other than v has odd outvalency.*

Proof. Let G be given an odd orientation relative to v. Let $u \in VG$, $u \neq v$. Then $|\delta^+\{u\}| \equiv r(\{u\}) + 1 = 1$ so that u has odd outvalency. Therefore every vertex of $VG - \{v\}$ has odd outvalency.

The proof of the other half of this theorem is contained in the proof of Theorem 1.

Remark. Whether or not v has odd outvalency of course is entirely dependent on $|VG|$ and $|EG|$.

It may be of some interest to point out how odd orientations are related to each other.

THEOREM 3. *Let G be a finite, connected, directed graph with an odd orientation R_1 relative to some vertex v. Then another orientation R_2 is odd relative to v if and only if every vertex has incident upon it an even number of edges whose orientations in R_1 and R_2 differ.*

Proof. For vertices in $VG - \{v\}$ this theorem is obvious from Theorem 2 and for v it is clear from the remark following Theorem 2.

THEOREM 4. *Let G be a finite, connected graph and let R be an odd orientation relative to some vertex v. Then R is odd relative to every vertex in G if and only if $|VG| \equiv |EG| \bmod 2$. If $|VG| \equiv |EG| + 1 \bmod 2$, then an odd orientation relative to a vertex $w \neq v$ is obtained by interchanging the positive and negative ends of every edge of some simple path P joining v and w.*

Proof. Since R is odd relative to v, every vertex of $VG - \{v\}$ has odd outvalency. If $|VG| \equiv |EG| \bmod 2$, then v also has odd outvalency, since the total outvalency summed over all vertices of G is $|EG|$. Hence every vertex has odd outvalency and R is odd relative to w for every $w \in VG$. However, if $|VG| \equiv |EG| + 1 \bmod 2$, then v has even outvalency and R is not odd relative to w. In this case, if the positive and negative ends of every edge of P are interchanged, w becomes the only vertex with even outvalency and thus the resulting orientation is odd relative to w.

3. A further generalization of Kasteleyn's theorem. In this section, we show that Lemma 1 of the previous section is actually a special case of a much more general phenomenon, given in Theorem 5. Thus Kasteleyn's theorem is seen in a more general setting.

Definition. Let G be a finite graph and A, B disjoint subsets of VG. Then a subset W of VG is called a *parity set relative to* (A, B) if for every vertex $v \in A \cup B$, $|\delta(\{v\}, W - \{v\})|$ is even if $v \in A$ and odd if $v \in B$.

THEOREM 5. *Let G be a finite graph and let A, B, X be subsets of VG such that $A \cap B = \emptyset$. Then a necessary and sufficient condition for the existence of a subset W of X such that W is a parity set relative to (A, B) is that there does not exist a subset S of $A \cup B$ such that (i) $|S \cap B|$ is odd, and (ii) $|\delta(\{v\}, S - \{v\})|$ is even for every $v \in X$.*

Proof. The theorem is an application of the following result in linear algebra. Given numbers a_{ij} $(i = 1, \ldots, m;\ j = 1, \ldots, n)$ and z_i $(i = 1, \ldots, m)$, the equations

$$\sum_{j=1}^{n} a_{ij} y_j = z_i \quad (i = 1, \ldots, m)$$

can be solved for $y_1, \ldots, y_n$ if and only if, for every sequence of coefficients $\lambda_1, \ldots, \lambda_m$ such that the vector

$$\sum_{i=1}^{m} \lambda_i (a_{i1}, a_{i2}, \ldots, a_{in})$$

vanishes, we have

$$\sum_{i=1}^{m} \lambda_i z_i = 0.$$

In our application of this theorem, all the scalars belong to the field of residue classes modulo 2. Let $X = \{x_1, x_2, \ldots, x_n\}$ and $A \cup B = \{v_1, v_2, \ldots, v_m\}$. If $1 \leqq i \leqq m$, $1 \leqq j \leqq n$, let a_{ij} be 1 if there is an edge joining v_i to x_j and 0 otherwise, and let z_i be 1 if $v_i \in B$ and 0 if $v_i \in A$. The scalars $y_1, \ldots, y_n$ correspond to W and $\lambda_1, \ldots, \lambda_m$ to S in an obvious way, and the theorem is proved.

A graph theoretical proof of this theorem can also be given, but it is considerably more complicated.

In order to show how Lemma 1 is a special case of this theorem, we need a corollary.

Definition. If G is a graph, a *spanning subgraph* of G is defined as a subgraph with vertex set VG.

COROLLARY 1. *Let G be a finite graph and A and B be disjoint subsets of VG. Then a necessary and sufficient condition for the existence of a spanning subgraph W of G such that every vertex of A has even valency in W and every vertex of B has odd valency in W, is that there does not exist a component C of G such that $VC \subseteq A \cup B$ and $|VC \cap B|$ is odd.*

Remark. This result can be proved simply without recourse to Theorem 5, but our purpose is to show it as a special case of Theorem 5.

Proof. Define a bipartite graph H as follows: Let

$$VH = VG \cup EG, \quad \delta_H EG = \delta_H VG = EH$$

and let vertices $v \in VG$ and $e \in EG$ be adjacent in H if and only if e is incident on v in G. Let $X = EG$. Then it is clear that the required subgraph W of G exists if and only if some subset W' of X is a parity set relative to (A, B). Therefore, by Theorem 5, if W does not exist, there must be a subset S of $A \cup B$ such that $|S \cap B| \equiv 1 \bmod 2$ and, for every $e \in X$, $|\delta(\{e\}, S - \{e\})| \equiv 0 \bmod 2$. The latter condition means that the ends of any edge e of G are either both in S or both in $VG - S$. Hence $\delta_G S = \emptyset$, so that S is the vertex set of a set T of components of G, and every vertex of every component in T must be in $A \cup B$ because $S \subseteq A \cup B$. Since $|S \cap B|$ is odd, some component $C \in T$ satisfies $VC \subseteq A \cup B$ and $|VC \cap B| \equiv 1 \bmod 2$.

The other half of this corollary is obvious from the fact that the total valency summed over all vertices in any component of a graph must be even.

Proof of Lemma 1. Let R be any orientation of G. Let A be the set of vertices of $VG - \{v\}$ of odd outvalency, and let $B = VG - A - \{v\}$. In order to arrange that every vertex of $VG - \{v\}$ have odd outvalency, we must interchange positive and negative ends of every edge of some spanning subgraph W of G with the property that every vertex of A has even valency in W and every vertex in B has odd valency in W. Since G has just one component and $VG \neq A \cup B$, such a subgraph exists by Corollary 1. The lemma is proved and Kasteleyn's theorem is seen as a consequence of Theorem 5.

A 1-*factor* of a graph G is defined to be a set T of edges such that for every $v \in VG$ exactly one edge of T is incident on v. The symmetric difference of two 1-factors is a set of circuits of even length, each of which is called an *alternating circuit.* Any circuit possesses two senses, which we call *clockwise* and *counterclockwise* respectively. If G is a directed graph, we say that the orientation of any edge of a circuit agrees or disagrees with the clockwise sense according as the direction of an arrow pointing from the positive to the negative end of the edge agrees with the clockwise or counterclockwise sense of the circuit. A circuit C is called *clockwise odd* if an odd number of edges of C are oriented to agree with the clockwise sense. Otherwise C is called *clockwise even.*

If G is a directed graph, let $F = \{f_1, \ldots, f_j\}$ be the set of 1-factors of G, and for all i write

$$f_i = \{(u_{i1}, w_{i1}), (u_{i2}, w_{i2}), \ldots, (u_{ik}, w_{ik})\}$$

where $k = \frac{1}{2}|VG|$ and $u_{ij}, w_{ij} \in VG$ for all j. Associate with f_i a plus sign if $u_{i1}w_{i1}u_{i2}w_{i2} \ldots u_{ik}w_{ik}$ is an even permutation of $u_{11}w_{11}u_{12}w_{12} \ldots u_{1k}w_{1k}$, and a minus sign otherwise. Let S_1 be the set of 1-factors that are thus given a plus sign and let $S_2 = F - S_1$. Let $VG = \{v_1, v_2, \ldots, v_{2k}\}$ and let $M = (m_{ij})$ be the matrix such that $m_{ij} = 1$ if $(v_i, v_j) \in EG$, $m_{ij} = -1$ if $(v_j, v_i) \in EG$, and $m_{ij} = 0$ otherwise. Let $\mathrm{Pf}M$ be the Pfaffian of M, and $|M|$ the determinant. Since M is a skew symmetric $2k \times 2k$ matrix, we have $|M| = \mathrm{Pf}^2 M$ by a theorem of Cayley [**2**; **3**]. Furthermore, $|\mathrm{Pf}M| = ||S_1| - |S_2||$, so that if all the 1-factors of G have the same sign, then $|F| = |\mathrm{Pf}M|$ and thus $|F|$ can readily be calculated. In addition, Kasteleyn [**1**] shows that the orientation whose existence for planar graphs is asserted in his theorem guarantees that all the 1-factors of G are given the same sign. The question of a characterization of those graphs that can be oriented to ensure that all 1-factors have the same sign thus arises. No practical characterization is known but Theorem 5 can be used to give the following interesting theoretical characterization that sheds some light on the nature of the problem.

A set S of spanning subgraphs of a graph G is said to be of *empty digital sum* if every edge of G is found in an even number of elements of S.

COROLLARY 2. *A necessary and sufficient condition for the existence of an orientation of G so that all* 1*-factors have the same sign is the non-existence of a set S of* 1*-factors such that S is of empty digital sum and an odd number of elements of S have a minus sign.*

Proof. Let H be the bipartite graph defined as follows. Let $VH = F \cup EG$, $\delta_H F = EH$, and let vertices $v \in F$ and $w \in EG$ be adjacent if and only if $w \in v$ in G. Let $A = S_1$, $B = S_2$, $X = EG$. Then the corollary is immediate upon applying Theorem 5 to H.

Kasteleyn [**1**] has also shown that all the 1-factors have the same sign if and only if the orientation of G is such that all alternating circuits are clockwise odd. By using the same result from linear algebra that we used, Pla [**4**] has given necessary and sufficient conditions for the existence of such an orientation. The reader should be warned, however, that Theorems 2 and 3 of Pla's paper are in general false. We now show that Pla's result is also a special case of Theorem 5, as is not surprising.

COROLLARY 3. *A necessary and sufficient condition for the existence of an orientation of G such that all alternating circuits of G are clockwise odd is the non-existence of a set S of alternating circuits such that S is of empty digital sum and an odd number of elements of S are clockwise even.*

Proof. The proof is identical to that of Corollary 2 except that F is replaced by the set of all alternating circuits of G.

4. Conclusions. We have shown that the theorem of Kasteleyn mentioned in the introduction is a special case of the more general Theorem 5. Corollaries 1, 2 and 3 of Theorem 5 illustrate the generality of that theorem and suggest that it may be a useful tool in solving a number of problems in which parity considerations play a rôle. It appears from these examples that applications of Theorem 5 to bipartite graphs show particular promise.

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