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AN EXTENSION OF MÜNTZ'S THEOREMS IN MULTIVARIABLES

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Müntz's theorems give necessary and sufficient conditions for a sequence of powers in one variable to be complete in the spaces of all real-valued continuous functions or square integrable functions with the usual norms.

The purpose of this paper is to give an extension of these theorems to multivariable cases. In other words, taking some sequences of generalized multivariable polynomials, we obtain some necessary and sufficient conditions for these sequences to be complete in function spaces analogous to the above.

Let N be the set of all natural numbers and $\{x^{a_i}\}_{i\in N}$ a sequence of powers with $a_i \in \mathbb{R}$. Then, under what circumstances can continuous functions or functions in L^2 be approximated by linear combinations of these powers? As is well known Müntz [4] studied these problems in depth and solved them. Nowadays these results are introduced as Müntz's Theorems in Cheney [1], Davis [2], Watson [5] and so on.

The aim of this paper is to study the analogous problems for multi-variables and to obtain more general results by replacing x^{a_i} by $\psi(x)^{a_i}$

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for some function $\psi(x)$. In section 1, we present some lemmas which are used later. In section 2, we study some necessary and sufficient conditions for any continuous function defined on $[0,1]^n \subset R^n$ to be approximated by a linear combination of some sequence of generalized powers in the uniform norm. In Section 3, we treat the same problem in the L^p-norm$(1 \leq p < +\infty)$.

1. Preliminaries

We first define the notion of completeness.

DEFINITION. *A set of vectors* $\{V_n\}$ *in a Banach space* B *is said to be complete in* B *, if every vector* X *of* B *can be approximated to any degree of accuracy by a linear combination of* $\{V_n\}$.

Let $C[0,1]$ be the vector space of all real-valued continuous functions, defined on $[0,1]$, with the uniform norm. Then, it is well known that, by Weierstrass' theorem, $\{1\} \cup \{x^n\}_{n \in \mathbb{N}}$ is complete in $C[0,1]$. Generalizing the above result, we consider the following problem. Let $\{a_i\}_{i \in \mathbb{N}}$ be a sequence of real numbers. Under what conditions is $\{x^{a_i}\}_{i \in \mathbb{N}}$ complete in $C[0,1]$ or in $L^2[0,1]$ with the usual norms? The answers to these problems, namely Müntz's Theorems (see Cheney [1] and Davis [2]), are as follows.

THEOREM A. $\{x^{a_i}\}_{i \in \mathbb{N}}$ *with* $-1/2 < a_n \to +\infty$ *is complete in* $L^2[0,1]$ *with the least-squares norm if and only if* $\sum_{a_i \neq 0} 1/a_i = +\infty$.

THEOREM B. $\{1\} \cup \{x^{a_i}\}_{i \in \mathbb{N}}$ *with* $0 < a_1 < a_2 < \dots$ *is complete in* $C[0,1]$ *with the uniform norm if and only if* $\sum_i 1/a_i = +\infty$.

Let N_0 denote $\mathbb{N} \cup \{0\}$, I the unit interval, $C(I^n)$ the space of all real-valued continuous functions in n variables $x_1, \dots, x_n$ on I^n and $C_0(I^n)$ the set $\{f(x_1, \dots, x_n) \in C(I^n) \mid f(0, \dots, 0) = 0\}$. Then we can easily prove the following lemma using Weierstrass' theorem.

LEMMA 1. $A = \{x_1^{i_1} \dots x_n^{i_n}\}_{i_1, \dots, i_n \in N_0,\ i_1^2 + \dots + i_n^2 \neq 0}$ *is complete in* $C_0(I^n)$ *with the uniform norm.*

Using the fact that $C(I^n)$ is a dense subspace of $L^p(I^n)$ with the L^p-norm, $1 \leqq p < +\infty$, we have

LEMMA 2. *The set* A *in Lemma 1 is complete in* $L^p(I^n)$ *with the* L^p*-norm,* $1 \leqq p < +\infty$.

2. Extended Müntz's theorem in $C(I^n)$ with the uniform norm.

We begin by introducing some notation denoting several sets of powers involving n variables $x_1, \dots, x_n$.

Notation 1. For the sake of simplicity, let us consider the case $n = 3$. For the variables x, y, z, $B_1^{(3)}$ is a set which is null or contains some of the three sequences $\{x^{a_i}\}$, $\{y^{b_i}\}$, $\{z^{c_i}\}$, $a_i, b_i, c_i \in \mathbb{R}$ for $i \in \mathbb{N}$. $B_2^{(3)}$ is a set which is null or includes some of the three double sequences $\{x^{d_i}y^{e_j}\}$, $\{y^{f_i}z^{g_j}\}$, $\{z^{h_i}x^{k_j}\}$, $d_i, e_j, \dots, k_j \in \mathbb{R}$ for $i, j \in \mathbb{N}$. $B_3^{(3)}$ is the null set or $\{x^{\ell_i}y^{m_j}z^{n_k}\}$, $\ell_i, m_j, n_k \in \mathbb{R}$ for $i, j, k \in \mathbb{N}$. Here each sequence $\{a_i\}, \dots, \{n_j\}$ is positive and strictly monotonic increasing. Finally the family $B^{(3)}$ or $B^{(3)}_{(x,y,z)}$ is defined by $B^{(3)} = B^{(3)}_{(x,y,z)} = B_1^{(3)} \cup B_2^{(3)} \cup B_3^{(3)}$. In the general case we introduce analogously the notation $B_1^{(n)}, \dots, B_n^{(n)}$ with respect to n variables $x_1, \dots, x_n$ and put $B^{(n)} = B_x^{(n)} = B_1^{(n)} \cup \dots \cup B_n^{(n)}$, where $x = (x_1, \dots, x_n)$.

Then we have

PROPOSITION 1. *In order that* $\{1\} \cup B_x^{(n)}$ *be complete in* $C(I^n)$ *with the uniform norm* $\|\cdot\|_\infty$, *it is necessary and sufficient that for*

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each i, $i = 1, 2, \ldots, n$.

(i) $B_i^{(n)}$ *includes all* i*-tuple sequences of the form in Notation 1 involving* i *variables of* $x_1, \ldots, x_n$ *and*

(ii) in each i*-tuple sequence of (i), for example* $\{x_1^{p_{j(1)}^{(1)}} \cdot \ldots x_i^{p_{j(i)}^{(i)}}\}$, $\sum_{j(k)=1}^{\infty} 1/p_{j(k)}^{(k)} = +\infty$ *for* $k = 1, 2, \ldots, i$.

Proof. For the sake of simplicity, we prove this proposition in the case $n = 3$ and the proof of the general case is analogous. By Weierstrass' theorem, $S = \{1\} \cup \{x^i, y^i, z^i, x^iy^j, y^iz^j, z^ix^j, x^iy^jz^k\}_{i,j,k \in \mathbb{N}}$ is complete in $C(I^3)$ with the uniform norm. If the conditions (i) and (ii) are fulfilled, we can easily show that, by means of Theorem B, any element of S can be approximated by a linear combination of $\{1\} \cup B^{(3)}$. Thus the conditions are sufficient.

Suppose that $\{1\} \cup B^{(3)}$ is complete in $C(I^3)$. Firstly we show that the condition (i) is satisfied. Assume that, for instance, $\{x^{a_i}\}$ is not in $B_1^{(3)}$. Then, for any $\varepsilon > 0$, there exists a function $p(x,y,z) \in \text{Span}(\{1\} \cup B^{(3)})$ such that $\|x - p(x,y,z)\|_\infty < \varepsilon$, which implies $\|x - p(x,0,0)\|_\infty < \varepsilon$. This is a contradiction, because $p(x,y,z)$ does not contain any term of x^{a_i} and $p(x,0,0)$ is a constant. Next suppose that, for example, $\{x^{d_i}y^{e_j}\}$ is omitted from $B_2^{(3)}$. Then, for any $\varepsilon > 0$, there is a function $q(x,y,z) \in \text{Span}(\{1\} \cup B^{(3)})$ such that $\|xy - q(x,y,z)\|_\infty < \varepsilon$, which implies

$$\|xy - q(x,y,0)\|_\infty < \varepsilon . \tag{2.1}$$

Since $q(x,y,0) = p_0 + \sum_{i=1}^{n} q_i x^{a_i} + \sum_{i=1}^{m} r_i y^{b_i}$, putting $x = y = 0$ in (2.1), we have $|p_0| < \varepsilon$ and substituting $x = 0$ into (2.1), we have

$|p_0 + \sum_{i=1}^{m} r_i y^{b_i}| < \varepsilon$ for $y \in [0,1]$. Therefore $\| \sum_{i=1}^{m} r_i y^{b_i} \|_\infty < 2\varepsilon$.

In the same way, we have $\| \sum_{i=1}^{n} q_i \cdot x^{a_i} \|_\infty < 2\varepsilon$. Thus $\|q(x,y,0)\|_\infty < 5\varepsilon$ and by (2.1) $\|xy\|_\infty < 6\varepsilon$, which leads to a contradiction. Other cases can be proved similarly. Secondly we are going to show that the condition (ii) is fulfilled under condition (i). For instance, let us consider $B_3^{(3)} = \{x^{\ell_i} y^{m_j} z^{n_k}\}$. Suppose that $\sum_{i=1}^{\infty} 1/\ell_i < +\infty$. Then, by Theorem B, there is a function $f(x) \in C_0(I)$ such that

$$\inf_{g \in \mathrm{Span}(\{1\} \cup \{x^{\ell_i}\})} \| f - g \|_\infty = \delta > 0 . \tag{2.2}$$

Set $h(x,y,z) = f(x)yz$. For any $\varepsilon > 0$, by the assumption of completeness of $\{1\} \cup B^{(3)}$ in $C(I^3)$, there exists a function $r(x,y,z) \in \mathrm{Span}(\{1\} \cup B^{(3)})$ such that $\|h(x,y,z) - r(x,y,z)\|_\infty < \varepsilon$. If we express the terms of the linear combination of $\{x^{\ell_i} y^{m_j} z^{n_k}\}$ in $r(x,y,z)$ as $\sum_{i,j,k} s_{ijk} x^{\ell_i} y^{m_j} z^{n_k}$, in the same way as in the proof of (i), we get

$$\| h(x,y,z) - \sum_{i,j,k} s_{ijk} x^{\ell_i} y^{m_j} z^{n_k} \|_\infty < 30\varepsilon . \tag{2.3}$$

Putting $y = z = 1$ in (2.3), $\|f(x) - \sum_{i,j,k} s_{ijk} x^{\ell_i} \|_\infty < 30\varepsilon$, which contradicts (2.2). Other cases can be proved analogously. This completes the proof.

To generalize Proposition 1, we need the following lemma.

LEMMA 3. *Let* $\psi(x)$ *be a non-negative function in* $C_0(I)$. *Then* $\{1\} \cup \{\psi(x)^{a_i}\}_{i \in \mathbb{N}}$ *with* $0 < a_1 < a_2 < \dots$ *is complete in* $C(I)$ *with the uniform norm* $\|\cdot\|_\infty$ *if and only if* $\psi(x)$ *is strictly monotonic and* $\sum_{i=1}^{\infty} 1/a_i = +\infty$.

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Proof. Assume that $\{1\} \cup \{\psi(x)^{a_i}\}$ is complete in $C(I)$ and that $\psi(x)$ is not monotonic. Then there exist two points α, $\beta(\alpha \neq \beta)$ such that $\psi(\alpha) = \psi(\beta)$ and, for any function $f(x) \in \mathrm{Span}(\{1\} \cup \{\psi(x)^{a_i}\})$, $f(\alpha) = f(\beta)$. For a function $g(x) \in C(I)$ such that $g(\alpha) \neq g(\beta)$ and for any function $f(x) \in \mathrm{Span}(\{1\} \cup \{\psi(x)^{a_i}\})$, $\|g(x) - f(x)\|_\infty \geqq 1/2 \cdot |g(\alpha) - g(\beta)| > 0$, which contradicts the assumption. Now suppose that $\{1\} \cup \{\psi(x)^{a_i}\}$ is complete in $C(I)$ and $\psi(x)$ is monotonic. Put $t = \psi(x)$ and $[0,a] = \psi(I)$. Then, by Theorem B, $\{1\} \cup \{t^{a_i}\}$ is complete in $C[0,a]$ with the uniform norm if and only if $\sum_{i=1}^{\infty} 1/a_i = +\infty$. Thus the assumption that $\{1\} \cup \{\psi(x)^{a_i}\}$ is complete in $C(I)$ implies that $\psi(x)$ is monotonic and $\sum_{i=1}^{\infty} 1/a_i = +\infty$. By putting $t = \psi(x)$ and by applying Theorem B, the converse is easily verified.

Let $\psi_i(x_i)$, $i = 1, 2, \ldots, n$ be a non-negative continuous function of the i-th of the n variables $x_1, \ldots, x_n$ with $\psi_i(0) = 0$. We write simply $\psi_i(x_i) \in C_0(I_i)$, $i = 1, 2, \ldots, n$. Then we introduce the following notation.

Notation 2. By $\tilde{B}_i^{(n)}$, $i = 1, 2, \ldots, n$, we denote the set which is obtained by replacing x_i in $B_i^{(n)}$ with $\psi_i(x_i)$ and we define $\tilde{B}_\psi^{(n)}$ by $\tilde{B}_1^{(n)} \cup \ldots \cup \tilde{B}_n^{(n)}$, where $\psi = (\psi_1, \ldots, \psi_n)$.

Then we have the first main theorem.

THEOREM 1. *In order that* $\{1\} \cup \tilde{B}_\psi^{(n)}$ *be complete in* $C(I^n)$ *with the uniform norm, it is necessary and sufficient that for each* i, $i = 1, 2, \ldots, n$,

(i) $\psi_i(x)$ *is strictly monotonic,*

(ii) $\tilde{B}_i^{(n)}$ *includes all* i*-tuple sequences involving* i *of* $\psi_1(x_1), \ldots, \psi_n(x_n)$ *and* *(iii) in each i-tuple sequence of (ii), for instance*

$$\{\psi_1(x_1)^{p^{(1)}_{j(1)}} \cdots \psi_i(x_i)^{p^{(i)}_{j(i)}}\}, \quad \sum_{j(k)=1}^{\infty} 1/p^{(k)}_{j(k)} = +\infty \quad \text{for } k = 1, 2, \ldots, i .$$

Proof. Suppose that $\{1\} \cup \tilde{B}^{(n)}_{\psi}$ is complete in $C(I^n)$ with the uniform norm. Restricting the arguments on I_i, $i = 1, 2, \ldots, n$, in the same way as in the proof of Lemma 3, we can prove condition (i). If we set $t_i = \psi_i(x_i)$, $i = 1, 2, \ldots, n$, then under condition (i) we obtain that $\{1\} \cup \tilde{B}^{(n)}_{\psi}$ is complete in $C(I^n)$ with the uniform norm if and only if $\{1\} \cup \tilde{B}^{(n)}_{t}$ is complete in $C([0,a_1] \times \ldots \times [0,a_n])$ with the uniform norm, where $a_i = \psi_i(1)$, $i = 1, 2, \ldots, n$ and $t = (t_1, \ldots, t_n)$. Hence from this fact and Proposition 1 conditions (ii) and (iii) follow. In the same way, by conditions (i), (ii), (iii) and Proposition 1, we can easily verify the converse.

3. Extended Müntz's Theorem in $L^p(I^n)$ with the L^p-norm

In the first place, we start with

LEMMA 4. *$\{x^{a_i}\}_{i \in \mathbb{N}}$ with $0 < a_1 < a_2 \ldots$ is complete in $L^p(I)$ with the L^p-norm, $1 \leqq p < +\infty$ if and only if $\sum_{i=1}^{\infty} 1/a_i = +\infty$.*

Proof. Assume that $\{x^{a_i}\}$ is complete in $L^p(I)$ with the L^p-norm. Then, for any $m \in \mathbb{N}$, x^m can be approximated by a linear combination of the family $\{x^{a_i+1}\}$ in the uniform norm. In fact, by Hölder's inequality, we obtain

$$\left|x^m - \sum_{i=1}^{n} \alpha_i x^{a_i+1}\right| = \left| \int_0^x (mt^{m-1} - \sum_{i=1}^{n} \alpha_i(a_i+1)t^{a_i})dt \right|$$

$$\leqq \int_0^1 \left|mt^{m-1} - \sum_{i=1}^{n} \alpha_i(a_i+1)t^{a_i}\right| dt$$

$$\leqq \left(\int_0^1 \left|mt^{m-1} - \sum_{i=1}^{n} \alpha_i(a_i+1)t^{a_i}\right|^p dt\right)^{1/p} .$$

Therefore, by Theorem B, we have $\sum_{i=1}^{\infty} 1/(a_i+1) = +\infty$ and consequently $\sum_{i=1}^{\infty} 1/a_i = +\infty$. Conversely, if $\sum_{i=1}^{\infty} 1/a_i = +\infty$, then we can conclude that $\{x^{a_i}\}$ is complete in $C_0(I)$ with the uniform norm. From this fact and Lemma 2, the converse follows immediately.

We also have

PROPOSITION 2. *$B^{(n)}$ is complete in $L^p(I^n)$ with the L^p-norm $\|\cdot\|_p$, $1 \leq p < +\infty$ if and only if*

(i) $B_n^{(n)}$ is not the null set and

(ii) in $B_n^{(n)} = \{x_1^{p_{j(1)}^{(1)}} \cdots x_n^{p_{j(n)}^{(n)}}\}$, $\sum_{j(i)=1}^{\infty} 1/p_{j(i)}^{(i)} = +\infty$ for $i = 1, 2, \ldots, n$.

Proof. For the sake of simplicity, we prove this proposition in case $n = 3$. Assume that $B_{(x,y,z)}^{(3)}$ is complete in $L^p(I^3)$ with the L^p-norm and that $B_3^{(3)}$ is the null set. Then any function $f(x,y,z) \in L^p(I^3)$ can be approximated in the L^p-norm by a linear combination of $B_1^{(3)} \cup B_2^{(3)}$. Consequently, $A_{(x,y,z)} = \{x^i, y^i, z^i, x^i y^j, y^i z^j, z^i x^j\}_{i,j \in \mathbb{N}}$ is complete in $L^1(I^3)$ with the L^1-norm. Then, for the function $xyz \in L^1(I^3)$ and for any $\varepsilon > 0$, there exists a polynomial $p(x,y,z) \in \mathrm{Span}(A_{(x,y,z)})$ such that

$$\| xyz - p(x,y,z) \|_1 < \varepsilon . \tag{3.1}$$

On the other hand, for any $(t_1, t_2, t_3) \in I^3$,

$$\begin{aligned} &\left| (1/8)\cdot t_1^2 t_2^2 t_3^2 - \int_0^{t_3}\int_0^{t_2}\int_0^{t_1} p(x,y,z)\,dx\,dy\,dz \right| \\ &\leq \int_0^{t_3}\int_0^{t_2}\int_0^{t_1} |xyz - p(x,y,z)|\,dx\,dy\,dz \qquad (3.2) \\ &\leq \| xyz - p(x,y,z) \|_1 . \end{aligned}$$

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If we put

(3.3) $$q(t_1,t_2,t_3) = \int_0^{t_3}\int_0^{t_2}\int_0^{t_1} p(x,y,z)\,dx\,dy\,dz ,$$

$q(t_1,t_2,t_3)$ has a factor $t_1t_2t_3$ and we can express it as

(3.4) $$q(t_1,t_2,t_3) = t_1t_2t_3\; r(t_1,t_2,t_3) ,$$

where $r(t_1,t_2,t_3) \in \mathrm{Span}(A_{(t_1,t_2,t_3)})$. By (3.2), (3.3) and (3.4) ,

(3.5) $$|(1/8)\cdot t_1^2t_2^2t_3^2 - q(t_1,t_2,t_3)|$$
$$= |t_1t_2t_3|\cdot|(1/8)\cdot t_1t_2t_3 - r(t_1,t_2,t_3)| < \varepsilon .$$

If we put $\tilde{I} = [0,1/2]$ and $t_i = s_i + 1/2$, $i = 1, 2, 3,$ since $|(s_1+1/2)(s_2+1/2)(s_3+1/2)| \geqq 1/8$ for any $(s_1,s_2,s_3) \in \tilde{I}^3$, by (3.5) we have

$$|(1/8)(s_1+1/2)(s_2+1/2)(s_3+1/2) - r(s_1+1/2,s_2+1/2,s_3+1/2)| < 8\varepsilon$$
$$\text{for any } (s_1,s_2,s_3) \in \tilde{I}^3 .$$

Then $\|(1/8)\cdot s_1s_2s_3 - k(s_1,s_2,s_3)\|_\infty < 8\varepsilon$ on $\tilde{I}^3$, where $k(s_1,s_2,s_3) \in \mathrm{Span}(\{1\} \cup A_{(s_1,s_2,s_3)})$. This contradicts Proposition 1, thus $B_3^{(3)}$ is not null.

Next we show that, in $B_3^{(3)} = \{x^{\ell_i}y^{m_j}z^{n_k}\}$, $\sum_{i=1}^{\infty} 1/\ell_i = +\infty$, $\sum_{j=1}^{\infty} 1/m_j = +\infty$ and $\sum_{k=1}^{\infty} 1/n_k = +\infty$. Suppose that $B^{(3)}_{(x,y,z)}$ is complete in $L^p(I^3)$ with the L^p-norm and that $\sum_{k=1}^{\infty} 1/n_k < +\infty$. Then $B_3^{(3)} \cup A_{(x,y,z)}$ is complete in $L^p(I^3)$ with the L^p-norm. First we show that, in $[1/2,1]^3$, every function of the form $(x-1/2)^a(y-1/2)^b(z-1/2)^c$, $a, b, c \in \mathbb{N}$ is approximated in the uniform norm by a linear combination of $\{x^{\ell_i}y^{m_j}z^{n_k}\}_{i,j,k\in\mathbb{N}_0}$, where $\ell_0 = m_0 = n_0 = 0$. By similar arguments to those in (3.1) to (3.5), each

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x^a, y^b, z^c, $a, b, c \in \mathbb{N}_0$ is approximated by a linear combination of $B_3^{(3)} \cup A_{(x,y,z)}$ uniformly on $[1/2,1]^3$. Therefore, for any $(x-1/2)^a (y-1/2)^b (z-1/2)^c$, $a, b, c \in \mathbb{N}$ and any $\varepsilon > 0$, there exists a function $h(x,y,z) \in \mathrm{Span}(\{1\} \cup B_3^{(3)} \cup A_{(x-1/2,y-1/2,z-1/2)})$ such that

$$(3.6) \qquad \| (x-1/2)^a (y-1/2)^b (z-1/2)^c - h(x,y,z) \|_\infty < \varepsilon \quad \text{on} \quad [1/2,1]^3 .$$

Further we set

$$(3.7) \qquad h(x,y,z) = \sum_{i,j,k} \alpha_{ijk} x^{\ell_i} y^{m_j} z^{n_k} + q_1(x,y) + q_2(y,z) + q_3(z,x) + p_1(x) + p_2(y) + p_3(z) + d ,$$

where $q_1(x,y) \in \mathrm{Span}(\{(x-1/2)^i (y-1/2)^j\}), \ldots, p_1(x) \in \mathrm{Span}(\{(x-1/2)^i\})$, ... and $d \in \mathbb{R}$. If we put $x = y = 1/2$ in (3.6), then by (3.7) we have

$$\Big| \sum_{i,j,k} \alpha_{ijk} (1/2)^{\ell_i} (1/2)^{m_j} z^{n_k} + p_3(z) + d \Big| < \varepsilon \qquad \text{for all} \quad z \in [1/2,1] .$$

Hence if we set $\tilde{p}_3(z) = -\sum_{i,j,k} \alpha_{ijk} (1/2)^{\ell_i} (1/2)^{m_j} z^{n_k} - d$, then $\|p_3(z) - \tilde{p}_3(z)\|_\infty < \varepsilon$ on $[1/2,1]^3$. In the other cases, by using the similar method to the proof of Proposition 1, we can approximate each polynomial of $\{p_1, \ldots, q_1, \ldots\}$ by a linear combination of $\{x^{\ell_i} y^{m_j} z^{n_k}\}_{i,j,k \in \mathbb{N}_0}$.

On the other hand, noting that $\{1\} \cup \{z^{n_k}\}_{k \in \mathbb{N}}$ is incomplete in $C[1/2,1]$ with the uniform norm (see Watson [5] p. 82), there exists a polynomial $(z-1/2)^{r_0}$, $r_0 \in \mathbb{N}$ such that

$$(3.8) \qquad \inf_{f(z) \in \mathrm{Span}(\{1\} \cup \{z^{n_k}\})} \| (z-1/2)^{r_0} - f(z) \|_\infty > \delta > 0 .$$

For a polynomial $(x-1/2)(y-1/2)(z-1/2)^{r_0}$ and for any $\varepsilon > 0$, there exists a function $u(x,y,z) \in \mathrm{Span}(\{x^{\ell_i}y^{m_j}z^{n_k}\}_{i,j,k\in\mathbb{N}_0})$ such that

(3.9) $\|(x-1/2)(y-1/2)(z-1/2)^{r_0} - u(x,y,z)\|_\infty < \varepsilon$ on $[1/2,1]^3$.

Putting $x = y = 1$ in (3.9), we have

$$|(1/4)\cdot(z-1/2)^{r_0} - u(1,1,z)| < \varepsilon \quad \text{for all} \quad z \in [1/2,1] ,$$

which contradicts (3.8). Thus the necessity follows.

Conversely, assume that $B_3^{(3)}$ is not null and in $B_3^{(3)} = \{x^{\ell_i}y^{m_j}z^{n_k}\}_{i,j,k\in\mathbb{N}}$, $\sum_{i=1}^{\infty} 1/\ell_i = +\infty$, $\sum_{j=1}^{\infty} 1/m_j = +\infty$ and $\sum_{k=1}^{\infty} 1/n_k = +\infty$.

Since by Lemma 4, each sequence of $\{x^{\ell_i}\}$, $\{y^{m_j}\}$ and $\{z^{n_k}\}$ is complete in $L^p(I)$ with the L^p-norm, clearly every monomial $x^ay^bz^c$, a, b, $c \in \mathbb{N}_0$ with $a^2+b^2+c^2 \neq 0$ is approximated by a linear combination of $B_3^{(3)}$ in the L^p-norm. Hence, by Lemma 2, $B_3^{(3)}$ satisfying condition (ii) is complete in $L^p(I^3)$. This completes the proof.

LEMMA 5. *Let $\psi(x)$ be a strictly monotonic increasing and absolutely continuous function in $C_0(I)$ and put $M = \operatorname{ess\,sup}_{x\in I} |\psi'(x)| < +\infty$. Then, $\{\psi(x)^{a_i}\}_{i\in\mathbb{N}}$ with $0 < a_1 < a_2 < \dots$ is complete in $L^p(I)$ with the L^p-norm, $1 \leq p < +\infty$ if and only if $\sum_{i=1}^{\infty} 1/a_i = +\infty$.*

Proof. Suppose that $\{\psi(x)^{a_i}\}$ is complete in $L^p(I)$ with the L^p-norm. If $\sum_{i=1}^{\infty} 1/a_i < +\infty$, then by Lemma 4 $\{t^{a_i}\}$ is not complete in $L^1[0,a]$ with the L^1-norm, where $\psi(1) = a > 0$. Consequently, there exists a positive integer n_0 such that $\inf_{p(t)\in\mathrm{Span}(\{t^{a_i}\})} \|t^{n_0} - p(t)\|_1 = \delta > 0$. It follows from this fact that by putting $t = \psi(x)$,

$$\delta \leqq \int_0^a |t^{n_0} - p(t)|dt = \int_0^1 |\psi(x)^{n_0} - p(\psi(x))| \cdot \psi'(x)dx$$

$$\leqq M \cdot \int_0^1 |\psi(x)^{n_0} - p(\psi(x))|\, dx .$$

Hence $\inf_{q(x)\in \mathrm{Span}(\{\psi(x)^{a_i}\})} \| \psi(x)^{n_0} - q(x)\|_1 \geqq \delta/M > 0$, which contradicts the completeness of $\{\psi(x)^{a_i}\}$. Conversely, assume that $\sum_{i=1}^{\infty} 1/a_i = +\infty$. Then, by Lemma 3, $\{1\} \cup \{\psi(x)^{a_i}\}$ is complete in $C(I)$ with the uniform norm and consequently each x^i, $i \in \mathbb{N}$, is approximated by a linear combination of $\{\psi(x)^{a_i}\}$ uniformly on I . Hence, by Lemma 2, $\{\psi(x)^{a_i}\}$ is complete in $L^p(I)$ with the L^p-norm.

Then we obtain the second main theorem.

THEOREM 2. *Let* $\psi_i(x_i)$, $i = 1, 2, \ldots, n$ *be functions satisfying the condition in Lemma 5. Then* $\tilde{B}_\psi^{(n)}$ *is complete in* $L^p(I^n)$ *with the* L^p*-norm,* $1 \leqq p < +\infty$ *if and only if*

(i) $\tilde{B}_n^{(n)}$ *is not null and*

(ii) *in* $\tilde{B}_n^{(n)} = \{\psi_1(x_1)^{p_{j(1)}^{(1)}} \cdots \psi_n(x_n)^{p_{j(n)}^{(n)}}\}$,

$$\sum_{j(i)=1}^{\infty} 1/p_{j(i)}^{(i)} = +\infty \quad \textit{for} \quad i = 1, 2, \ldots, n .$$

Proof. By Proposition 2 and Lemma 5, we can easily verify that $\tilde{B}_\psi^{(n)}$ is complete in $L^p(I^n)$ with the L^p-norm if and only if $\tilde{B}_t^{(n)}$ is complete in $L^p([0,a_1] \times \ldots \times [0,a_n])$ with the L^p-norm, where $t_i = \psi_i(x_i)$, $a_i = \psi_i(1)$, $i = 1, 2, \ldots, n$ and $t = (t_1, \ldots, t_n)$. Hence the conclusion follows immediately.

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