Appendix C

Annihilation and creation operators

C.1 The simple harmonic oscillator

The reader may well have met annihilation and creation operators in treating the quantum mechanics of the simple harmonic oscillator. In this context, an operator a and its Hermitian conjugate $a^\dagger$ are constructed. These satisfy the commutation relations

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = 1 \tag{C.1}$$

and also of course

$$[a, a] = 0, \quad [a^\dagger, a^\dagger] = 0.$$

The operator $N = a^\dagger a$ is Hermitian. We denote by $|n\rangle$ the normalised eigenstate of N with eigenvalue n. Since $n = \langle n|a^\dagger a|n\rangle$ is the modulus squared of the state $a|n\rangle$, n is real and ≥ 0, and equal to 0 only if $a|n\rangle = 0$.

It follows from the commutation relations that the lowest eigenstate of n is $n = 0$, corresponding to the ground state $|0\rangle$. This is because

$$Na|n\rangle \;=\; a^\dagger aa|n\rangle = (aa^\dagger - 1)a|n\rangle = (n-1)a|n\rangle.$$

Thus $a|n\rangle$ is, apart from normalisation, an eigenstate of N with eigenvalue $(n - 1)$, unless $a|n\rangle = 0$. Similarly $a|n - 1\rangle$ is an eigenstate of N with eigenvalue $(n - 2)$, and so on. The process must terminate at the eigenstate $|0\rangle$ with eigenvalue 0, and $a|0\rangle = 0$, since otherwise we would be able to violate the condition $n \geq 0$.

Similarly $a^\dagger|n\rangle$ is, apart from normalisation, an eigenstate of N with eigenvalue $(n+1)$. Thus the eigenvalues of the *number operator* N are the integers 0, 1, 2, 3 . . .

Since $\langle n|a^\dagger a|n\rangle = n$, we have

$$a|n\rangle = n^{1/2}|n - 1\rangle. \tag{C.2}$$

Also, $\langle n|aa^\dagger|n\rangle = \langle n|a^\dagger a + 1|n\rangle = n + 1$, so that

$$a^\dagger|n\rangle = (n + 1)^{1/2}|n + 1\rangle. \tag{C.3}$$

We call a an *annihilation operator* and $a^\dagger$ a *creation operator.*

Written in terms of a and $a^\dagger$, the simple harmonic oscillator Hamiltonian becomes

$$H = \left(a^\dagger a + \frac{1}{2}\right)\hbar\omega = \left(N + \frac{1}{2}\right)\hbar\omega, \tag{C.4}$$

where ω is the frequency of the corresponding classical oscillator (Problem C.1). The term $\frac{1}{2}\hbar\omega$ is the zero-point energy. Since in field theory only energy differences are of physical

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significance, it is usually convenient to redefine H, dropping the zero-point energy and taking $H = a^\dagger a\hbar\omega$. We may then reinterpret the state $|n\rangle$ as a state in which there are n identical 'particles' each of energy $\hbar\omega$, associated with the oscillator, and say that a and $a^\dagger$ annihilate and create particles.

In the Heisenberg representation (Section 8.2),

$$a(t) = \mathrm{e}^{\mathrm{i}Ht} a \mathrm{e}^{-\mathrm{i}Ht} = \mathrm{e}^{\mathrm{i}N\omega t}\, a \mathrm{e}^{-\mathrm{i}N\omega t} = \mathrm{e}^{-\mathrm{i}\omega t} a. \tag{C.5}$$

This may be seen by considering the effect of $a(t)$ acting on a state $|n\rangle$, and noting that, since

$$\mathrm{e}^{\pm \mathrm{i}N\omega t}|n\rangle = \mathrm{e}^{\pm n\omega t}|n\rangle,$$

the two expressions for $a(t)$ give the same result. Similarly,

$$a^\dagger(t) = \mathrm{e}^{\mathrm{i}\omega t} a^\dagger. \tag{C.6}$$

C.2 An assembly of bosons

A similar operator formalism may be developed for assemblies of identical particles. We set out first the formalism when the particles are bosons.

Let $u_i(\xi)$ be a complete set of single particle states, where ξ stands for the space and spin coordinate of a particle. We define annihilation and creation operators a_i and $a_i^\dagger$ for each state, satisfying the commutation relations

$$[a_i, a_j{}^\dagger] = \delta_{ij}, \qquad [a_i,\ a_j] = 0, \qquad [a_i{}^\dagger, a_j{}^\dagger] = 0. \tag{C.7}$$

Any state of the system can be constructed by operating on the *vacuum state* $|0\rangle$, in which there are no particles present, and $a_i|0\rangle = 0$ for all i. For example, a three-particle state having two particles in the state u_1 and one particle in the state u_2 is given (apart from normalisation) by $a_1^\dagger a_1^\dagger a_2^\dagger|0\rangle$. Evidently such a state is symmetric in the interchange of any two particles since the creation operators all commute, and the particles will obey Bose–Einstein statistics.

It follows from the commutation relations that the number operator $N_i = a_i^\dagger a_i$ gives the number of particles in the state u_i. In the case of non-interacting bosons, the $u_i(\xi)$ can be taken as the single particle energy eigenstates and the Hamiltonian operator is then

$$H_0 = \sum_i a_i^\dagger a_i \varepsilon_i = \sum_i N_i \varepsilon_i, \tag{C.7}$$

where the ε_i are the single particle energy levels.

In the Heisenberg representation and with the free particle Hamiltonian H_0, the time dependence of the annihilation and creation operators is like that of simple harmonic oscillator operators, and follows by a similar argument:

$$a_i(t) = \mathrm{e}^{-\mathrm{i}\varepsilon_i t} a_i, \qquad a_i^\dagger(t) = \mathrm{e}^{\mathrm{i}\varepsilon_i t} a_i^\dagger. \tag{C.8}$$

C.3 An assembly of fermions

In the case of an assembly of identical fermions, we define annihilation and creation operators b_i and $b_i{}^\dagger$ for each single particle state $u_i(\xi)$, which are *anticommuting:*

$$\{b_i, b_j{}^\dagger\} = b_i b_j{}^\dagger + b_j{}^\dagger b_i = \delta_{ij}, \qquad \{b_i, b_j\} = 0, \qquad \{b_i{}^\dagger, b_j{}^\dagger\} = 0. \tag{C.9}$$

In particular,

$$(b_i)^2 = 0,\ (b_j{}^\dagger)^2 = 0. \tag{C.10}$$

Thus two fermions cannot be annihilated from the same state, or created in the same state, in accord with the Pauli principle.

The number operator $N_i = b_i{}^\dagger b_i$ satisfies

$$N_i^2 = b_i{}^\dagger b_i b_i{}^\dagger b_i = b_i{}^\dagger(1 - b_i{}^\dagger b_i)\, b_i = b_i{}^\dagger b_i = N_i,$$

or

$$N_i(N_i - 1) = 0,$$

so that the eigenvalues of N_i are 0 and 1. This, again, is in accord with the Pauli principle. A many-particle fermion state can be constructed by operating on the vacuum state $|0\rangle$ with creation operators. For example $b_1{}^\dagger b_2{}^\dagger b_5{}^\dagger|0\rangle$ is a state with a fermion in each of the states u_1, u_2, u_5. Such a state is antisymmetric under particle exchange, and the particles obey Fermi–Dirac statistics.

In the case of an assembly of non-interacting fermions, the Hamiltonian operator is

$$H_0 = \sum_i b_i{}^\dagger b_i \varepsilon_i, \tag{C.11}$$

and in the Heisenberg representation

$$b_i(t) = \mathrm{e}^{-\mathrm{i}\varepsilon_i t} b_i, \qquad b_i{}^\dagger(t) = \mathrm{e}^{\mathrm{i}\varepsilon_i t} b_i{}^\dagger. \tag{C.12}$$

Problems

C.1 With rescaling of coordinates,

$$P = p/(m\hbar\omega)^{1/2}, \qquad X = x(m\omega/\hbar)^{1/2},$$

the simple harmonic oscillator Hamiltonian

$$H = (p^2/2m) + (m\omega^2 x^2/2)$$

becomes

$$H = (\hbar\omega/2)(P^2 + X^2),$$

and

$$[X, P] = \mathrm{i}$$

Show that if $a = (1/\sqrt{2})(X + \mathrm{i}P)$, $a^\dagger = (1/\sqrt{2})(X - \mathrm{i}P)$, then

$$[a, a^\dagger] = 1 \quad \text{and} \quad H = (a^\dagger a + \frac{1}{2})\hbar\omega.$$

C.2 Show that the normalised ground state wave function of the simple harmonic oscillator is $(m\omega/\pi\hbar)^{1/4}\exp(-m\omega x^2/2\hbar)$.

C.3 Using the commutation relations for fermions show that the state $b_i{}^\dagger|0\rangle$ is an eigenstate of $N_i = b_i{}^\dagger b_i$ with eigenvalue 1.

C.4 Show that the matrices

$$\mathbf{b} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{b}^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

satisfy the commutation relations for fermion annihilation and creation operators.