ON GROUP RINGS

D. B. COLEMAN

Introduction. Let R be a commutative ring with unity and let G be a group. The group ring RG is a free R-module having the elements of G as a basis, with multiplication induced by

$$G\colon \left(\sum_g \alpha_g g\right)\left(\sum_g \beta_g g\right) = \sum_{h,g} \alpha_h \beta_{h^{-1}g} g.$$

The first theorem in this paper deals with idempotents in RG and improves a result of Connell. In the second section we consider the Jacobson radical of RG, and we prove a theorem about a class of algebras that includes RG when G is locally finite and R is an algebraically closed field of characteristic zero. The last theorem shows that if R is a field and G is a finite nilpotent group, then RG determines RP for every Sylow subgroup P of G, regardless of the characteristic of R.

1. For a subgroup H of G, let wH denote the augmented left ideal of H; that is, wH is the left ideal in RG generated by elements $h - 1$ for $h \in H$. It is easy to see that if $\{g_i\}_{i \in I}$ is a complete set of left coset representatives for G modulo H, then the elements $g_i(h-1)$, with $i \in I$ and $h \neq 1$, form an R-basis for wH.

In [**3**] it was shown that wG is a direct summand of RG if and only if G has finite order n and n is a unit in R, i.e., $n \cdot 1$ has an inverse in R. It was also noted there that if H is a subgroup of G and wH is a direct summand, then H has finite order. We see now that in this case this order must also be a unit in R.

THEOREM 1. *Let H be a subgroup of G. Then wH is a direct summand of RG (as a left ideal) if and only if H has finite order m, and m is a unit in R. Moreover, in this case the right unity element of wH is unique if and only if H is normal.*

Proof. Suppose that H has finite order m, and let $H^* = \sum_H h$. Then clearly $1 - m^{-1}H^*$ is a right unity element for wH.

Conversely, suppose that wH has a right unity element e. Let $\{g_i\}_{i \in I}$ and $\{\bar{g}_i\}_{i \in I}$ be complete sets of left and right coset representatives for G modulo H, respectively. Take $1 \in I$ and $g_1 = \bar{g}_1 = 1$, the identity in G. As noted above, H has finite order m. Then the elements $H^*\bar{g}_i$, $i \in I$, form an R-basis for the right annihilator $(wH)^r$ of wH.

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Let $H = \{h_1 = 1, h_2, \ldots, h_m\}$, and write

$$e = \sum_{i,j} \alpha_{ij} g_i (h_j - 1); \qquad \alpha_{ij} \in R, \quad j \neq 1.$$

Then $1 - e \in (wH)^r$, and thus

$$1 - e = \sum_i \beta_i H^* \bar{g}_i. \tag{1}$$

Equating coefficients in these expressions, we see that $\beta_1 = 1 + \sum_{j=2}^{m} \alpha_{1j}$. Fix $k > 1$ and consider $h_k - 1$. Letting $h_k^{-1} = h_{k'}$, we have

$$h_k - 1 = (h_k - 1)e = \sum \alpha_{ij}(h_k - 1) g_i (h_j - 1).$$

Now equating coefficients of the identity and using (1) we have

$$-1 = \sum_{j=2}^{m} \alpha_{1j} + \alpha_{1k'} = (\beta_1 - 1) + \alpha_{1k'},$$

so that $\beta_1 = -\alpha_{1k'}$. Now

$$\beta_1 = 1 + \sum_{j=2}^{m} \alpha_{1j} = 1 + \sum_{k=2}^{m} \alpha_{1k'} = 1 - (m - 1)\beta_1;$$

thus $m\beta_1 = 1$. Hence m is a unit.

In case m is a unit and $e = 1 - m^{-1}H^*$, it is easy to see that for each $i \in I$ and $j = 2, \ldots, m$, $g_i h_j g_i^{-1} \in H$ if and only if $eg_i(h_j - 1) = g_i(h_j - 1)$. Since the uniqueness of a right unity is equivalent to its being a two-sided unity, the result follows. Note that the commutativity of R was not needed.

An open question is the following. If R is an integral domain, and if RG has an idempotent different from 0 or 1, is it true that G has a finite non-trivial subgroup whose order is a unit in R?

It is of interest here that if G has finite order n and if n is a unit in R, then Maschke's theorem holds for R-representations of G. That is, if

$$\mu(g) = \begin{pmatrix} \mu_1(g) & * \\ 0 & \mu_2(g) \end{pmatrix}, \qquad g \in G,$$

gives a representation of G by unimodular matrices over R, then there is a matrix D over R such that

$$\begin{pmatrix} I & D \\ 0 & I \end{pmatrix} \mu(g) \begin{pmatrix} I & -D \\ 0 & I \end{pmatrix} = \begin{pmatrix} \mu_1(g) & 0 \\ 0 & \mu_2(g) \end{pmatrix}, \qquad g \in G.$$

This can be seen by an almost exact duplication of the material immediately surrounding [**4**, Theorem (73.22)]. Conversely, if n is not a unit in R, then Maschke's theorem fails for G. For in this case wG does not have a G-invariant complement in RG. These remarks give a matrix analogue to [**3**, corollary to Theorem 3].

2. Let $J(A)$ denote the Jacobson radical of a ring A.

H. K. Farahat has asked: Under what conditions does $J(RG) = J(R)G$? We give some partial answers.

Again, let R be a commutative ring with unity. Let $\bar{R} = R/J(R)$.

A *locally finite group* is a group in which every finitely generated subgroup is finite.

LEMMA. *If G is locally finite, then $J(RG) = J(R)G$ if and only if $J(\bar{R}G) = 0$.*

Proof. By [**3**, Proposition 9], $J(R) = J(RG) \cap R$ if G is locally finite. Hence $J(R)G \subset J(RG)$. Thus it suffices to notice that $\bar{R}G = RG/J(R)G$.

LEMMA. *If G is finite of order n, then $J(RG) = J(R)G$ if and only if n is not a zero divisor in $\bar{R}$.*

Proof. By [**3**, Theorem 7], $J(\bar{R}G) = 0$ if and only if n is not a zero divisor in $\bar{R}$. This suffices, using the previous lemma.

Note that the above condition on n and R means that every element of finite order in $(\bar{R}, +)$ has order prime to n. This is equivalent to the condition: For every $x \notin J(R)$, there is a maximal ideal $\mathscr{A}$ in R such that $x \notin \mathscr{A}$ and $n \notin \mathscr{A}$.

If every finitely generated subgroup of G has a semisimple group ring, then so does G; see [**1**]. Thus the above lemmas yield the following.

THEOREM 2. *Suppose that G is a locally finite group such that no element in G has order that is a zero divisor in $\bar{R}$. Then $J(R)G = J(RG)$.*

The converse of Theorem 2 is false. For example, take R to be an algebraically closed field of characteristic 2 and let $G = \langle A, \sigma \rangle$, where A is an infinite abelian group without elements of order 2, $\sigma^{-1}x\sigma = x^{-1}$ for $x \in A$, and $\sigma^2 = 1$. Then according to [**7**, Theorem 1], $J(RG) = J(R)G = 0$. But the condition on R fails at 2, of course.

However, we do have the following result.

PROPOSITION 3. *Let R be a commutative ring with unity. Then the following conditions are equivalent.*

(1) $(\bar{R}, +)$ *is a torsion-free abelian group.*

(2) $J(RG) = J(R)G$ *for every finite group G.*

(3) $J(RG) = J(R)G$ *for every locally finite group G.*

The proof is clear from the above.

Let K be a field, and let A be an algebra over K. Then $J(A)$ is the intersection of the kernels of all the irreducible representations of A as linear transformations on vector spaces over K. If $\mu: A \to \mathrm{Hom}_K(V, V)$ is such a representation, let $C(\mu) = \mathrm{Hom}_A(V, V)$, the commuting algebra of μ. Let us define $J^*(A) = \bigcap \mathrm{Ker}\, \mu$, where μ is taken over those irreducible representations such that $C(\mu)$ consists of scalar multiples of the identity. Then clearly $J^*(A) \supset J(A)$.

If K is algebraically closed, and if G is either an abelian or a locally finite group, then $J^*(KG) = J(KG)$. The first part follows from [**6**, Lemma 2] and the second since locally finite groups have algebraic group algebras. Complex Banach algebras also have $J^*(A) = J(A)$.

THEOREM 4. *Suppose that K is a field and A and B are algebras over K such that* $J^*(A) = 0$ *and* $J(B) = 0$. *Then* $J(A \otimes_K B) = 0$.

Proof. Let $\{a_i\}$ and $\{b_j\}$ be K-bases for A and B, respectively. Suppose that $c = \sum_{i,j} \alpha_{ij} a_i \otimes b_j$, $\alpha_{ij} \in K$, is a non-zero member of $A \otimes B$. For convenience, label b_1 so that some $\alpha_{i1} \neq 0$. Since $J^*(A) = 0$, there is an irreducible A-module V whose commuting algebra is K and such that $(\sum \alpha_{i1} a_i) V \neq 0$.

By a theorem of Azumaya and Nakayama [**5**, p. 113], if W is an irreducible B-module with commuting algebra C, then $V \otimes W$ is an irreducible $(A \otimes B)$-module with commuting algebra C. Thus it suffices to show that for some irreducible B-module W, we have that $c \cdot V \otimes W \neq 0$.

Let $\{x_\lambda\}$ be a K-basis for V. Put $a_i x_\lambda = \sum_k \beta^i_{k\lambda} x_k$ for each i, λ. There is λ_1 such that

$$\left(\sum_i \alpha_{i1} a_i\right) x_{\lambda_1} = \sum_k \left(\sum_i \alpha_{i1} \beta^i_{k\lambda_1}\right) x_k \neq 0.$$

Fix k_1 so that $\sum_i \alpha_{i1} \beta^i_{k_1\lambda_1} \neq 0$. Let $\rho_j = \sum_i \alpha_{ij} \beta^i_{k_1\lambda_1}$; note that $\rho_1 \neq 0$.

Choose an irreducible B-module W so that $(\sum_j \rho_j b_j) W \neq 0$. Let $\{y_\mu\}$ be a K-basis for W and fix μ_1 such that

$$\left(\sum_j \rho_j b_j\right) y_{\mu_1} \neq 0.$$

Put $b_j y_{\mu_1} = \sum_\nu \gamma^j_\nu y_\nu$ for each j. We claim that $c \cdot x_{\lambda_1} \otimes y_{\mu_1} \neq 0$. For suppose not; then

$$\begin{aligned} 0 = c \cdot x_{\lambda_1} \otimes y_{\mu_1} &= \sum_{ij} \alpha_{ij} (a_i x_{\lambda_1}) \otimes (b_j y_{\mu_1}) \\ &= \sum_{i,j} \alpha_{ij} \left(\sum_k \beta^i_{k\lambda_1} x_k\right) \otimes \left(\sum_\nu \gamma^j_\nu y_\nu\right) = \sum_{k,\nu} \left(\sum_{i,j} \alpha_{ij} \beta^i_{k\lambda_1} \gamma^j_\nu\right) x_k \otimes y_\nu. \end{aligned}$$

Since $\{x_k \otimes y_\nu\}_{k,\nu}$ is a basis for $V \otimes W$, it follows that for each pair k, ν we have

$$\sum_{i,j} \alpha_{ij} \beta^i_{k\lambda_1} \gamma^j_\nu = 0.$$

In particular, for each ν,

$$\sum_j \rho_j \gamma^j_\nu = \sum_{i,j} \alpha_{ij} \beta^i_{k_1\lambda_1} \gamma^j_\nu = 0.$$

But for each j,

$$\rho_j b_j y_{\mu_1} = \sum_\nu \rho_j \gamma^j_\nu y_\nu;$$

hence summing over j, we have

$$\left(\sum_j \rho_j b_j\right) y_{\mu 1} = \sum_\nu \left(\sum_j \rho_j \gamma_\nu^j\right) y_\nu = 0.$$

This is a contradiction.

Hence $c \cdot V \otimes W \neq 0$, and the proof is complete.

COROLLARY. *If* $J^*(A) = 0$ *and* $J^*(B) = 0$, *then* $J^*(A \otimes_K B) = 0$.

Proof. In the proof of the theorem, take W to have trivial commuting albegra.

3. If G is a finite nilpotent group, and K is a field whose characteristic does not divide $|G|$, then KG determines KP for every Sylow subgroup P of G; see [**8**; **2**]. We are able to drop the condition on the characteristic of K.

A_n denotes the ring of $n \times n$ matrices over a ring A.

THEOREM 5. *Let* $G = P \times H$, *where* P *is a finite* p-*group and* H *is a finite group whose order is prime to* p, *and let* K *be a field of characteristic* p. *Then* KG *determines* KP *and* KH.

Proof. KH is semisimple; thus suppose that $KH = \bigoplus \sum_i D_{n_i}{}^{(i)}$, with $D^{(i)}$ a division algebra over K. Then

$$KG \cong KP \otimes KH \cong \bigoplus \textstyle\sum_i KP \otimes D_{n_i}{}^{(i)} \cong KP \oplus \ldots \oplus KP \oplus (\sum KP \otimes D_{n_i}{}^{(i)}),$$

this last summation is taken over those summands with $n_i > 1$ or $D_i \neq K$. Now $KP \otimes D_{n_i}{}^{(i)} \cong (D^{(i)}P)_{n_i}$ as algebras over K; thus

$$KG \cong KP \oplus \ldots \oplus KP \oplus \sum (D^{(i)}P)_{n_i}.$$

Each of these $D^{(i)}P$ is indecomposable as a direct sum of two-sided ideals. Hence $(D^{(i)}P)_{n_i}$ is indecomposable and its dimension over K exceeds $|P|$. Thus KP appears as a two-sided ideal component of KG of minimal dimension over K. By the Krull-Schmidt theorem, KP is uniquely determined. Since $KG/wP = KG/J(KG) \cong KH$, we have that KH is uniquely determined.

COROLLARY. *If* G_1 *and* G_2 *are finite nilpotent groups and* K *is a field, then* $KG_1 \cong KG_2$ *if and only if for each prime* p, $KP_1 \cong KP_2$, *where* P_i *is the Sylow* p-*subgroup of* G_i, $i = 1, 2$.

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University of Kentucky,
Lexington, Kentucky