Can. J. Math., Vol. XXIX, No. 4, 1977, pp. 806–818

# ON UPCROSSING PROBABILITIES

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**1. Introduction.** In [**1**] a simple but ingenious technique was developed for calculating hitting probabilities for submartingales (or martingales or supermartingales) subject to various constraints. This technique is extended here in order to find sharp bounds on upcrossing probabilities for submartingales subject to constraints. The general results in Section 2 are applied to submartingales $\{X_n\}_{n=1}^{\infty}$ such that $E[(X_n - a)^+]^p \leqq L$ (a constant) for all $n$, $p \geqq 1$ and we find the probability of at least $k$ upcrossings of $[a, b]$ is at most

$$\frac{L - [(m-a)^+]^p}{(b-a)^p} \frac{(p-1)^{p-1}}{\left(k + \frac{(m-a)^+}{b-a}(p-1) - 1\right)^p + (p-1)^{p-1}},$$

where $m = EX_1$. For $p = 1$ this bound collapses to $(L - (m-a)^+)/((b-a)k)$ (taking $(p-1)^{p-1} = 1$ when $p = 1$). A simple corollary is that Doob's upcrossing inequality is sharp. A second example gives Dubins' sharp bounds on upcrossing probabilities for bounded martingales.

**2. General Results.** Keeping the notation established in [**1**] let $R$ be the set of real numbers; $B$ be the Borel subsets of $R$; $R^{\infty} = R \times R \times \ldots$; and $B^{\infty} = B \times B \times \ldots$. Let $\{X_n\}_{n=1}^{\infty}$ be the coordinate process on $R^{\infty}$. A submartingale (or martingale or supermartingale) may be regarded as a probability measure $P$ on $B^{\infty}$. $\{X_n\}_{n=1}^{\infty}$ defined on $\{R^{\infty}, B^{\infty}, P\}$ is a submartingale in the usual sense.

Let $\mu$ be a probability measure on $(R, B)$; we define the following classification:

*Definition* 1a.) $\mu$ satisfies a *condition of type* $(\phi, \underline{r}, \bar{r})$ if there is a family $\phi$ of convex, increasing, Borel functions from $R$ to $R$ and mappings $\underline{r}$ and $\bar{r}$ of $\phi$ to $R \cup \{-\infty, \infty\}$ such that for all $\theta \in \phi$,

$$\underline{r}(\theta) \leqq \int \theta(x)\mu(dx) \leqq \bar{r}(\theta).$$

($(\phi, \bar{r})$ means $\underline{r}(\theta) = -\infty$ for all $\theta \in \phi$.)

1.b) $\mu$ satisfies a *condition of type* $(L, U)$ if there exist two constants $L < U$ such that $\mu\{[L, U]\} = 1$.

*Definition* 2.a) A probability measure $Q$ on $(R^{\infty}, B^{\infty})$ satisfies a *condition of*

Received September 8, 1976 and in revised form, February 22, 1977. This work was done at Cornell University under a post-doctoral scholarship from the Government of Quebec.

*type* $(\phi, \underline{r}, \bar{r})$ if for all $\theta \in \phi$ and all $n$,

$$\underline{r}(\theta) \leqq \int \theta(X_n)dQ \leqq \bar{r}(\theta).$$

2.b) $Q$ satisfies a *condition of type* $(L, U)$ if $Q\{L \leqq X_n \leqq U\} = 1$ for all $n$.

LEMMA 1. *Let $M$ be the collection of all submartingales satisfying conditions of type $(\phi, \underline{r}, \bar{r})$ and/or type $(L, U)$ (that is certain conditions of these types are satisfied). If $\tau_1 \leqq \tau_2 \leqq \ldots \leqq \tau_m$ are bounded stopping times and $P \in M$, then the measure $Q$ on $B^\infty$ defined by*

$$(1) \quad Q\{X_1 \in B_1, X_2 \in B_2, \ldots, X_m \in B_m, X_{m+1} \in B_{n+1}, \ldots, X_{m+l} \in B_{m+l}\}$$
$$P\{X_{\tau_1} \in B_1, X_{\tau_2} \in B_2, \ldots, X_{\tau m} \in B_m, X_{\tau m} \in B_{m+1}, \ldots, X_{\tau m} \in B_{m+l}\}$$

*belongs to $M$.*

*Proof.* $Q$ is well defined since the stopping times are bounded. Next consider any cylinder set measurable with respect to $X_1, X_2, \ldots, X_k$; say

$$C = \{X_1 \in B_1, X_2 \in B_2, \ldots, X_k \in B_k\}.$$

$$(2) \quad \int_C E_Q\{X_{k+1}|X_k, \ldots, X_1\}dQ = \int_C X_{k+1}\,dQ = \int_{C'} X_{\tau_{k+1}}\,dP \text{ by (1)},$$

where $C' = \{X_{\tau_1} \in B_1, X_{\tau_2} \in B_2, \ldots, X_{\tau_k} \in B_k\}$. Since $P$ is a submartingale,

$$\int_{C'} X_{\tau_{k+1}}\,dP \geqq \int_{C'} X_{\tau_k}\,dP = \int_C X_k\,dQ.$$

Since the $\sigma$-algebra of sets measurable with respect to $X_1, X_2, \ldots, X_k$ $(\sigma(X_1, \ldots, X_k))$ is generated by sets of the form $C$, we have

$$\int_A E_Q\{X_{k+1}|X_k, \ldots, X_1\}dQ \geqq \int_A X_k\,dQ,$$

where $A \in \sigma(X_1, \ldots, X_k)$. Therefore $Q$ is a submartingale. If $\theta \in \phi$ then $\{\theta(X_n)\}_{n=1}^\infty$ defined on $\{R^\infty, B^\infty, P\}$ is a submartingale. Hence

$$\underline{r}(\theta) \leqq \int \theta(X_1)dP \leqq \int \theta(X_{\tau_k})dP = \int \theta(X_k)dQ \quad \text{for all } k.$$

Moreover if $l$ is an integer such that $\tau_m \leqq l$ then

$$\int \theta(X_k)dQ = \int \theta(X_{\tau_k})dP \leqq \int \theta(X_l)dP \leqq \bar{r}(\theta).$$

Therefore $Q$ satisfies condition $(\phi, \underline{r}, \bar{r})$. Condition $(L, U)$ follows trivially. Therefore $Q \in M$. This completes the proof.

Let $B_1, B_2, \ldots, B_m$ be Borel sets. Let

$$T = \{X_{n_1} \in B_1 \text{ for some } n_1, X_{n_2} \in B_2 \text{ for some } n_2 \geqq n_1, \ldots, X_{n_m} \in P_m \text{ for some } n_m \geqq n_{m-1}\}.$$

THEOREM 1. *Let $M$ be the collection of all submartingales satisfying certain conditions of type $(\phi, \underline{r}, \bar{r})$ and/or type $(L, U)$. Then*

$$\sup_{P\in M} P\{T\} = \sup_{P\in M} P\{X_1 \in B_1, \ldots, X_m \in B_m\}.$$

*Proof*. For $\omega = (x_1, x_2, \ldots) \in R^\infty$, let

$$\tau_1 = \begin{cases} \text{least } n_1 \text{ (if any) such that } x_{n_1} \in B_1, \\ \infty \quad \text{if there is no such } n_1; \end{cases}$$

$$\tau_2 = \begin{cases} \text{least } n_2 \geqq n_1 \text{ (if any) such that } x_{n_2} \in B_2, \\ \infty \quad \text{if there is no such } n_2; \end{cases}$$

$$\tau_m = \begin{cases} \text{least } n_m \geqq n_{m-1} \text{ (if any) such that } x_{n_m} \in B_m, \\ \infty \quad \text{if there is no such } n_m. \end{cases}$$

Therefore $\tau_1 \leqq \tau_2 \leqq \ldots \leqq \tau_m$ and

$$P\{T\} = P\{\tau_m < \infty\} = \lim_{n\to\infty} P\{\tau_m \leqslant n\}$$
$$= \lim_{n\to\infty} P\{X_{\tau_1 \wedge n} \in B_1, \ldots, X_{\tau_m \wedge n} \in B_m.$$

However, by Lemma 1.a),

$$P\{X_{\tau_1\wedge n} \in B_1, \ldots, X_{\tau_m\wedge n} \in B_m\} = Q\{X_1 \in B_1, \ldots, X_m \in B_m\}$$

for some $Q \in M$. Hence

$$\lim_{n\to\infty} P\{X_{\tau_1 \wedge n} \in B_1, \ldots, X_{\tau_m \wedge n} \in B_m\} \leqq \sup_{P\in M} P\{X_1 \in B_1, \ldots, X_m \in B_m\},$$

and we have

$$\sup_{P\in M} P\{T\} \leqq \sup_{P\in M} P\{X_1 \in B_1, \ldots, X_m \in B_m\}.$$

The reverse inequality is immediate, completing the proof.

Theorem 1 provides a prescription for obtaining sharp bounds for upcrossing probabilities. Essentially it says stopping times are unnecessary.

For any pair of real numbers $a < b$ and any $\omega = (x_1, x_2, \ldots) \in R^\infty$ define $\gamma_{ab}$ to be the number of upcrossings of the interval $[a, b]$. Define

$$S_n = \{X_1 \leqq a, X_2 \geqq b, \ldots, X_{2n-1} \leqq a\} \quad \text{and}$$
$$T_n = \{X_1 \leqq a, X_2 \geqq b, \ldots, X_{2n-1} \leqq a, X_{2n} \geqq b\}$$

for all $n$.

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LEMMA 2.a. *Let* $a$, $c$, $r$ *and* $W$ *be reals such that* $a < c$ *and* $W \in [0, 1)$; *let* $l \in R$ *be defined by* $Wa + (1 - W)l = c$ *and let* $\beta$ *be the two point probability* $\beta = W\delta_a + (1 - W)\delta_l$ ($\delta_a$ *and* $\delta_l$ *are point probabilities at* $a$ *and* $l$ *respectively*). *Then among all probabilities* $\mu$ *on* $R$ *such that* $\mu((-\infty, a]) \geqq W$ *and* $\int x\mu(dx) \geqq c$, $\beta$ *minimizes*

$$\theta(r) \cdot \mu(-\infty, r] + \int_{x>r} \theta(x)\, \mu(dx) = \int \theta(r) \vee \theta(x)\, \mu(dx)$$

*whatever convex, increasing function* $\theta$ *may be.*

*Proof.* Let $\mathscr{C}$ be the class of convex, increasing polygonal functions with a finite number of vertices. It is clear that for any convex increasing function $\theta$,

$$\int \theta(r) \vee \theta(dx) = \sup_{\substack{g\in\mathscr{C} \\ g\leqq\theta}} \int \theta(r) \vee g(x)\, \mu(dx).$$

Thus to show $\int \theta(r) \vee \theta(x)\ \mu(dx) \geqq \int \theta(r) \vee \theta(x)\ \beta(dx)$ it suffices to show $\int \theta(r) \vee g(x)\ \mu(dx) \geqq \int \theta(r) \vee g(x)\ \beta(dx)$ for all $g \in \mathscr{C}$. Now if $g \in \mathscr{C}$, then $\theta(r) \vee g$ may be represented in the following form:

$$\theta(r) \vee g(x) = \theta(r) + d_1(x \vee r - r) + (d_2 - d_1)(x \vee x_1 - x_1) + \ldots + (d_n - d_{n-1})(x \vee x_n - x_n),$$

where $\{(r, \theta(r)), (x_1, g(x_1)), \ldots, (x_n, g(x_n))\}$ are the vertices of $\theta(r) \vee g$ and $0 < d_1 < d_2 \ldots < d_n$. By linearity then, to show $\int \theta(r) \vee g(x)\ \ \mu(dx) \geqq \int \theta(r) \vee g(x)\ \beta(dx)$ it is enough to show $\int y \vee x\, \mu(dx) \geqq \int y \vee x\, \beta(dx)$ for all $y \in R$.

When $y \in [a, l]$,

$$\begin{aligned}\int x \vee y\, \mu(dx) &\geqq \int x\, \mu(dx) + (y - a)\, \mu((-\infty, a]) \\ &\geqq c + (y - a)W \\ &= yW + l(1 - W) = \int x \vee y\, \beta(dy).\end{aligned}$$

When $y \leqq a$,

$$\int x \vee y\, \mu(dx) \geqq \int x\, \mu(dx) \geqq c = \int x \vee y\, \beta(dx).$$

When $y \geqq l$,

$$\int x \vee y\mu(dx) \geqq y \geqq l = \int x \vee y\beta(dx).$$

LEMMA 2.b. *With* $a$, $c$, $W$, $l$, $\mu$ *and* $\beta$ *as in Lemma* 2.a *we have*

$$\int_{a^+}^{\infty} \theta(x)\, \mu(dx) \geqq \int_{a^+}^{\infty} \theta(x)\, \beta(dx)$$

*for all convex, increasing functions* $\theta$.

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*Proof.* Let $\mu((-\infty, a]) = \tilde{W}$. Define $\tilde{l}$ by $\tilde{W}a + (1 - \tilde{W})\tilde{l} = c$ and let $\tilde{\beta} = \tilde{W}\delta_a + (1 - \tilde{W})\delta_{\tilde{l}} = c$. By Lemma 2.a, taking $r = a$

$$\theta(a)\,\mu(-\infty, a] + \int_{a^+}^{\infty} \theta(x)\,\mu(dx) \geqq \theta(a)\,\tilde{\beta}(-\infty, a] + \int_{a^+}^{\infty} \theta(x)\,\tilde{\beta}(dx)$$

for all convex, increasing functions $\theta$. However $\mu(-\infty, a] = \tilde{\beta}(-\infty, a]$, hence

$$\int_{a^+}^{\infty} \theta(x)\,\mu(dx) \geqq \int_{a^+}^{\infty} \theta(x)\,\tilde{\beta}(dx) = \theta(\tilde{l})(1 - \tilde{W})$$

$$= (1 - \tilde{W})\theta\left(\frac{c - \tilde{W}(a)}{1 - \tilde{W}}\right) = (1 - \tilde{W})\theta\left(\frac{c - a}{1 - \tilde{W}} + a\right).$$

However, $(1 - s)\theta\ ((c - a)/(1 - s) + a)$ is an increasing function in $s \in [0, 1]$ (by a supporting hyperplane argument), and $\tilde{W} \geqq W$ by Lemma 2.a. Therefore,

$$\int_{a^+}^{\infty} \theta(x)\,\mu(dx) \geqq (1 - \tilde{W})\theta\left(\frac{c - a}{1 - \tilde{W}} + a\right)$$

$$\geqq (1 - W)\theta\left(\frac{c - a}{1 - W} + a\right)$$

$$= \int_{a^+}^{\infty} \theta(x)\,\beta(dx).$$

THEOREM 2. *Let $P$ be a submartingale satisfying a condition of type $(\phi, \bar{r})$ and such that*

$$\int X_1\, dP \geqq m.$$

*Let the submartingale $Q^n$ on $(R^\infty, B^\infty)$ be defined by:*

$$Q^n\{X_1 = m\} = 1 - q_0$$
$$Q^n\{X_1 = a\} = q_0 - q_1$$
$$Q^n\{X_1 = l_1\} = q_1$$
$$Q^n\{X_2 = X_1 | X_1 \neq a\} = 1$$
$$Q^n\{X_2 = b | X_1 = a\} = 1$$

$$(A)\ \begin{cases} Q^n\{X_3 = X_2 | X_2 \neq b\} = 1 \\ Q^n\{X_3 = a | X_2 = b\} = 1 - q_2 \\ Q^n\{X_3 = l_2 | X_2 = b\} = q_2 \end{cases}$$

$$(B)\ \begin{cases} Q^n\{X_4 = X_3 | X_3 \neq a\} = 1 \\ Q^n\{X_4 = b | X_3 = a\} = 1 \end{cases}$$

*and so on repeating $(A)$ and $(B)$ for $X_5, X_6, X_7, X_8, \ldots, X_{2n-1}, X_{2n}$. $X_{k+1} = X_k$ for $k \geqq 2n$. There exist $0 \leqq q_0, q_1, \ldots, q_n \leqq 1$ and $b \leqq l_1, \ldots, l_n$ such that*

(a) $q_1 = 0$ *if* $m \leqq a$

$q_1 l_1 + (1 - q_1)a = m, q_0 = 1$ *if* $m > a$

$q_k l_k + (1 - q_k)a = b$ *if* $1 < k \leqq n$;

(b) *Q satisfies* $(\phi, \bar{r})$;

(c) $\int X_1 dQ \geqq m$; *and*

(d) $P\{T_n\} = Q\{T_n\}$.

(We remark that the trajectories are a.s. $-Q$ of the form (if $m \geqq a$):

$(l_1, l_1, l_1, \ldots)$ w.p. $q_1$;

$(a, b, a, b, \ldots a, b, l_k, l_k, \ldots)$ w.p. $(1 - q_1)(1 - q_2) \ldots (1 - q_{k-1})q_k$

for $1 \leqq k \leqq n$; and

$(a, b, a, b, \ldots a, b, a, b, b, \ldots)$ w.p. $(1 - q_1) \ldots (1 - q_n)$.

If $m < a$, the trajectory $(l_1, l_1, \ldots)$ is replaced by the trajectory $(m, m, \ldots)$ having probability $1 - q_0$).

*Proof.* We proceed by induction. Suppose the theorem is true for $k \leqq n - 1$. Then there exists a submartingale $Q^{n-1}$ of the above form (along with $q_0, q_1, \ldots, q_{n-1}$ and $l_1, l_2, \ldots, l_{n-1}$) such that $\int \theta(X_k)dQ^{n-1} \leqq \int \theta(X_k)dP$ for $1 \leqq k \leqq 2n - 2$, and $Q^{n-1}\{T_{n-1}\} = P\{T_{n-1}\}$. Now define $1 - q_n = P\{T_n | T_{n-1}\}$ and $l_n$ by $q_n l_n + (1 - q_n)a = b$. Define $Q^n$ using $q_0, q_1, \ldots q_n$ and $l_1, l_2, \ldots, l_n$. Now consider the probability $\mu(dx) = P\{X_{2n-1} \in dx | T_{n-1}\}$. By Lemma 2.a taking $r = -\infty$, the two point probability $\beta = q_n \delta_{l_n} + (1 - q_n)\delta_a$ satisfies

$$\frac{1}{P\{T_{n-1}\}} \int_{T_{n-1}} \theta(X_{2n-1})dP = \int \theta(x)\, \mu(dx) \geqq \int \theta(x)\, \beta(dx)$$
$$= \frac{1}{P\{T_n\}} \int_{T_{n-1}} \theta(X_{2n-1})dQ^n.$$

Hence

$$(1) \quad \int_{T_{n-1}} \theta(X_{2n-1})dP \geqq \int_{T_{n-1}} \theta(X_{2n-1})dQ^n.$$

Next,

$$(2) \quad \int_{T_{n-1}^c} \theta(X_{2n-1})dP \geqq \int_{T_{n-1}^c} \theta(X_{2n-2})dP \geqq \int_{T_{n-1}^c} \theta(X_{2n-2})dQ^{n-1}$$
$$= \int_{T_{n-1}^c} \theta(X_{2n-1})dQ^n.$$

So

$$\int \theta(X_{2n-1})dP = \int_{T_{n-1}} \theta(X_{2n-1})dP + \int_{T_{n-1}^c} \theta(X_{2n-1})dP$$
$$\geqq \int \theta(X_{2n-1})dQ^n$$

by (1) and (2).

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Next,

$$\int \theta(X_{2n})dP = \int_{T_n} \theta(X_{2n})dP + \int_{S_n - T_n} \theta(X_{2n})dP + \int_{T_{n-1}-S_n} \theta(X_{2n})dP$$

$$(3)\quad + \int_{T_{n-1}^c} \theta(X_{2n})dP \geqq bP\{T_n\} + \int_{T_{n-1}-S_n} \theta(X_{2n-1})dP + \int_{T_{n-1}^c} \theta(X_{2n-2})dP.$$

Again defining $\mu(dx) = P\{X_{2n-1} \in dx | T_{n-1}\}$,

$$\int_{T_{n-1}-S_n} \theta(X_{2n-1})dP = P\{T_{n-1}\} \int_{a^+}^{\infty} \theta(x)\, \mu(dx)$$

$$\geqq P\{T_{n-1}\} \left( \int_{a^+}^{\infty} \theta(x)\, \beta(dx) \right) = \int_{T_{n-1}-S_n} \theta(X_{2n-1})dQ^n \quad \text{(by Lemma 2.b).}$$

Hence from (3),

$$\int \theta(X_{2n})dP \geqq bP\{T_n\} + \int_{T_{n-1}-S_n} \theta(X_{2n-1})dQ^n + \int_{T_{n-1}^c} \theta(X_{2n-2})dQ^n$$

$$= \int \theta(X_{2n})dQ^n.$$

Therefore, $\int \theta(X_k)dQ^n \leqq \int \theta(X_k)dP$ for $1 \leqq k \leqq 2n$; $Q^n\{T_n\} = P\{T_n\}$, and by construction, $Q^n$ is a submartingale.

**3. Applications.** Theorems 1 and 2 provide an algorithm for obtaining sharp upcrossing probabilities. Denote $(x^+)^p$ by $[x]_+^p$.

PROPOSITION 1. *If $M$ is the collection of submartingales such that*

$$\int [X_n - a]_+^p \, dP \leqq L \quad \textit{for all } n, \textit{ and}$$

$$\int X_1 \, dP \geqq m, \quad \textit{where } p \geqq 1, \textit{ and if}$$

$$B_{ab}(k) = \frac{L - [m-a]_+^p}{(b-a)^p} \cdot \frac{(p-1)^{p-1}}{\left\{k + \frac{(m-a)^+}{b-a}(p-1) - 1\right\}^p + (p-1)^{p-1}}$$

*(if $p = 1$, set $(p-1)^{p-1} = 1$), then*

$$\sup_{P \in M} P\{\gamma_{ab} \geqq k\} \leqq B_{ab}(k) \quad \textit{for all } k, \textit{ and}$$

$$\sup_{P \in M} P\{\gamma_{ab} \geqq k\} \sim B_{ab}(k).$$

*Proof.* Let $\theta_1(x) = [x-a]_+^p$, $\bar{r}(\theta_1) = L$ and $\underline{r}(\theta_1) = -\infty$. Let $\theta_2(x) = x - m$, $\underline{r}(\theta_2) = 0$ and $\bar{r}(\theta_2) = \infty$. Let $\phi = \{\theta_1, \theta_2\}$ and $\bar{M}$ be the collection of all submartingales satisyfing condition $(\phi, \underline{r}, \bar{r})$. We check that $\bar{M} = M$. Setting

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$B_1 = (-\infty, a]$, $B_2 = [b, \infty), \ldots, B_{2n-1} = (-\infty, a]$, $B_{2n} = [b, \infty)$ and applying Theorem 1, we have

$$\sup_{Q\in M} Q\{\gamma_{ab} \geqq n\} = \sup_{Q\in M} Q\{X_1 \leqq a, \ldots, X_{2n} \geqq b\}.$$

Next, by Theorem 2,

$$\sup_{Q\in M} Q\{X_1 \leqq a, \ldots, X_{2n} \geqq b\} = \sup_{Q\in \tilde{M}} Q\{X_1 \leqq a, \ldots, X_{2n} \geqq b\},$$

where $\tilde{M}$ is the collection of submartingales in $M$ also having the form given in Theorem 2. Let $Q \in \tilde{M}$. Let

$$Q\{X_1 = m\} = 1 - p_0$$
$$Q\{S_k\} = p_k, \quad k = 1, \ldots, n.$$

Therefore, by the submartingale property, $m \leqq (1 - p_0)m + p_1 a + (p_0 - p_1)l_1$. Clearly equality is best (for satisfying $(\phi, \bar{r})$) so

$$l_1 = \frac{p_0 m - p_1 a}{p_0 - p_1}.$$

Similarly $p_k b = p_{k+1}a + (p_k - p_{k+1})l_{k+1}$, $k = 1, \ldots, n - 1$, so

$$l_{k+1} = \frac{p_k b - p_{k+1}a}{p_k - p_{k+1}}.$$

Next

$$\begin{aligned} L &\geqq \int [X_{2n} - a]_+^p dQ \\ &\geqq (1 - p_0)[m - a]_+^p + (p_0 - p_1)\left[\frac{p_0 m - p_1 a}{p_0 - p_1} - a\right]_+^p \\ &\quad + (p_1 - p_2)\left[\frac{p_1 b - p_2 a}{p_1 - p_2} - a\right]_+^p + \ldots \\ &\quad + (p_{n-1} - p_n)\left[\frac{p_{n-1}b - p_n a}{p_{n-1} - p_n} - a\right]^p + p_n(b - a)^p \\ &= (1 - p_0)[m - a]_+^p + \frac{p_0^p[m - a]_+^p}{(p_0 - p_1)^{p-1}} + \frac{p_1^p}{(p_1 - p_2)^{p-1}}(b - a)^p + \ldots \\ &\qquad + \frac{p_{n-1}^p}{(p_{n-1} - p_n)^{p-1}}(b - a)^p + p_n(b - a)^p. \end{aligned}$$

Let

$$\tilde{L} = \frac{L - [m - a]_+}{(b - a)^p}, \quad \alpha_0 = p_0, \alpha_1 = \frac{p_1}{p_0}, \ldots, \alpha_n = \frac{p_n}{p_{n-1}}.$$

Therefore

$$\begin{aligned} \tilde{L} \geqq \frac{-\alpha_0[m - a]_+^p}{(b - a)^p} + \frac{\alpha_0[m - a]_+^p}{(b - a)^p \cdot (1 - \alpha_1)^{p-1}} + \frac{\alpha_0\alpha_1}{(1 - \alpha_2)^{p-1}} + \ldots \\ + \frac{\alpha_0 \ldots \alpha_{n-1}}{(1 - \alpha_n)^{p-1}} + \alpha_0 \ldots \alpha_n. \end{aligned} \tag{1}$$

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We must now maximize $p_n = \alpha_0 \ldots \alpha_n$ subject to the constraint (1) and $0 \leqq \alpha_0, \ldots, \alpha_n \leqq 1$. Clearly the maximum occurs when (1) is an equality. Solving for $\alpha_0$ we must maximize

$$(2) \quad \left\{\frac{\tilde{L}\alpha_1 \ldots \alpha_n}{\frac{-[m-a]_+^p}{(b-a)^p} + \frac{[m-a]_+^p}{(b-a)^p(1-\alpha_1)^{p-1}} + \ldots + \frac{\alpha_1 \ldots \alpha_{n-1}}{(1-\alpha_n)^{p-1}} + \alpha_1 \ldots \alpha_n}\right\}.$$

Equivalently, we can minimize

$$\frac{1}{\tilde{L}}\left\{\frac{[m-a]_+^p}{(b-a)^p\alpha_1 \ldots \alpha_n}\left(\frac{1}{(1-\alpha_1)^{p-1}} - 1\right) + \frac{1}{(1-\alpha_2)^{p-1}\alpha_2 \ldots \alpha_k} + \ldots + \frac{1}{(1-\alpha_k)^{p-1}\alpha_k} + 1\right\}.$$

Set

$$\gamma_1 = \frac{[m-a]_+^p}{(b-a)^p\alpha_1} \cdot \left(\frac{1}{(1-\alpha_1)^{p-1}} - 1\right).$$

For $m \geqq 2$ set

$$\gamma_m = \frac{[m-a]_+^p}{(b-a)^p\alpha_1 \ldots \alpha_m}\left(\frac{1}{(1-\alpha_1)^{p-1}} - 1\right) + \frac{1}{(1-\alpha_2)^{p-1}\alpha_2 \ldots \alpha_m} + \ldots + \frac{1}{(1-\alpha_m)^{p-1}\alpha_m}.$$

Therefore for $k \geqq 2$,

$$(3) \quad \gamma_k = \frac{\gamma_{k-1}}{\alpha_k} + \frac{1}{(1-\alpha_k)^{p-1}\alpha_k}.$$

We wish to minimize $\gamma_n$ by choosing $\bar{\alpha}_1, \ldots, \bar{\alpha}_n$. However, $\gamma_{n-1}$ depends only on $\alpha_1, \ldots, \alpha_{n-1}$; therefore, at the minimum,

$$0 = \left.\frac{d\gamma_n}{d\alpha_n}\right|_{\bar{\alpha}_n} = \frac{-\gamma_{n-1}}{\bar{\alpha}_n^{\,2}} - \frac{1}{(1-\bar{\alpha}_n)^{p-1}\bar{\alpha}_n^{\,2}} + \frac{p-1}{(1-\bar{\alpha}_n)^p\bar{\alpha}_n}.$$

This gives

$$\gamma_{n-1} = \frac{p\bar{\alpha}_n - 1}{(1-\bar{\alpha}_n)^p}.$$

Substituting back into (3) also gives $\gamma_n = (p-1)/(1-\bar{\alpha}_n)^p$. Now at the minimum,

$$0 = \frac{d\gamma_{n-1}}{d\alpha_{n-1}} = \frac{d\gamma_{n-1}}{d\alpha_{n-2}} = \ldots = \frac{d\gamma_{n-1}}{d\alpha_1},$$

so the above relations hold for each level. Hence, (henceforth $\gamma_k$ represents the minimum value)

(4) $\gamma_{k-1} = \dfrac{p\bar{\alpha}_k - 1}{(1 - \bar{\alpha}_k)^p}$, and

(5) $\gamma_k = \dfrac{p - 1}{(1 - \bar{\alpha}_k)^p}$ for $k \geqq 2$.

We remark that

$$\left.\frac{\tilde{L}}{\gamma_k}\right|_{\bar{\alpha}_1, \ldots, \bar{\alpha}_k} \wedge 1$$

yields the maximum probability of $k$ upcrossings under our constraints for all $k \geqq 1$. With our recurrence relations we now examine the asymptotic behavior of $\gamma_k$.

$$\frac{\gamma_{k-1}}{\gamma_k} = \frac{p\bar{\alpha}_k - 1}{p - 1} = 1 - \frac{p(1 - \bar{\alpha}_k)}{p - 1}.$$

Also from (5), $(1 - \bar{\alpha}_k) = (p - 1)^{1/p}\gamma_k^{-1/p}$; hence

(6) $\dfrac{\gamma_{k-1}}{\gamma_k} = 1 - \dfrac{p}{(p - 1)^{(p-1)/p}}\gamma_k^{-1/p}$, or

(7) $\gamma_k - \gamma_{k-1} = C\gamma_k^{1-1/p}$; $C = \dfrac{p}{(p - 1)^{(p-1)/p}}$.

Now consider the equation $d\gamma(t)/dt = C\gamma(t)^{1-1/p}$. Solutions are of the form $\gamma(t) = ((Ct)/p) + C_1)^p$ where $C_1$ is a constant. Also

$$\begin{aligned}\gamma(k) - \gamma(k - 1) &= \dot{\gamma}(s) \quad \text{for some } k - 1 \leqq s \leqq k,\\ &= C\gamma(s)^{1-1/p}\\ &\leqq C\gamma(k)^{1-1/p},\end{aligned}$$

since solutions are increasing. Therefore $\gamma(k)$ increases slower than $\gamma_k$.

Next from, (6) we have $\lim_{k\to\infty} \gamma_{k-1}/\gamma_k = 1$, so for all $\delta > 1$ there exists an $n_0$ such that for $k \geqq n_0$,

$$\gamma_k^{1-1/p} \leqq \delta\gamma_{k-1}^{1-1/p} \quad (\delta - 1 \text{ is small}).$$

Hence, $\gamma_k - \gamma_{k-1} \leqq \delta C\gamma_{k-1}^{1-1/p}$. Now let $d\tilde{\gamma}(t)/dt = \delta C\tilde{\gamma}(t)^{1-1/p}$. Hence

$$\begin{aligned}\tilde{\gamma}(k) - \tilde{\gamma}(k - 1) &= \dot{\tilde{\gamma}}(s) \quad \text{for some } k - 1 \leqq s \leqq k\\ &= \delta C\tilde{\gamma}(s)^{1-1/p}\\ &\geqq \delta C\tilde{\gamma}(k - 1)^{1-1/p}.\end{aligned}$$

Therefore $\tilde{\gamma}(k)$ increases faster than $\gamma_k$. Also as before $\tilde{\gamma}(t) = ((\delta Ct)/p) + C_2)^p$, where $C_2$ is a positive constant. Hence $\gamma_k$ is $o(k^p)$. Moreover, we can solve explicitly for $\gamma_1$.

$$\gamma_1 = \frac{[m - a]_+^p}{(b - a)^p}\frac{1}{\alpha_1}\left(\frac{1}{(1 - \alpha_1)^{p-1}} - 1\right)$$

https://doi.org/10.4153/CJM-1977-083-0 Published online by Cambridge University Press

is increasing in $0 \leqq \alpha_1 \leqq 1$. Hence the minimum is

$$\lim_{\alpha_1 \to 0} \gamma_1 = \frac{[m-a]_+^p}{(b-a)^p}(p-1).$$

If we set $\gamma(1) = \gamma_1$ we have

$$\left(\frac{C}{p} + C_1\right)^p = \frac{[m-a]_+^p}{(b-a)^p}(p-1)$$

and after substitution

$$\gamma(t) = \frac{1}{(p-1)^{p-1}}\left(t + \frac{(m-a)^+}{(b-a)}(p-1) - 1\right)^p.$$

Hence

$$\frac{\tilde{L}}{\gamma_k + 1} \leqq \frac{\tilde{L}}{\gamma(k) + 1} = \frac{L - [m-a]_+^p}{(b-a)^p} \times \frac{(p-1)^{p-1}}{\left(k + \frac{(m-a)^+}{b-a}(p-1) - 1\right)^p + (p-1)^{p-1}}.$$

Since the probability of $k$ upcrossings is at most $\tilde{L}/(\gamma_k + 1)$ we have our bound.

We now set $\tilde{\gamma}(n_0) = \gamma_{n_0}$, thereby determining $C_2$. Hence

$$\frac{\gamma_k}{\gamma(k)} \leqq \frac{\tilde{\gamma}(k)}{\gamma(k)} \quad \text{for } k \geqq n_0$$
$$\leqq \frac{\left(\frac{\delta C k}{p} + C_2\right)^p}{\left(\frac{Ck}{p} + C_1\right)^p}.$$

Therefore

$$\overline{\lim_{k\to\infty}} \frac{\gamma_k}{\gamma(k)} \leqq \delta^p;$$

but $\delta - 1$ is arbitrarily small. Therefore

$$\overline{\lim_{k\to\infty}} \frac{\gamma_k}{\gamma(k)} \leqq 1.$$

Hence

$$\lim_{k\to\infty} \frac{\dfrac{\tilde{L}}{\gamma(k)+1}}{\dfrac{\tilde{L}}{\gamma_k + 1}} = 1,$$

so

$$\sup_{P \in M} P\{\gamma_{ab} \geqq k\} \sim B_{ab}(k).$$

For $p = 1$ it is easiest to maximize the expression (2). The maximum is $\tilde{L}/k$. This completes the proof.

We could generalize Proposition 1 by supposing $EX_1 = m$ and $E\theta(X_n - a) \leqq L$ for all $n$ where $\theta$ is an increasing convex function with derivative $\theta'$. The above proof goes through and (7) becomes $\gamma_k - \gamma_{k-1} = \theta' \circ \alpha(\gamma_k)$ where $\alpha \circ (\theta'(x)x - \theta(x)) = x$. In a particular case we may be able to proceed (as above) from here.

COROLLARY 1. *Doob's upcrossing inequality is sharp.*

*Proof.* For any submartingale $P$ such that $\int (X_n - a)^+ dP \leqq L$ for all $n$ and $\int X_1 = m$, Doob's inequality says

$$\int \gamma_{ab}\, dP \leqq \frac{L - (m-a)^+}{b-a}.$$

Applying Chebyschev's inequality, we have

$$P\{\gamma_{ab} \geqq k\} \leqq \frac{L - (m-a)^+}{k(b-a)}$$

which is precisely the bound given in Proposition 1 for $p = 1$ (Prof. David Heath pointed this out). Proposition 1 provides the construction of a submartingale (almost) attaining this bound (in fact $\alpha_2 = \alpha_3 = \ldots = \alpha_k = 1$ means $l_1 = l_2 = \ldots = l_k = \infty$ so at best by taking $l_1, \ldots, l_k$ large we may almost attain the bound). Hence Doob's inequality must also be sharp.

It is in fact possible to obtain Doob's upcrossing inequality directly by these methods (see [**2**]).

*Example* 2. (Dubins' inequality—see [**3**, p. 27]).

PROPOSITION 2. *If $P$ is a submartingale such that $P\{L \leqq X_n \leqq U\} = 1$ for constants $L$, $U$ ($L \leqq U$) for all $n$, and $\int X_1\, dP \geqq m$, then*

$$P\{\gamma_{ab} \geqq k\} \leqq \left(\frac{U - m \vee a}{U-a}\right)\left(\frac{U-b}{U-a}\right)^{k-1}$$

*for $L \leqq a < b \leqq U$, where $m \vee a$ =* max $\{m, a\}$.

*Proof.* Define $\theta(x) = x$, $\underline{r}(\theta) = m$, $\bar{r}(\theta) = \infty$ and $\phi = \{\theta\}$. Let $M$ be the class of all submartingales satisfying conditions ($\phi$, $r$, $\bar{r}$) and ($L$, $U$). It is clear that $M$ consists of exactly those submartingales satisfying our hypotheses. Therefore by Theorem 1,

$$\sup_{P \in M} P\{\gamma_{ab} \geqq k\} = \sup_{P \in M} P\{T_k\}.$$

For any $P \in M$,

$$\int_{T_{k-1}} X_{2k-2}\,dP \leqq \int_{T_{k-1}} X_{2k-1}\,dP = \int_{S_k} X_{2k-1}\,dP + \int_{T_{k-1} \cap S_k{}^c} X_{2k-1}\,dP.$$

So $bP\{T_{k-1}\} \leqq P\{S_k\}a + (P\{T_{k-1}\} - P\{S_k\})U$. Hence

$$P\{T_k\} \leqq P\{S_k\} \leqq \frac{U-b}{U-a} P\{T_{k-1}\}.$$

Next $m \leqq \int X_1\,dP = \int_{S_1} X_1\,dP + \int_{S_1{}^c} X_1\,dP$, so $m \leqq P\{S_1\}a + (1 - P\{S_1\})U$. Hence

$$P\{T_1\} \leqq P\{S_1\} \leqq \left(\frac{U-m}{U-a}\right) \wedge 1.$$

By iteration we have

$$P\{T_k\} \leqq \left(\frac{U - m \vee a}{U-a}\right)\left(\frac{U-b}{U-a}\right)^{k-1}.$$

Again this bound is sharp. A martingale with precisely these upcrossing probabilities is given as an Exercise II-2 in [**3**].

I thank Prof. Harry Kesten for his help with the asymptotic analysis in Proposition 1. Thanks also to the referee for the current improved version of Lemma 2.a.b.

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