MINIMAL MODAL LOGICS, CONSTRUCTIVE MODAL LOGICS AND THEIR RELATIONS

TIZIANO DALMONTE
Free University of Bozen-Bolzano

Abstract. We present a family of minimal modal logics (namely, modal logics based on minimal propositional logic) corresponding each to a different classical modal logic. The minimal modal logics are defined based on their classical counterparts in two distinct ways: (1) via embedding into fusions of classical modal logics through a natural extension of the Gödel–Johansson translation of minimal logic into modal logic S4; (2) via extension to modal logics of the multi- vs. single-succedent correspondence of sequent calculi for classical and minimal logic. We show that, despite being mutually independent, the two methods turn out to be equivalent for a wide class of modal systems. Moreover, we compare the resulting minimal version of K with the constructive modal logic CK studied in the literature, displaying tight relations among the two systems. Based on these relations, we also define a constructive correspondent for each minimal system, thus obtaining a family of constructive modal logics which includes CK as well as other constructive modal logics studied in the literature.

§1. Introduction. Although modal logics are usually defined as extensions of classical logic, significant attention has been also devoted to the analysis of modalities over non-classical basis, such as relevant [7, 17, 19, 39, 55, 56], linear [21, 40, 51] or other substructural logics [10, 30, 53]. In this context, a major role is played by intuitionistic logic, many modal extensions of which have been studied with motivations ranging from philosophical or legal reasoning to computer science applications.

A crucial difference between classical and intuitionistic modal logics is that, by analogy with intuitionistic connectives, in the latter systems the modalities $\Box$ and $\Diamond$ are assumed to be not interdefinable. This peculiarity allows for the definition of systems validating independent $\Box$- and $\Diamond$-principles, as well as with different interactions between the two modalities. Such a notable freedom has led to the question of how to define meaningful intuitionistic (or constructive) counterparts of a given classical modal logic. Clearly, the meaning of 'intuitionistic counterpart' is not univocal, and different interpretations have produced different intuitionistic modal systems. This situation is particularly evident in the case of intuitionistic correspondents of classical modal logic K, with a wide variety of counterparts of it studied in the literature.[1]

Received: September 5, 2023.
2020 *Mathematics Subject Classification*: Primary 03B45, 03F03, Secondary 03B20.
Key words and phrases: constructive modal logic, minimal modal logic, modal companion, sequent calculus, neighbourhood semantics.

[1] Several intuitionistic variants of K have been defined over the last few years, some examples (in addition to the ones mentioned in this introduction) can be found in [4, 5, 13, 25, 31]. Intuitionistic monomodal versions of K have been also studied (see [59] for a survey).

doi:10.1017/S1755020325000097

We mention here the three most relevant such systems: Intuitionistic K, denoted I.K [49, 57–59]; Wijesekera's K (the propositional fragment of Wijesekera's Constructive Concurrent Dynamic Logic [65, 66]), that we call W.K, and Constructive K, denoted C.K [6, 41, 48]. These systems have increasing strength, from the weakest C.K to the strongest I.K, and are definable axiomatically extending intuitionistic propositional logic (IPL) as follows:

$$\begin{aligned}&\mathsf{C.K} := \mathsf{IPL} + \Box(A \supset B) \supset (\Box A \supset \Box B), \Box(A \supset B) \supset (\Diamond A \supset \Diamond B), \ \frac{A}{\Box A}\\ &\mathsf{W.K} := \mathsf{C.K} + \neg\Diamond\bot\\ &\mathsf{I.K} := \mathsf{W.K} + \Diamond(A \vee B) \supset \Diamond A \vee \Diamond B, (\Diamond A \supset \Box B) \supset \Box(A \supset B).\end{aligned}$$

In particular, I.K and C.K belong each to one of the two most studied families of modal logics based on IPL: so-called *intuitionistic modal logics*, that stem from the works of Fischer Servi [57, 58], Plotkin and Stirling [49], and Ewald [15], and so-called *constructive modal logics*, whose definition goes back to Fitch [18] and has been further developed by de Paiva, Mendler and Ritter [1, 41, 48] among others. The latter logics have found several applications in the constructivisation of description [42] and dynamic [66] logic, the formalisation of provability and consistency in Heyting arithmetic [34], contextual reasoning [41], and also serve as type languages for programming in structured knowledge bases and for computation stages [6, 14, 44].

Once an intuitionistic correspondent of a classical modal logic is defined, the question arises of how to define analogous counterparts of further classical modal logics. To this aim, general theories of intuitionistic modal logics have been proposed in the literature. An elegant characterisation of this type of systems was provided by Simpson [59] via reduction to validities in first-order intuitionistic logic (FOIL): Given a classical modal logic L, its intuitionistic counterpart I.L is defined as the set of formulas A such that

$$A \in \mathsf{I.L} \text{ if and only if } \mathscr{T} \vdash_{\mathsf{FOIL}} \forall x(A^x),$$

where the superscript x denotes the standard translation of modal formulas into first-order sentences with respect to the variable x, and $\mathscr{T}$ is a set of first-order sentences that express the frame conditions corresponding to the modal axioms of L in the relational semantics of classical modal logics (e.g., reflexivity, transitivity, symmetry for T, 4, B). The same logics have been also shown to be reducible to products of modal logics [20] via suitable translations of modal formulas [57, 67].

Furthermore, a general characterisation of W.K and related systems was proposed in [11] where so-called *Wijesekera-style constructive modal logics* were defined in two distinct but ultimately equivalent ways: (1) by restricting classical modal sequent calculi to sequents with at most one formula in the conclusion, thus extending to modal logics a relation that is known to hold between classical and intuitionistic sequent calculi since Gentzen [23]; (2) by means of a simple generalisation of the satisfaction clauses of modal formulas in relational models in order to fit with the preorder semantics of intuitionistic logic. The second way provides quite immediately a reduction of each system W.L, counterpart of the classical logic L, to the fusion of S4 and L [20] via a natural extension of Gödel's translation of S4 into IPL [24].

By contrast, no such uniform characterisation of constructive modal logics has been proposed so far in the literature. Constructive counterparts for the whole modal cube from K to S5 obtained by the combinations of the axioms D, T, 4, B and 5 have been presented in [1, 2, 43] and endowed with nested sequent calculi [2] and some of them also with relational semantics [1, 41], sequent calculi [6, 34, 36] and 2-sequent calculi [43]. These logics are defined based on their axiomatic systems by extending C.K with pairs of corresponding $\Box$- and $\Diamond$-axioms, like $T_\Box$ $\Box A \supset A$, $T_\Diamond$ $A \supset \Diamond A$ and $4_\Box$ $\Box A \supset \Box\Box A$, $4_\Diamond$ $\Diamond\Diamond A \supset \Diamond A$, that are both needed due to the loss of duality between $\Box$ and $\Diamond$ (the axiom D is an exception as it is taken in the usual formulation $\Box A \supset \Diamond A$ only). However, the validity of corresponding $\Box$- and $\Diamond$-principles in constructive modal logics does not hold in general, so that it is not clear how to extend this family of systems from a purely axiomatical point of view, especially when it comes to logics weaker than K. For instance, C.K validates $\Box A \wedge \Box B \supset \Box(A \wedge B)$ and the necessitation rule $A/\Box A$, but does not validate the corresponding $\Diamond$-principles $\Diamond(A \vee B) \supset \Diamond A \vee \Diamond B$ and $\neg\Diamond\bot$. Moreover, $\Box$- and $\Diamond$-versions also exist for the axiom D, namely $\neg(\Box A \wedge \Box\neg A)$ and $\Diamond A \vee \Diamond\neg A$, but neither of them is derivable in C.KD.

In this paper, we address this problem by presenting a systematisation of constructive modal logics that includes the systems C.K, C.KD and C.KT already studied in the literature as well as new constructive counterparts of further classical modal logics. The starting point of our analysis is the observation that constructive modal logics share some similarities with *minimal* propositional logic (MPL) and possible modal extensions thereof, both from a semantical and from a proof-theoretical perspective.

Our approach toward the systematisation of constructive modal logics can be summarised as follows. First of all, we introduce a family of minimal modal logics M.L. These logics correspond each to a different classical modal logic, and are defined following an approach similar to the one of [11] for the definition of Wijesekera-style modal logics: we define the logics M.L by means of a reduction to fusions of classical modal logics of the form $\mathsf{S4} \oplus \mathsf{L}$ through a natural extension of the Gödel–Johansson translation of MPL into S4 (in turn, this translation is a combination of Johansson's translation of MPL into IPL [29] and Gödel's translation of IPL into S4). The logics M.L will coincide with the sets of modal formulas A such that $A^t \in \mathsf{S4} \oplus \mathsf{L}$, where t is the aforementioned translation. To this aim, we provide the minimal modal logics with a modular semantic characterisation. We then show that the same minimal modal logics can be equivalently obtained by restricting sequent calculi for classical modal logics to sequents with *exactly one* formula in the succedent. This extends to modal logics a relation that holds between sequent calculi for classical and MPL firstly observed by Johansson [29] (see [61] for an extended presentation of sequent calculi for CPL, IPL and MPL and corresponding bounds on the cardinality of succedents of sequents). To our knowledge, this is the first study of modal logics based on MPL.

Furthermore, we observe that M.K, the resulting minimal counterpart of K, is strictly related to constructive C.K. In particular, the two systems share exactly the same modal principles, despite over a different propositional base. This means that C.K coincides with the extension of M.K with the principle of ex falso quodlibet $\bot \supset A$ (exactly as $\mathsf{IPL} = \mathsf{MPL} + \bot \supset A$). We show that equally tight relations between M.K and C.K can be also observed in terms of their semantics and sequent calculi. More precisely, C.K can be semantically characterised by means of a simple restriction of the models of M.K that ensures the validity of $\bot \supset A$, and the sequent calculus for C.K defined in [6] can be re-obtained by adding the modal sequent rules for M.K to an intuitionistic sequent calculus. By extending these relations to the other minimal systems, we then define a constructive correspondent for each minimal and classical modal logic, thus obtaining an entire family of constructive modal logics with corresponding semantics and sequent calculi.

In short, the aim of this paper is threefold: (1) provide a uniform characterisation of some constructive modal logics already existing in the literature, (2) extend this family of logics by defining constructive counterparts of further classical modal logics coherently with our characterisation, (3) display the tight relations between minimal and constructive modalities.

The paper is organised as follows. In Sections 1.1 and 1.2, we present some preliminary notions needed throughout the paper. In Section 2, we define M.K, our minimal counterpart of K, via a reduction into $\mathsf{S4} \oplus \mathsf{K}$, and also provide an adequate semantics for it. In Section 3, we apply the single-succedent restriction to a sequent calculus for K and show that the resulting logic coincides with M.K. In Section 4, we show how the constructive logic C.K relates to our M.K, both from the point of view of the semantics and of the sequent calculi. In Section 5, we apply the same two methods to further classical modal logics, thus obtaining an entire family of corresponding minimal systems. Moreover, by extending to these systems the relations just observed between M.K and C.K, we define in Section 6 an analogous family of constructive modal logics. Finally, Section 7 contains some discussion of the results.

1.1. Syntactic preliminaries. Given a countable set $Atm = \{p_0, p_1, p_2, \ldots\}$ of propositional variables and a finite set $\mathbb{M}$ of unary modal operators, the language $\mathcal{L}_{\mathbb{M}}^{Atm}$ is defined by the following BNF grammar, where $p \in Atm$, $\circ \in \{\wedge, \vee, \supset\}$, and $\heartsuit \in \mathbb{M}$:

$$A ::= p \mid \bot \mid A \circ A \mid \heartsuit A.$$

In the following, we use p, q, r as metavariables for elements of Atm, and A, B, C, D as metavariables for formulas. Moreover, we define $\top := \bot \supset \bot$, $\neg A := A \supset \bot$, and $A \supset\subset B := (A \supset B) \wedge (B \supset A)$. Minimal modal logics will be defined in a language $\mathcal{L}_{\{\Box,\Diamond\}}^{Atm}$ containing the modalities $\Box$ and $\Diamond$. For the sake of simplicity, we denote $\mathcal{L}_{\{\Box,\Diamond\}}^{Atm}$ as $\mathcal{L}$.

We consider the following axiomatisation for MPL, formulated in $\mathcal{L}$ (see e.g., [54])[2]:

$$\begin{array}{lll}
A \wedge B \supset A & (A \supset B) \supset ((A \supset C) \supset (A \supset B \wedge C)) & \\
A \wedge B \supset B & (A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C)) & \dfrac{A \supset B \qquad A}{B} \\
A \supset A \vee B & (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C)) & \\
B \supset A \vee B & A \supset (B \supset A). &
\end{array}$$

As usual, we can define IPL as the extension of MPL with *ex falso quodlibet*[3]:

$$\mathsf{IPL} := \mathsf{MPL} + \bot \supset A,$$

and classical propositional logic (CPL) as the extension of IPL with *excluded middle*:

$$\mathsf{CPL} := \mathsf{IPL} + A \vee \neg A.$$

[2] In this paper we consider axiomatic systems to be defined by axiom schemata and rule schemata. For the sake of simplicity, we simply refer to axiom schemata and rule schemata as axioms and rules.

[3] Given two axiomatic systems L and L′ and an axiom A, we denote $\mathsf{L} + A$ the axiomatic extension of L with the axiom A, and $\mathsf{L} \oplus \mathsf{L}'$ the fusion of L and L′ (cf. Definition 2.1).

In this paper, we shall define minimal and constructive counterparts of classical modal logics. Classical modal logics are defined as usual as extensions of CPL, formulated in $\mathcal{L}$, with modal axioms and rules. For instance, the logic K is defined extending CPL with

$$K_{\Box}\ \Box(A \supset B) \supset (\Box A \supset \Box B) \qquad dual\ \Box A \supset\subset \neg\Diamond\neg A \qquad nec\ \frac{A}{\Box A},$$

and S4 is defined extending K with $T_{\Box}\ \Box A \supset A$ and $4_{\Box}\ \Box A \supset \Box\Box A$.[4] Further classical modal logics will be introduced in Section 5.

For any logic defined in the following (no matter whether based on classical, intuitionistic or minimal logic), we consider the standard notions of derivability: Given a logic L formulated in $\mathcal{L}_{\mathbb{M}}^{Atm}$ and formulas $A, B_1, \dots, B_n$ of $\mathcal{L}_{\mathbb{M}}^{Atm}$, the rule $B_1, \dots, B_n/A$ is *derivable* in L if there is a finite sequence of formulas ending with A in which every formula is an (instance of an) axiom of L, or it belongs to $\{B_1, \dots, B_n\}$, or it is obtained from previous formulas by the application of a rule of L. A formula A is *derivable* in L, written $\mathsf{L} \vdash A$, if the rule $\emptyset/A$ is derivable in L. Finally, A is (locally) *derivable* in L from a set of formulas Φ of $\mathcal{L}_{\mathbb{M}}^{Atm}$, written $\Phi \vdash_{\mathsf{L}} A$, if there is a finite set $\{B_1, \dots, B_n\} \subseteq \Phi$ such that $\mathsf{L} \vdash B_1 \wedge \dots \wedge B_n \supset A$.

1.2. Semantic preliminaries. We shall define semantics for minimal modal logics by suitably extending relational models for MPL. We consider to this purpose relational models for MPL as defined in [54]: A *minimal relational model* is a tuple $\mathcal{M} = \langle \mathcal{W}, \leq, \mathbb{F}, \mathcal{V} \rangle$, where $\mathcal{W}$ is a non-empty set of worlds, $\leq$ is a reflexive and transitive binary relation on $\mathcal{W}$, $\mathbb{F} \subseteq \mathcal{W}$ is a $\leq$-upward closed set (that is, if $w \in \mathbb{F}$ and $w \leq v$, then $v \in \mathbb{F}$) of so-called *fallible worlds*, and $\mathcal{V} : Atm \longrightarrow \mathcal{P}(\mathcal{W})$ is a *hereditary* valuation function (that is, if $w \in \mathcal{V}(p)$ and $w \leq v$, then $v \in \mathcal{V}(p)$). We write $v \geq w$ for $w \leq v$. The forcing relation $\mathcal{M}, w \Vdash A$ is inductively defined as follows:

$\mathcal{M}, w \Vdash p$	iff	$w \in \mathcal{V}(p)$;
$\mathcal{M}, w \Vdash \bot$	iff	$w \in \mathbb{F}$;
$\mathcal{M}, w \Vdash B \wedge C$	iff	$\mathcal{M}, w \Vdash B$ and $\mathcal{M}, w \Vdash C$;
$\mathcal{M}, w \Vdash B \vee C$	iff	$\mathcal{M}, w \Vdash B$ or $\mathcal{M}, w \Vdash C$;
$\mathcal{M}, w \Vdash B \supset C$	iff	for all $v \geq w$, $\mathcal{M}, v \Vdash B$ implies $\mathcal{M}, v \Vdash C$.

Given a formula A, we say that A is *valid in a model* $\mathcal{M}$, written $\mathcal{M} \models A$, if $\mathcal{M}, w \Vdash A$ for all worlds w of $\mathcal{M}$. The same definition of validity applies to all kinds of models considered in this paper. In the following, we simply write $w \Vdash A$ when $\mathcal{M}$ is clear from the context.

Minimal relational models are a generalisation of the well-known intuitionistic relational models firstly introduced by Kripke [33]. In particular, a minimal relational model is an *intuitionistic relational model* if $\mathbb{F} = \emptyset$. We point out that an alternative semantics for IPL can be obtained from minimal relational models by preserving the fallible worlds but assuming the condition $\mathbb{F} \subseteq \mathcal{V}(p)$ for all $p \in Atm$ which ensures the

[4] As usual, classical monomodal logics could be equivalently defined based on a single modality, considering the other one to be defined via duality. We prefer to assume here both $\Box$ and $\Diamond$ as primitive in order to simplify the comparison of the classical systems with the minimal and constructive ones where the modalities are not interdefinable.

validity of ex falso quodlibet $\bot \supset A$. We will consider this latter kind of restriction in Section 4.1.

§2. Minimal K via bimodal companion. Our first approach toward the definition of minimal modal logics is based on reductions into fusions of classical modal logics. We consider to this purpose the following notions of fusion, Gödel–Johansson translation and bimodal companion.

DEFINITION 2.1 (Fusion). *Let* L_1 *and* L_2 *be* classical *modal logics respectively defined in the languages* $\mathcal{L}^{Atm}_{\{\Box_1,\Diamond_1\}}$ *and* $\mathcal{L}^{Atm}_{\{\Box_2,\Diamond_2\}}$ *sharing the same propositional variables and propositional connectives but with disjoint sets of modalities. The* fusion $\mathsf{L}_1 \oplus \mathsf{L}_2$ *of* L_1 *and* L_2 *is the smallest logic in the language* $\mathcal{L}^{Atm}_{\{\Box_1,\Diamond_1,\Box_2,\Diamond_2\}}$ *containing* $\mathsf{L}_1 \cup \mathsf{L}_2$ *and closed under the rules of* L_1 *and* L_2.

DEFINITION 2.2 (Extended Gödel–Johansson translation). *Let* $Atm' = Atm \cup \{f\}$, *with* $f \notin Atm$, *and* $\mathcal{L}'_{1,2}$ *denote* $\mathcal{L}^{Atm'}_{\{\Box_1,\Diamond_1,\Box_2,\Diamond_2\}}$. *The* extended Gödel–Johansson translation $t : \mathcal{L} \longrightarrow \mathcal{L}'_{1,2}$ *is inductively defined as follows*:

$$\begin{array}{rcl} \bot^t & = & \Box_1 f \\ p^t & = & \Box_1 p \\ (A \wedge B)^t & = & A^t \wedge B^t \\ (A \vee B)^t & = & A^t \vee B^t \\ (A \supset B)^t & = & \Box_1(A^t \supset B^t) \\ (\Box A)^t & = & \Box_1 \Box_2 A^t \\ (\Diamond A)^t & = & \Box_1 \Diamond_2 A^t. \end{array}$$

DEFINITION 2.3 (Bimodal companion). *For any logic* M *formulated in the language* $\mathcal{L}$, *we say that a fusion of classical modal logics* $\mathsf{L}_1 \oplus \mathsf{L}_2$ *in the language* $\mathcal{L}'_{1,2}$ *is the* Gödel–Johansson bimodal companion (*or just* bimodal companion) *of* M *if it holds*:

$$\mathsf{M} \vdash A \text{ if and only if } \mathsf{L}_1 \oplus \mathsf{L}_2 \vdash A^t.$$

The above translation t is based on Gödel's [24] reduction of IPL into S4. The clauses for the modal formulas extend the translation in the trivial way, and are considered for instance in [16, 67], while the reduction of $\bot$ into a distinguished propositional constant f goes back to Johansson [29]. A similar translation that employs Johannson's solution was already applied for the embedding of a constructive modal logic into a classical multimodal logic in [16].

Given a classical modal logic L, we shall define its minimal counterpart $\mathsf{M.L}$ by considering modal companions of the form $\mathsf{S4} \oplus \mathsf{L}$.

DEFINITION 2.4 (Minimal counterpart of a classical logic). *Given a classical modal logic* L, *the* minimal counterpart *of* L *is the logic* $\mathsf{M.L}$ *in* $\mathcal{L}$ *such that* $\mathsf{S4} \oplus \mathsf{L}$ *is the bimodal companion of* $\mathsf{M.L}$.

In other words, the minimal counterpart $\mathsf{M.L}$ of L is the solution of the equation

$$(*) \quad \mathsf{M.L} \vdash A \text{ if and only if } \mathsf{S4} \oplus \mathsf{L} \vdash A^t.$$

The solution of $(*)$, if it exists, is unique (modulo equivalent axiomatisations). Indeed, if both $\mathsf{M.L}$ and $\mathsf{M.L}'$ are solutions to $(*)$, then $\mathsf{M.L} \vdash A$ iff $\mathsf{S4} \oplus \mathsf{L} \vdash A^t$ iff $\mathsf{M.L}' \vdash A$, hence $\mathsf{M.L} = \mathsf{M.L}'$. We start by presenting the minimal counterpart of the classical modal logic K.

Definition 2.5 (Minimal K). *The minimal modal logic* $\mathsf{M.K}$ *is defined extending* MPL *with the following axioms and rule*:

$$K_\Box\ \Box(A \supset B) \supset (\Box A \supset \Box B) \qquad K_\Diamond\ \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) \qquad nec\ \frac{A}{\Box A}.$$

In order to prove that $\mathsf{M.K}$ is our minimal counterpart of K (that is, the solution of $(*)$ for L replaced with K), we first provide a semantics for $\mathsf{M.K}$, which is defined by suitably extending minimal relational models for MPL (cf. Section 1.2) with an additional relation dealing with the modalities.

Definition 2.6 (Minimal birelational semantics). *A* minimal birelational model *is a tuple* $\mathcal{M} = \langle \mathcal{W}, \leq, \mathbb{F}, \mathcal{R}, \mathcal{V} \rangle$, *where* $\langle \mathcal{W}, \leq, \mathbb{F}, \mathcal{V} \rangle$ *is a minimal relational model, and* $\mathcal{R}$ *is a binary relation on* $\mathcal{W}$. *The forcing relation* $\mathcal{M}, w \Vdash A$ *is inductively defined extending the clauses for* $p, \bot, \wedge, \vee, \supset$ *in Section* 1.2 *with the following clauses for the modalities*:

$$\begin{array}{lll} \mathcal{M}, w \Vdash \Box B & \text{iff} & \text{for all } v \geq w, \text{ for all } u, \text{ if } v\mathcal{R}u, \text{ then } \mathcal{M}, u \Vdash B; \\ \mathcal{M}, w \Vdash \Diamond B & \text{iff} & \text{for all } v \geq w, \text{ there is } u \text{ such that } v\mathcal{R}u \text{ and } \mathcal{M}, u \Vdash B. \end{array}$$

The semantics in Definition 2.6 essentially coincides with Wijesekera's semantics for $\mathsf{W.K}$ [65] with the only difference of the addition of the fallible worlds, which is due to the fact that the base models are minimal rather than intuitionistic.

The generalisation of the standard clauses for $\Box, \Diamond$ in the relational semantics to all $\leq$-successors is the simplest way to preserve the hereditary property of minimal relational models.

Proposition 2.1 (Hereditary property). *Given a minimal birelational model* $\mathcal{M}$ *and a formula* A *of* $\mathcal{L}$, *for every worlds* w *and* v *of* $\mathcal{M}$ *it holds*: *If* $w \Vdash A$ *and* $w \leq v$, *then* $v \Vdash A$.

Proof. Immediate by induction on the construction of A. □

Theorem 2.2 (Soundness). *For all* $A \in \mathcal{L}$, *if* A *is derivable in* $\mathsf{M.K}$, *then* A *is valid in every minimal birelational model.*

Proof. By showing that all the axioms and rules of $\mathsf{M.K}$ are valid, respectively validity preserving, in every model for $\mathsf{M.K}$. We consider the modal principles.

($K_\Box$) Suppose that $w \Vdash \Box(A \supset B)$ and $w \Vdash \Box A$. Then for all v, u, if $w \leq v$ and $v\mathcal{R}u$, then $u \Vdash A \supset B$ and $u \Vdash A$, hence $u \Vdash B$. Thus, $w \Vdash \Box B$. Therefore $\mathcal{M} \models \Box(A \supset B) \supset (\Box A \supset \Box B)$.

($K_\Diamond$) Suppose that $w \Vdash \Box(A \supset B)$ and $w \Vdash \Diamond A$. Then for all v, if $w \leq v$, then there is u such that $v\mathcal{R}u$ and $u \Vdash A$. Moreover, $u \Vdash A \supset B$. Thus $u \Vdash B$, hence $w \Vdash \Diamond B$. Therefore $\mathcal{M} \models \Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$.

(nec) Suppose that $\mathcal{M} \models A$. Then for all w, v, u, if $w \leq v$ and $v\mathcal{R}u$, then $u \Vdash A$, hence $w \Vdash \Box A$. Therefore $\mathcal{M} \models \Box A$. □

We now present a completeness proof for M.K with respect to the minimal birelational semantics by means of the canonical model technique. The proof adapts the completeness proof for W.K by Wijesekera [65] with the addition of the impossible worlds. For every logic L in $\mathcal{L}$, we call L-*full* any set Φ of formulas of $\mathcal{L}$ such that if $\Phi \vdash_{\mathsf{L}} A$, then $A \in \Phi$ (closure under derivation), and if $A \vee B \in \Phi$, then $A \in \Phi$ or $B \in \Phi$ (disjunction property). Moreover, for every set of formulas Φ, we denote $\Box^{-}\Phi$ the set $\{A \mid \Box A \in \Phi\}$. The following holds.

LEMMA 2.3 (Lindenbaum). *For every set Φ of formulas of $\mathcal{L}$, there is a* M.K*-full set Ψ such that $\Phi \subseteq \Psi$. Moreover, if $\Phi \nvdash_{\mathsf{M.K}} A$, then there is a* M.K*-full set Ψ such that $\Phi \subseteq \Psi$ and $A \notin \Psi$.*

Proof. The proof extends the one of [54] for MPL and IPL to the modal language $\mathcal{L}$ without essential modifications. Consider an enumeration $B_0, B_1, B_2, \ldots$ of all formulas of $\mathcal{L}$. The set Ψ is constructed as follows: $\Phi_0 = \Phi$; $\Phi_{n+1} = \Phi_n \cup \{B_n\}$ if $\Phi_n \nvdash_{\mathsf{M.K}} B_n \supset A$, and $\Phi_{n+1} = \Phi_n$ otherwise; $\Psi = \bigcup_{n\in\mathbb{N}} \Phi_n$. It follows that $\Psi \nvdash_{\mathsf{M.K}} A$, otherwise there are $C_1, .., C_k \in \Psi$ such that $\vdash_{\mathsf{M.K}} C_1 \wedge \cdots \wedge C_k \supset A$, which means that there is $n \in \mathbb{N}$ such that $C_1, .., C_k \in \Phi_n$, hence $\Phi_n \vdash_{\mathsf{M.K}} A$, against the construction hypothesis. Then, $A \notin \Psi$ (given the validity of $A \supset A$ in MPL). Moreover, Ψ is deductively closed: suppose $\Psi \vdash_{\mathsf{M.K}} C$ and $C \notin \Psi$. Then $C = B_n$ for some $n \in \mathbb{N}$, and, by construction, $\Phi_n \vdash_{\mathsf{M.K}} C \supset A$. Since $\Phi_n \subseteq \Psi$, it follows $\Psi \vdash_{\mathsf{M.K}} C \supset A$, hence $\Psi \vdash_{\mathsf{M.K}} A$, giving a contradiction. Finally, Ψ satisfies the disjunction property: suppose $C \vee D \in \Psi$, $C \notin \Psi$ and $D \notin \Psi$. Then $C = B_i$ and $D = B_j$ for some $i, j \in \mathbb{N}$. Let $n = max\{i, j\}$. Then by construction, $\Phi_n \vdash_{\mathsf{M.K}} C \supset A$ and $\Phi_n \vdash_{\mathsf{M.K}} D \supset A$, thus $\Phi_n \vdash_{\mathsf{M.K}} C \vee D \supset A$. It follows $\Psi \vdash_{\mathsf{M.K}} C \vee D \supset A$, and since $C \vee D \in \Psi$, we have $\Psi \vdash_{\mathsf{M.K}} A$, therefore $A \in \Psi$, giving a contradiction. □

DEFINITION 2.7. *For every logic* L *in $\mathcal{L}$, an* L-relational segment, *or just* segment, *is a pair $(\Phi, \mathscr{U})$, where Φ is an* L*-full set, and $\mathscr{U}$ is a set of* L*-full sets such that*:

- *if $\Box A \in \Phi$, then for all $\Psi \in \mathscr{U}$, $A \in \Psi$; and*
- *if $\Diamond A \in \Phi$, then there is $\Psi \in \mathscr{U}$ such that $A \in \Psi$.*

The following holds.

LEMMA 2.4. *For every* M.K*-full set Φ, there exists an* M.K*-relational segment $(\Phi, \mathscr{U})$.*

Proof. Given a M.K-full set Φ, we define $\mathscr{U} = \{\Psi$ M.K-full $\mid \Box^{-}\Phi \subseteq \Psi$ and $B \in \Psi$ for some $\Diamond B \in \Phi\}$. Then by definition, for all $\Box A \in \Phi$ and all $\Psi \in \mathscr{U}$, $A \in \Psi$. Moreover, suppose that $\Diamond A \in \Phi$. By Lemma 2.3, there is an M.K-full set Ψ such that $\Box^{-}\Phi \cup \{A\} \subseteq \Psi$, then $A \in \Psi$ and $\Psi \in \mathscr{U}$. Hence $(\Phi, \mathscr{U})$ is an M.K-segment. □

DEFINITION 2.8. *For every logic* L *in $\mathcal{L}$, the* canonical birelational model *for* L *is the tuple $\mathcal{M} = \langle \mathcal{W}, \leq, \mathbb{F}, \mathcal{R}, \mathcal{V}\rangle$, where*:

- *$\mathcal{W}$ is the set of all* L*-relational segments*;
- *for all $(\Phi, \mathscr{U}), (\Psi, \mathscr{V}) \in \mathcal{W}$, $(\Phi, \mathscr{U}) \leq (\Psi, \mathscr{V})$ if and only if $\Phi \subseteq \Psi$*;
- *for all $(\Phi, \mathscr{U}) \in \mathcal{W}$, $(\Phi, \mathscr{U}) \in \mathbb{F}$ if and only if $\bot \in \Phi$*;
- *for all $(\Phi, \mathscr{U}), (\Psi, \mathscr{V}) \in \mathcal{W}$, $(\Phi, \mathscr{U})\mathcal{R}(\Psi, \mathscr{V})$ if and only if $\Psi \in \mathscr{U}$*;
- *for all $(\Phi, \mathscr{U}) \in \mathcal{W}$, $(\Phi, \mathscr{U}) \in \mathcal{V}(p)$ if and only if $p \in \Phi$.*

It is easy to see that the canonical birelational model for M.K is a minimal birelational model (Definition 2.6). We prove the following lemma.

LEMMA 2.5. *Let* $\mathcal{M} = \langle \mathcal{W}, \leq, \mathbb{F}, \mathcal{R}, \mathcal{V} \rangle$ *be the canonical birelational model for* M.K. *Then for all* $(\Phi, \mathscr{U}) \in \mathcal{W}$ *and all* $A \in \mathcal{L}$, $(\Phi, \mathscr{U}) \Vdash A$ *if and only if* $A \in \Phi$.

Proof. By induction on the construction of A. For the base case $A = p$ and the inductive cases $A = B \wedge C, B \vee C$ the proof is immediate. We show the other cases, writing $\vdash$ for $\vdash_{\mathsf{M.K}}$.

($A = \bot$) $(\Phi, \mathscr{U}) \Vdash \bot$ iff $(\Phi, \mathscr{U}) \in \mathbb{F}$ iff, by definition, $\bot \in \Phi$.

($A = B \supset C$) Suppose that $B \supset C \in \Phi$, and assume $(\Phi, \mathscr{U}) \leq (\Psi, \mathscr{V})$ and $(\Psi, \mathscr{V}) \Vdash B$. By definition, $\Phi \subseteq \Psi$, thus $B \supset C \in \Psi$. Moreover by i.h., $B \in \Psi$, hence $C \in \Psi$, thus by i.h., $(\Psi, \mathscr{V}) \Vdash C$. Therefore $(\Phi, \mathscr{U}) \Vdash B \supset C$. Now suppose that $B \supset C \notin \Phi$. Then $\Phi \not\vdash B \supset C$, thus $\Phi \cup \{B\} \not\vdash C$. By Lemma 2.3, there is Ψ M.K-full such that $\Phi \cup \{B\} \subseteq \Psi$ and $C \notin \Psi$. Then by Lemma 2.4 and Definition 2.8, there is an M.K-segment $(\Psi, \mathscr{V}) \in \mathcal{W}$. Thus by definition, $(\Phi, \mathscr{U}) \leq (\Psi, \mathscr{V})$, and by i.h., $(\Psi, \mathscr{V}) \Vdash B$ and $(\Psi, \mathscr{V}) \nVdash C$. Therefore $(\Phi, \mathscr{U}) \nVdash B \supset C$.

($A = \Box B$) Suppose that $\Box B \in \Phi$. Then for all $(\Psi, \mathscr{V}) \geq (\Phi, \mathscr{U})$, $\Box B \in \Psi$. Moreover by definition, if $(\Psi, \mathscr{V})\mathcal{R}(\Theta, \mathscr{Z})$, then $\Theta \in \mathscr{V}$, hence by definition of segment, $B \in \Theta$. Then by i.h., $(\Theta, \mathscr{Z}) \Vdash B$, therefore $(\Phi, \mathscr{U}) \Vdash \Box B$. Now suppose that $\Box B \notin \Phi$. Then $\Box^{-}\Phi \not\vdash B$ (indeed, if $\Box^{-}\Phi \vdash B$, then $\vdash C_1 \wedge \cdots \wedge C_n \supset B$ for some $\Box C_1, \dots, \Box C_n \in \Phi$, then by *nec*, $\vdash \Box(C_1 \wedge \cdots \wedge C_n \supset B)$, and by $K_{\Box}$, $\vdash \Box(C_1 \wedge \cdots \wedge C_n) \supset \Box B$; since $\vdash \Box C_1 \wedge \cdots \wedge \Box C_n \supset \Box(C_1 \wedge \cdots \wedge C_n)$, we have $\vdash \Box C_1 \wedge \cdots \wedge \Box C_n \supset \Box B$, hence $\Phi \vdash \Box B$, therefore $\Box B \in \Phi$, against the assumption). By Lemma 2.3, there is Ψ M.K-full such that $\Box^{-}\Phi \subseteq \Psi$ and $B \notin \Psi$. We define $\mathscr{V} = \{\Psi\} \cup \{\Theta \text{ M.K-full} \mid \Box^{-}\Phi \subseteq \Theta \text{ and } C \in \Theta \text{ for some } \Diamond C \in \Phi\}$. Given that, by Lemma 2.3, such a set Θ exists for every $\Diamond C \in \Phi$, we have that $(\Phi, \mathscr{V})$ is an M.K-segment. Furthermore, by Lemma 2.4, there exists an M.K-segment $(\Psi, \mathscr{Z})$, hence since $\Psi \in \mathscr{V}$, by definition, $(\Phi, \mathscr{V})\mathcal{R}(\Psi, \mathscr{Z})$. Moreover, since $B \notin \Psi$, by i.h., $(\Psi, \mathscr{Z}) \nVdash B$, then since $(\Phi, \mathscr{U}) \leq (\Phi, \mathscr{V})$, we have $(\Phi, \mathscr{U}) \nVdash \Box B$.

($A = \Diamond B$) Suppose that $\Diamond B \in \Phi$. Then for all $(\Psi, \mathscr{V}) \geq (\Phi, \mathscr{U})$, $\Diamond B \in \Psi$. Thus by Definition 2.7, there is $\Theta \in \mathscr{V}$) such that $B \in \Theta$, and by Lemma 2.4, there is a segment $(\Theta, \mathscr{Z}) \in \mathcal{M}$. Moreover, by definition, $(\Psi, \mathscr{V})\mathcal{R}(\Theta, \mathscr{Z})$, and by i.h., $(\Theta, \mathscr{Z}) \Vdash B$. It follows that $(\Phi, \mathscr{U}) \Vdash \Diamond B$. Now suppose that $\Diamond B \notin \Phi$. Then for every $\Diamond C \in \Phi$, $\Box^{-}\Phi \cup \{C\} \not\vdash B$ (indeed, if $\Box^{-}\Phi \cup \{C\} \vdash B$, then $\vdash D_1 \wedge \cdots \wedge D_n \wedge C \supset B$ for some $\Box D_1, \dots, \Box D_n \in \Phi$, thus $\vdash D_1 \wedge \cdots \wedge D_n \supset (C \supset B)$, hence $\vdash \Box(D_1 \wedge \cdots \wedge D_n) \supset \Box(C \supset B)$, then by $K_{\Diamond}$ and valid principles, $\vdash \Box D_1 \wedge \cdots \wedge \Box D_n \supset (\Diamond C \supset \Diamond B)$, so $\vdash \Box D_1 \wedge \cdots \wedge \Box D_n \wedge \Diamond C \supset \Diamond B$, which implies $\Phi \vdash \Diamond B$, hence, finally, $\Diamond B \in \Phi$, against the assumption). We define $\mathscr{V} = \{\Psi \text{ M.K-full} \mid \Box^{-}\Phi \subseteq \Psi, B \notin \Psi \text{ and } C \in \Psi \text{ for some } \Diamond C \in \Phi\}$. By Lemma 2.3, such a set Ψ exists for every $\Diamond C \in \Psi$. It is easy to see that $(\Phi, \mathscr{V})$ is a M.K-segment, hence $(\Phi, \mathscr{V}) \in \mathcal{W}$. Moreover, by definition, for all $(\Psi, \mathscr{Z})$ such that $(\Phi, \mathscr{V})\mathcal{R}(\Psi, \mathscr{Z})$, $B \notin \Psi$, thus by i.h., $(\Psi, \mathscr{Z}) \nVdash B$. Given that $(\Phi, \mathscr{U}) \leq (\Phi, \mathscr{V})$, we obtain $(\Phi, \mathscr{U}) \nVdash \Diamond B$. □

THEOREM 2.6 (Completeness). *For all* $A \in \mathcal{L}$, *if* A *is valid in every minimal birelational model, then* A *is derivable in* M.K.

Proof. Suppose that $\mathsf{M.K} \nvdash A$. Then by Lemma 2.3, there is an M.K-full set Ψ such that $A \notin \Psi$, and by Lemma 2.4, there exists an M.K-segment $(\Psi, \mathscr{U})$. By Definition 2.8, $(\Psi, \mathscr{U})$ belongs to the canonical model $\mathcal{M}$ for M.K, then by Lemma 2.5, $(\Psi, \mathscr{U}) \nVdash A$. Since $\mathcal{M}$ is a minimal birelational model, we conclude that it is not the case that A is valid in all models for M.K. □

Based on this semantic characterisation, we now show that M.K is the solution to the equation $(*)$ for L replaced with K, hence, according to our criterion, it is the minimal counterpart of classical K.

THEOREM 2.7. *For all $A \in \mathcal{L}$, A is derivable in* M.K *if and only if A^t is derivable in* $\mathsf{S4} \oplus \mathsf{K}$.

Proof. We recall that $\mathsf{S4} \oplus \mathsf{K}$ is sound and complete with respect to the class of all classical birelational models $\langle \mathcal{W}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{V} \rangle$, where $\mathcal{R}_1$ and $\mathcal{R}_2$ are binary relations on $\mathcal{W}$ and $\mathcal{R}_1$ is reflexive and transitive.

($\Rightarrow$) Suppose that $\mathsf{S4} \oplus \mathsf{K} \nvdash A^t$. Then there are a model $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{V} \rangle$ for $\mathsf{S4} \oplus \mathsf{K}$ and a world w such that $\mathcal{M}, w \nVdash A^t$. We define $\mathcal{M}' = \langle \mathcal{W}, \leq, \mathbb{F}, \mathcal{R}, \mathcal{V}' \rangle$ over the same set $\mathcal{W}$ of $\mathcal{M}$, where $\leq\ = \mathcal{R}_1$, $\mathcal{R} = \mathcal{R}_2$, for all $p \in Atm$, $\mathcal{V}'(p) = \{v \mid \text{for all } u, v\mathcal{R}_1 u \text{ implies } u \in \mathcal{V}(p)\}$, and $\mathbb{F} = \{v \mid \text{for all } u, v\mathcal{R}_1 u \text{ implies } u \in \mathcal{V}(f)\}$. It is easy to verify that $\mathcal{M}'$ is a minimal birelational model, in particular $\mathcal{V}(p)$ and $\mathbb{F}$ are $\leq$-upward closed. We show that for all $v \in \mathcal{W}$ and all $B \in \mathcal{L}$ it holds:

$$\mathcal{M}', v \Vdash B \ \text{ if and only if } \ \mathcal{M}, v \Vdash B^t,$$

from which it follows that $\mathcal{M}', w \nVdash A$, therefore $\mathsf{M.K} \nvdash A$. The proof is by induction on the construction of B. The cases $B = C \wedge D$ and $B = C \vee D$ are immediate by i.h. We consider the other cases.

- $(B = p)$ $\mathcal{M}', v \Vdash p$ iff $v \in \mathcal{V}'(p)$ iff (by definition of $\mathcal{V}'$) for all u, $v\mathcal{R}_1 u$ implies $u \in \mathcal{V}(p)$; iff for all u, $v\mathcal{R}_1 u$ implies $\mathcal{M}, u \Vdash p$; iff $\mathcal{M}, v \Vdash \Box_1 p$.
- $(B = \bot)$ $\mathcal{M}', v \Vdash \bot$ iff $v \in \mathbb{F}$ iff (by definition of $\mathbb{F}$) for all u, $v\mathcal{R}_1 u$ implies $u \in \mathcal{V}(f)$; iff for all u, $v\mathcal{R}_1 u$ implies $\mathcal{M}, u \Vdash f$; iff $\mathcal{M}, v \Vdash \Box_1 f$.
- $(B = C \supset D)$ $\mathcal{M}', v \Vdash C \supset D$ iff for all $u \geq v$, $\mathcal{M}', u \Vdash C$ implies $\mathcal{M}', u \Vdash D$; iff (by definition of $\leq$ and i.h.) for all u, if $v\mathcal{R}_1 u$, then $\mathcal{M}, u \Vdash C^t$ implies $\mathcal{M}, u \Vdash D^t$; iff for all u, if $v\mathcal{R}_1 u$, then $\mathcal{M}, u \Vdash C^t \supset D^t$; iff $\mathcal{M}, v \Vdash \Box_1(C^t \supset D^t)$.
- $(B = \Box C)$ $\mathcal{M}', v \Vdash \Box C$ iff for all $u \geq v$, for all z, if $u\mathcal{R}z$, then $\mathcal{M}', z \Vdash C$; iff (by definition of $\leq$ and $\mathcal{R}$ and i.h.) for all u, z, if $v\mathcal{R}_1 u$ and $u\mathcal{R}_2 z$, then $\mathcal{M}, z \Vdash C^t$; iff $\mathcal{M}, v \Vdash \Box_1\Box_2 C^t$.
- $(B = \Diamond C)$ $\mathcal{M}', v \Vdash \Diamond C$ iff for all $u \geq v$, there is z such that $u\mathcal{R}z$ and $\mathcal{M}', z \Vdash C$; iff (by definition of $\leq$ and $\mathcal{R}$ and i.h.) for all u, if $v\mathcal{R}_1 u$, then there is z such that $u\mathcal{R}_2 z$ and $\mathcal{M}, z \Vdash C^t$; iff $\mathcal{M}, v \Vdash \Box_1\Diamond_2 C^t$.

($\Leftarrow$) Suppose that $\mathsf{M.K} \nvdash A$. Then there are a model $\mathcal{M} = \langle \mathcal{W}, \leq, \mathbb{F}, \mathcal{R}, \mathcal{V} \rangle$ for M.K and a world w such that $\mathcal{M}, w \nVdash A$. We define $\mathcal{M}'' = \langle \mathcal{W}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{V}'' \rangle$ over the same set $\mathcal{W}$ of $\mathcal{M}$, where $\mathcal{R}_1 = \leq$, $\mathcal{R}_2 = \mathcal{R}$, for all $p \in Atm$, $\mathcal{V}''(p) = \mathcal{V}(p)$, and $\mathcal{V}''(f) = \mathbb{F}$. $\mathcal{M}''$ is a model for $\mathsf{S4} \oplus \mathsf{K}$. We show that for all $v \in \mathcal{W}$ and all $B \in \mathcal{L}$ it holds:

$$\mathcal{M}, v \Vdash B \ \text{ if and only if } \ \mathcal{M}'', v \Vdash B^t,$$

from which it follows that $\mathcal{M}'', w \not\Vdash A$, therefore $\mathsf{S4} \oplus \mathsf{K} \not\vdash A^t$. The proof is by induction on the construction of B. The cases $B = C \wedge D$ and $B = C \vee D$ are immediate by i.h. We consider the other cases.

($B = p$) $\mathcal{M}, v \Vdash p$ iff $v \in \mathcal{V}(p)$ iff (since $\mathcal{V}$ is $\leq$-closed) for all $u \geq v$, $u \in \mathcal{V}(p)$; iff (by definition of $\mathcal{R}_1$ and $\mathcal{V}''$) for all u, if $v\mathcal{R}_1 u$, then $u \in \mathcal{V}''(p)$; iff $\mathcal{M}'', v \Vdash \Box_1 p$.

($B = \bot$) $\mathcal{M}, v \Vdash \bot$ iff $v \in \mathbb{F}$ iff (since $\mathbb{F}$ is $\leq$-closed) for all $u \geq v$, $u \in \mathbb{F}$; iff (by definition of $\mathcal{R}_1$ and $\mathcal{V}''$) for all u, if $v\mathcal{R}_1 u$, then $u \in \mathcal{V}''(f)$; iff $\mathcal{M}'', v \Vdash \Box_1 f$.

($B = C \supset D$) $\mathcal{M}, v \Vdash C \supset D$ iff for all $u \geq v$, $\mathcal{M}, u \Vdash C$ implies $\mathcal{M}, u \Vdash D$; iff (by definition of $\mathcal{R}_1$ and i.h.) for all u, if $v\mathcal{R}_1 u$, then $\mathcal{M}'', u \Vdash C^t$ implies $\mathcal{M}'', u \Vdash D^t$; iff $\mathcal{M}'', v \Vdash \Box_1(C^t \supset D^t)$.

($B = \Box C$) $\mathcal{M}, v \Vdash \Box C$ iff for all $u \geq v$, for all z, if $u\mathcal{R}z$, then $\mathcal{M}, z \Vdash C$; iff (by definition of $\mathcal{R}_1$ and $\mathcal{R}_2$ and i.h.) for all u, z, if $v\mathcal{R}_1 u$ and $u\mathcal{R}_2 z$, then $\mathcal{M}'', z \Vdash C^t$; iff $\mathcal{M}'', v \Vdash \Box_1\Box_2 C^t$.

($B = \Diamond C$) $\mathcal{M}, v \Vdash \Diamond C$ iff for all $u \geq v$, there is z such that $u\mathcal{R}z$ and $\mathcal{M}, z \Vdash C$; iff (by definition of $\mathcal{R}_1$ and $\mathcal{R}_2$ and i.h.) for all u, if $v\mathcal{R}_1 u$, then there is z such that $u\mathcal{R}_2 z$ and $\mathcal{M}'', z \Vdash C^t$; iff $\mathcal{M}'', v \Vdash \Box_1\Diamond_2 C^t$. □

§3. Minimal K via sequent calculus. From the point of view of the sequent calculi, classical and MPL stay in a clear and neat relation: given a suitable sequent calculus SC for CPL, a calculus for MPL can be obtained by restricting the rules of SC to *single-succedent* sequents, namely sequents with exactly one formula in the succedent. This relation is particularly evident in G1-style sequent calculi [61]. In this section, we extend this relation to modal logics and define a minimal version of K by restricting a standard G1-calculus for K to single-succedent sequents. Most importantly, we show that the resulting logic is precisely M.K introduced in the previous section. We consider the following standard definitions.

DEFINITION 3.1. *A* sequent *is a pair* $\Gamma \Rightarrow \Delta$, *where* Γ *and* Δ (*respectively, the* antecedent *and the* succedent *of the sequent*) *are finite, possibly empty multisets of formulas of* $\mathcal{L}$. *A sequent* $\Gamma \Rightarrow \Delta$ *is interpreted as a formula of* $\mathcal{L}$ *via the* formula interpretation ι *as* $\bigwedge \Gamma \supset \bigvee \Delta$ *if* Γ *is non-empty, and as* $\bigvee \Delta$ *if* Γ *is empty, where* $\bigvee \emptyset$ *is interpreted as* $\bot$. *A* sequent calculus SC *is a set of initial sequents and sequent rules.*[5] *A* derivation *of a sequent S in a calculus* SC *is a tree where each node is labelled by a sequent, the root is labelled by S, the leaves are labelled by initial sequents and each node is obtained by the immediate predecessor(s) by the application of a rule of* SC. *A sequent S is* derivable *in a calculus* SC *if there is a derivation of S in* SC. *A formula A is derivable in* SC *if the sequent* $\Rightarrow A$ *is derivable in* SC. *A sequent calculus* SC *is a calculus for a logic* L *if for every formula A, A is derivable in* SC *if and only if it is derivable in* L.

In order to analyse the sequent calculi, we also consider the following standard notions regarding the sequent rules.

DEFINITION 3.2. *A rule R is* admissible *in a calculus* SC *if whenever the premisses of R are derivable in* SC, *the conclusion is also derivable in* SC. *A formula is* principal *in the*

[5] More precisely, as for axiomatic systems, we rather consider sequent and rule schemata, omitting this specification throughout the text.

Sequent calculus G1-CPL **for** CPL.

$$\mathsf{init}^{c\ell}\ A \Rightarrow A \qquad \wedge^{c\ell}_{\mathsf{L}}\ \frac{\Gamma, A_i \Rightarrow \Delta}{\Gamma, A_1 \wedge A_2 \Rightarrow \Delta}\ (i = 1, 2) \qquad \wedge^{c\ell}_{\mathsf{R}}\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta} \qquad \bot^{c\ell}_{\mathsf{L}}\ \bot \Rightarrow$$

$$\vee^{c\ell}_{\mathsf{L}}\ \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \qquad \vee^{c\ell}_{\mathsf{R}}\ \frac{\Gamma \Rightarrow A_i, \Delta}{\Gamma \Rightarrow A_1 \vee A_2, \Delta}\ (i = 1, 2) \qquad \supset^{c\ell}_{\mathsf{L}}\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta}$$

$$\supset^{c\ell}_{\mathsf{R}}\ \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta} \qquad \mathsf{wk}^{cl}_{\mathsf{L}}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad \mathsf{wk}^{cl}_{\mathsf{R}}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta} \qquad \mathsf{ctr}^{cl}_{\mathsf{L}}\ \frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad \mathsf{ctr}^{cl}_{\mathsf{R}}\ \frac{\Gamma \Rightarrow A, A, \Delta}{\Gamma \Rightarrow A, \Delta}$$

Sequent calculus G1-MPL **for** MPL.

$$\mathsf{init}^{m}\ A \Rightarrow A \qquad \wedge^{m}_{\mathsf{L}}\ \frac{\Gamma, A_i \Rightarrow C}{\Gamma, A_1 \wedge A_2 \Rightarrow C}\ (i = 1, 2) \qquad \wedge^{m}_{\mathsf{R}}\ \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\vee^{m}_{\mathsf{L}}\ \frac{\Gamma, A \Rightarrow C \qquad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C} \qquad \vee^{m}_{\mathsf{R}}\ \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}\ (i = 1, 2) \qquad \supset^{m}_{\mathsf{R}}\ \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B}$$

$$\supset^{m}_{\mathsf{L}}\ \frac{\Gamma \Rightarrow A \qquad \Gamma, B \Rightarrow C}{\Gamma, A \supset B \Rightarrow C} \qquad \mathsf{wk}^{m}_{\mathsf{L}}\ \frac{\Gamma \Rightarrow C}{\Gamma, A \Rightarrow C} \qquad \mathsf{ctr}^{m}_{\mathsf{L}}\ \frac{\Gamma, A, A \Rightarrow C}{\Gamma, A \Rightarrow C}$$

Figure 1. Sequent calculi G1-CPL and G1-MPL.

application of a rule if it occurs in the conclusion and not in the premiss(es), while it is active *if it occurs in (at least) one premiss and not in the conclusion. Structural rules are an exception to these definitions: the principal and active formula of a contraction (respectively, weakening) rule is the one formula A which has $n > 0$ occurrences in the conclusion and $n + 1$ (respectively, $n - 1$) occurrences in the premiss. All formulas which are neither principal nor active are the* context.

The well-known G1-sequent calculi G1-CPL and G1-MPL [61] for CPL and MPL are displayed in Figure 1. It is easy to see that G1-MPL corresponds to the single-succedent restriction of G1-CPL. In particular, the initial sequent $\bot^{c\ell}_{\mathsf{L}}$ and the rule $\mathsf{ctr}^{cl}_{\mathsf{R}}$ are dropped in G1-MPL as they have respectively no formula in the succedent, and two occurrences of the active formula in the succedent of the premiss. $\mathsf{wk}^{cl}_{\mathsf{R}}$ is also dropped as it requires either no formula in the succedent of the premiss or at least two formulas in the succedent of the conclusion. Concerning the other rules, the right context is removed from the sequents with an active or principal formula in the succedent, this is the case for instance of initial sequents init^m and of the rule $\wedge^m_{\mathsf{R}}$, as well as of the left premiss of the rule $\supset^m_{\mathsf{L}}$. In the other rules, the right context is converted from an arbitrary multiset Δ to a single formula C.

In order to extend the multi- vs. single-succedent relation to modal logics, we consider the G1-sequent calculus G1-K for K, defined extending G1-CPL with the modal rules $\mathsf{K}^{c\ell}_{\Box}$ and $\mathsf{K}^{c\ell}_{\Diamond}$ in Figure 2. In these rules and the following, given a multiset $\Gamma = A_1, \dots, A_n$, we denote $\Box\Gamma$ and $\Diamond\Gamma$ the multisets $\Box A_1, \dots, \Box A_n$ and $\Diamond A_1, \dots, \Diamond A_n$, respectively. As for axiomatic systems, sequent calculi for K are more commonly defined in terms of $\Box$ only. Here we consider a formulation of the calculus with both $\Box$ and $\Diamond$ explicit in order to better display the relation with minimal modal logics, where $\Box$ and $\Diamond$ are not

$$\mathsf{K}^{c\ell}_{\Box}\ \frac{\Sigma \Rightarrow A, \Pi}{\Box\Sigma \Rightarrow \Box A, \Diamond\Pi} \qquad \mathsf{K}^{c\ell}_{\Diamond}\ \frac{\Sigma, A \Rightarrow \Pi}{\Box\Sigma, \Diamond A \Rightarrow \Diamond\Pi} \qquad \mathsf{K}^{m}_{\Box}\ \frac{\Sigma \Rightarrow A}{\Box\Sigma \Rightarrow \Box A} \qquad \mathsf{K}^{m}_{\Diamond}\ \frac{\Sigma, A \Rightarrow B}{\Box\Sigma, \Diamond A \Rightarrow \Diamond B}$$

Figure 2. Modal rules for G1-K and G1-M.K.

1. $A \supset (B \supset A \wedge B)$	(MPL)	1. $C_1 \wedge C_2 \supset A$	(premiss of $\mathsf{K}_\Box^m$)
2. $\Box(A \supset (B \supset A \wedge B))$	(1, *nec*)	2. $\Box(C_1 \wedge C_2 \supset A)$	(1, *nec*)
3. $\Box A \supset \Box(B \supset A \wedge B)$	(2, $K_\Box$)	3. $\Box(C_1 \wedge C_2) \supset \Box A$	(2, $K_\Box$, mp)
4. $\Box A \supset (\Box B \supset \Box(A \wedge B))$	(3, $K_\Box$)	4. $\Box C_1 \wedge \Box C_2 \supset \Box(C_1 \wedge C_2)$	(derivable)
5. $\Box A \wedge \Box B \supset \Box(A \wedge B)$	(4, MPL)	5. $\Box C_1 \wedge \Box C_2 \supset \Box A$	(3, 4)

1. $C_1 \wedge C_2 \wedge A \supset B$	(premiss of $\mathsf{K}_\Diamond^m$)
2. $C_1 \wedge C_2 \supset (A \supset B)$	(1, MPL)
3. $\Box(C_1 \wedge C_2 \supset (A \supset B))$	(2, *nec*)
4. $\Box(C_1 \wedge C_2) \supset \Box(A \supset B)$	(3, $K_\Box$, mp)
5. $\Box C_1 \wedge \Box C_2 \supset \Box(C_1 \wedge C_2)$	(derivable)
6. $\Box C_1 \wedge \Box C_2 \supset \Box(A \supset B)$	(4, 5)
7. $\Box C_1 \wedge \Box C_2 \supset (\Diamond A \supset \Diamond B)$	(6, $K_\Diamond$)
8. $\Box C_1 \wedge \Box C_2 \wedge \Diamond A \supset \Diamond B$	(7, MPL)

Figure 3. Derivations in M.K.

interdefinable. The rules $\mathsf{K}_\Box^{c\ell}$ and $\mathsf{K}_\Diamond^{c\ell}$ for K with explicit $\Box$ and $\Diamond$ can be found e.g., in [28].

On the basis of G1-K, we now define the calculus G1-M.K as the single-succedent restriction of G1-K. As a result, G1-M.K contains the rules of G1-MPL and the modal rules $\mathsf{K}_\Box^m$ and $\mathsf{K}_\Diamond^m$. Indeed, in the rule $\mathsf{K}_\Box^{c\ell}$, the succedent of the conclusion must have a $\Box$-formula $\Box A$ and can have additional $\Diamond$-formulas. Then, its single-succedent restriction only preserves $\Box A$. Concerning $\mathsf{K}_\Diamond^{c\ell}/\mathsf{K}_\Diamond^m$, the consequent of the conclusion of $\mathsf{K}_\Diamond^{c\ell}$ has an arbitrary number of $\Diamond$-formulas. Correspondingly, the consequent of the conclusion of $\mathsf{K}_\Diamond^m$ has exactly one $\Diamond$-formula.

In the remaining part of this section, we show that G1-M.K is equivalent to the logic M.K defined in the previous section. The proof is based on the following theorem, which entails that the addition of the cut rule to G1-M.K does not extend the set of derivable sequents. To do its length, the proof of Theorem 3.1 is presented in the appendix.

THEOREM 3.1. *The following rule* cut *is admissible in* G1-M.K:

$$\mathsf{cut}\ \frac{\Gamma \Rightarrow A \qquad \Sigma, A \Rightarrow C}{\Gamma, \Sigma \Rightarrow C}$$

Proof. The proof is in the appendix. □

THEOREM 3.2. *For all $A \in \mathcal{L}$, A is derivable in* G1-M.K *if and only if A is derivable in* M.K.

Proof. ($\Rightarrow$) $\imath(A \Rightarrow A) = A \supset A$ is derivable in M.K, moreover we can show that for all rules $S_1, \dots, S_n/S$ of G1-M.K, the rule $\imath(S_1), \dots, \imath(S_n)/\imath(S)$ is derivable in M.K, where $\imath$ is the formula interpretation of sequents as defined in Definition 3.2. For the propositional rules the proof is standard. We show in Figure 3 the derivations of the modal rules, considering the representative cases where $\Sigma = C_1, C_2$. The cases where Σ contains less or more formulas are a simplification or a generalisation of these cases.

($\Leftarrow$) The proof consists in showing that all axioms and rules of M.K are derivable, respectively admissible in G1-M.K. We omit the derivations of the propositional axioms which are standard. The derivations of the modal axioms and rule are displayed in Figure 4. □

$$\dfrac{\dfrac{\dfrac{(*)\ A \supset B, A \Rightarrow B}{\Box(A \supset B), \Box A \Rightarrow \Box B}\ \mathsf{K}^m_\Box}{\Box(A \supset B) \Rightarrow \Box A \supset \Box B}\ \supset^m_\mathsf{R}}{\Rightarrow \Box(A \supset B) \supset (\Box A \supset \Box B)}\ \supset^m_\mathsf{R} \qquad \dfrac{\dfrac{\dfrac{(*)\ A \supset B, A \Rightarrow B}{\Box(A \supset B), \Diamond A \Rightarrow \Diamond B}\ \mathsf{K}^m_\Diamond}{\Box(A \supset B) \Rightarrow \Diamond A \supset \Diamond B}\ \supset^m_\mathsf{R}}{\Rightarrow \Box(A \supset B) \supset (\Diamond A \supset \Diamond B)}\ \supset^m_\mathsf{R} \qquad \dfrac{\Rightarrow A}{\Rightarrow \Box A}\ \mathsf{K}^m_\Box$$

$$\dfrac{\Rightarrow A \qquad \dfrac{\Rightarrow A \supset B \qquad (*)\ A \supset B, A \Rightarrow B}{A \Rightarrow B}\ \mathsf{cut}}{\Rightarrow B}\ \mathsf{cut} \qquad \dfrac{A \Rightarrow A \qquad \dfrac{B \Rightarrow B}{B, A \Rightarrow B}\ \mathsf{wk}^m_\mathsf{L}}{(*)\ A \supset B, A \Rightarrow B}\ \supset^m_\mathsf{L}$$

Figure 4. Derivations in G1-M.K.

§4. Relating minimal K and constructive K. As one can easily notice from their axiomatisations, our minimal logic M.K is strictly related with the constructive logic C.K studied in the literature, in particular C.K coincides with the extension of M.K with ex falso quodlibet $\bot \supset A$ (exactly as IPL amounts to MPL $+ \bot \supset A$). This means that the two systems share exactly the same modal principles, despite over a different propositional base. We now show that analogously tight relations between the two logics can be also observed based on their semantics and sequent calculi.

4.1. Semantics. As recalled in Section 1.2, disregarding the modalities, there are two ways to transform relational models for MPL into relational models for IPL: (1) assuming $\mathbb{F} = \emptyset$, thus obtaining Kripke's intuitionistic relational models, or (2) preserving the fallible worlds but ensuring the validity of ex falso quodlibet by assuming $\mathbb{F} \subseteq \mathcal{V}(p)$ for all $p \in Atm$. Interestingly, the two ways are equivalent for propositional logic, as they both provide a semantics for IPL, but they are not equivalent in presence of the modalities. In particular, if applied to minimal birelational models, the restriction (1) gives relational models for W.K as defined in [65]. By contrast, a suitable adaptation of (2) which ensures the validity of $\bot \supset A$ also in presence of the modalities gives the following birelational models for C.K.

Definition 4.1 (Constructive birelational semantics). *A minimal birelational model* $\mathcal{M} = \langle \mathcal{W}, \leq, \mathbb{F}, \mathcal{R}, \mathcal{V} \rangle$ *is a* constructive birelational model *if for all* $w \in \mathbb{F}$ *it holds*:

- (i) $w \in \mathcal{V}(p)$ *for all* $p \in Atm$;
- (ii) *if* $w\mathcal{R}v$, *then* $v \in \mathbb{F}$;
- (iii) *there is* v *such that* $w\mathcal{R}v$.

Analogous constructive birelational models for C.K were defined in [41],[6] and a completeness proof for C.K was also provided. We show here how our canonical model construction extends to these models.

Theorem 4.1. *For all* $A \in \mathcal{L}$, *if* A *is derivable in* C.K, *then* A *is valid in every constructive birelational model* $\mathcal{M}$. *In particular,* $\mathcal{M} \models \bot \supset B$ *for every* $B \in \mathcal{L}$.

Proof. The proof extends the proof of Theorem 2.2 by showing that $\mathcal{M} \models \bot \supset A$ for every A. Suppose that $w \Vdash \bot$. Then $w \in \mathbb{F}$. We show by induction on the construction of A that $w \Vdash A$. $(A = p)$ By Definition 4.1, item (i), $w \in \mathcal{V}(p)$, then $w \Vdash p$.

[6] The models in Definition 4.1 slightly differ from those of [41] because of the latter condition (iii) which is not considered in [41]. We observe however that this (or a similar) condition is necessary in order to ensure the validity of ex falso quodlibet over the whole language $\mathcal{L}$. To see this, consider a model $\mathcal{M}$ satisfying (i) and (ii) but not (iii), where $\mathcal{W} = \mathbb{F} = \{w\}$, $w \leq w$ and not $w\mathcal{R}w$. It is easy to verify that $w \Vdash \bot$ but $w \not\Vdash \Diamond p$, and hence $\mathcal{M} \not\models \bot \supset \Diamond p$.

$$\mathsf{init}^i\ A \Rightarrow A \qquad \bot^i_\mathsf{L}\ \bot \Rightarrow \qquad \wedge^i_\mathsf{L}\ \frac{\Gamma, A_i \Rightarrow \delta}{\Gamma, A_1 \wedge A_2 \Rightarrow \delta}\ (i = 1, 2) \qquad \wedge^i_\mathsf{R}\ \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\vee^i_\mathsf{L}\ \frac{\Gamma, A \Rightarrow \delta \qquad \Gamma, B \Rightarrow \delta}{\Gamma, A \vee B \Rightarrow \delta} \qquad \vee^i_\mathsf{R}\ \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}\ (i = 1, 2) \qquad \supset^i_\mathsf{R}\ \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B}$$

$$\supset^i_\mathsf{L}\ \frac{\Gamma \Rightarrow A \qquad \Gamma, B \Rightarrow \delta}{\Gamma, A \supset B \Rightarrow \delta} \qquad \mathsf{wk}^i_\mathsf{L}\ \frac{\Gamma \Rightarrow \delta}{\Gamma, A \Rightarrow \delta} \qquad \mathsf{wk}^i_\mathsf{R}\ \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow A} \qquad \mathsf{ctr}^i_\mathsf{L}\ \frac{\Gamma, A, A \Rightarrow \delta}{\Gamma, A \Rightarrow \delta}$$

Figure 5. Sequent calculus G1-IPL.

$(A = \bot)$ By hypothesis. $(A = B \wedge C, B \vee C)$ Immediate by applying the i.h. $(A = B \supset C)$ Immediate by i.h. and $\leq$-upward closure of $\mathbb{F}$. $(A = \Box B)$ Suppose $w \leq v$. Since $\mathbb{F}$ is $\leq$-upward closed, $v \in \mathbb{F}$. Then by Definition 4.1, item (ii), for all u such that $v\mathcal{R}u$, $u \in \mathbb{F}$. Then by i.h., $u \Vdash B$, therefore $w \Vdash \Box B$. $(A = \Diamond B)$ Suppose $w \leq v$. Since $\mathbb{F}$ is $\leq$-upward closed, $v \in \mathbb{F}$. Then by Definition 4.1, item (iii), there is u such that $v\mathcal{R}u$, and by item (ii), $u \in \mathbb{F}$. Then by i.h., $u \Vdash B$, therefore $w \Vdash \Diamond B$. □

We now prove that C.K is complete with respect to constructive birelational models. First, note that Lemmas 2.3 and 2.4 also hold for C.K (in particular, for Lemma 2.4 the proof is the same, uniformly replacing M.K with C.K). We additionally prove the following lemma.

Lemma 4.2. *Let $\mathcal{M} = \langle \mathcal{W}, \leq, \mathbb{F}, \mathcal{R}, \mathcal{V} \rangle$ be the canonical birelational model for* C.K *(Definition 2.8). Then for all $(\Phi, \mathscr{U}) \in \mathcal{W}$ and all $A \in \mathcal{L}$, $(\Phi, \mathscr{U}) \Vdash A$ if and only if $A \in \Phi$. Moreover, $\mathcal{M}$ is a constructive birelational model.*

Proof. The first claim is proved exactly as Lemma 2.5. For the second claim, we show that $\mathcal{M}$ satisfies the conditions of Definition 4.1. Suppose that $(\Phi, \mathscr{U}) \in \mathbb{F}$. Then $\bot \in \Phi$. Since Φ is closed under derivation, by ex falso quodlibet we obtain $\Phi = \mathcal{L}$, which entails the following. (i) For all $p \in Atm$, $p \in \Phi$, hence by definition, $(\Phi, \mathscr{U}) \in \mathcal{V}(p)$. (ii) $\Box\bot \in \Phi$, hence by Definition 2.7, $\bot \in \Psi$ for all $\Psi \in \mathscr{U}$. Then $(\Phi, \mathscr{U})\mathcal{R}(\Psi, \mathscr{V})$ entails $\bot \in \Psi$, thus $(\Psi, \mathscr{V}) \in \mathbb{F}$. (iii) $\Diamond\bot \in \Phi$, hence by Definition 2.7, there is $\Psi \in \mathscr{U}$ such that $\bot \in \Psi$. By Lemma 2.4 (which holds for C.K as well), there exists a C.K-segment $(\Psi, \mathscr{V})$. Then $(\Phi, \mathscr{U})\mathcal{R}(\Psi, \mathscr{V})$ and $(\Psi, \mathscr{V}) \in \mathbb{F}$. □

As a consequence of the lemma, we obtain the completeness of C.K (cf. proof of Theorem 2.6).

Theorem 4.3. *For all $A \in \mathcal{L}$, A is derivable in* C.K *if and only if A is valid in every constructive birelational model.*

4.2. Sequent calculus. We have considered in Section 3 the multi- vs. single-succedent correspondence between the sequent calculi G1-CPL and G1-MPL. A similar relation holds between sequent calculi for CPL and IPL. In particular, the calculus G1-IPL for IPL can be defined by restricting G1-CPL to sequents with *at most one* formula in the succedent (cf. [61]). The resulting calculus is displayed in Figure 5, where $0 \leq |\delta| \leq 1$. If we apply the same restriction to G1-K, we obtain the calculus

$$\mathsf{G1\text{-}W.K} := \mathsf{G1\text{-}IPL} + \mathsf{K}^m_\Box + \mathsf{K}^m_\Diamond + \frac{\Sigma, A \Rightarrow}{\Box\Sigma, \Diamond A \Rightarrow}$$

$mon_\Box$	$\dfrac{A \supset B}{\Box A \supset \Box B}$	$dual$	$\Box A \supset\subset \neg\Diamond\neg A$	$N_\Box$	$\Box\top$
		$K_\Box$	$\Box(A \supset B) \supset (\Box A \supset \Box B)$	$N_\Diamond$	$\neg\Diamond\bot$
$mon_\Diamond$	$\dfrac{A \supset B}{\Diamond A \supset \Diamond B}$	$K_\Diamond$	$\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$	$T_\Box$	$\Box A \supset A$
		$C_\Box$	$\Box A \wedge \Box B \supset \Box(A \wedge B)$	$T_\Diamond$	$A \supset \Diamond A$
nec	$\dfrac{A}{\Box A}$	$C_\Diamond$	$\Diamond(A \vee B) \supset \Diamond A \vee \Diamond B$	$P_\Box$	$\neg\Box\bot$
		D	$\Box A \supset \Diamond A$	$P_\Diamond$	$\Diamond\top$

Figure 6. Modal axioms and rules.

for W.K defined in [65]. By contrast, in order to obtain a calculus for C.K, we need to extend G1-IPL with the minimal modal rules only:

$$\mathsf{G1\text{-}C.K} := \mathsf{G1\text{-}IPL} + \mathsf{K}^m_\Box + \mathsf{K}^m_\Diamond.$$

In this way, we re-obtain the sequent calculus for C.K defined and proved to be cut-free in [6].

§5. A family of minimal modal logics. We have seen that the two considered methods, respectively based on bimodal companion and sequent calculus restriction, define the same minimal counterpart of K. In this section, we show that this result is not a peculiarity of K only: we apply the same procedure to a family of 14 standard classical modal logics, and show that for each of them the two methods construct the same logic, thus obtaining a minimal counterpart for all these classical systems.

In order to apply our sequent-based approach, we need to restrict to classical modal logics enjoying standard cut-free Gentzen calculi (this restriction excludes well-known modal logics for which such calculi are not available, such as S5[7]). We also require the logics to have a uniform semantic characterisation, we consider to this purpose a neighbourhood semantics that uniformly covers all considered systems, that include both normal and non-normal modal logics.

Specifically, we consider 14 classical modal logics that are axiomatically defined in the language $\mathcal{L}$ extending CPL, formulated in $\mathcal{L}$, with the following modal axioms and rules from Figure 6:

$\mathsf{M} := dual, mon_\Box$	$\mathsf{MD} := \mathsf{M} + D$	$\mathsf{MT} := \mathsf{M} + T_\Box$
$\mathsf{MN} := \mathsf{M} + N_\Box$	$\mathsf{MND} := \mathsf{MN} + D$	$\mathsf{MNT} := \mathsf{MN} + T_\Box$
$\mathsf{MC} := \mathsf{M} + C_\Box$	$\mathsf{MCD} := \mathsf{MC} + D$	$\mathsf{MCT} := \mathsf{MC} + T_\Box$
$\mathsf{K} := \mathsf{M} + N_\Box, C_\Box$	$\mathsf{KD} := \mathsf{K} + D$	$\mathsf{KT} := \mathsf{K} + T_\Box$
$\mathsf{MP} := \mathsf{M} + P_\Box$		
$\mathsf{MNP} := \mathsf{MN} + P_\Box$		

We adopt the standard naming convention of monotonic non-normal modal logics (cf. e.g., [26, 38]): each classical modal logic is denoted $\mathsf{M}\Sigma$, where $\Sigma \subseteq \{\mathsf{C}, \mathsf{N}, \mathsf{P}, \mathsf{D}, \mathsf{T}\}$ corresponds to the list of axioms among $C_\Box$, $N_\Box$, $P_\Box$, D, $T_\Box$ extending M. The only exceptions to this notation are K, KD and KT for which we use the usual names. Note

7 Here we only refer to label-free, two-sided Gentzen-style sequent calculi. Many alternative sequent calculi for S5 and other modal logics have been defined by adding labels [45, 62–64] or enriching the sequent structure [3, 8, 28, 37, 50, 52].

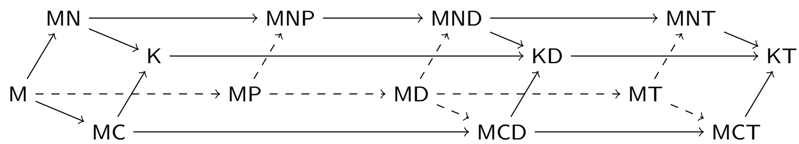

Figure 7. Diagram of classical modal logics.

however that K amounts to MCN, this axiomatisation of K is equivalent to the more standard one with *nec* and $K_\Box$ considered in Section 1.1 (cf. e.g., [9]). As usual, given the duality between $\Box$ and $\Diamond$ in classical logics, the above systems can be equivalently defined by replacing $mon_\Box$, $N_\Box$, $C_\Box$, $P_\Box$, and $T_\Box$, with their $\Diamond$-versions $mon_\Diamond$, $N_\Diamond$, $C_\Diamond$, $P_\Diamond$, and $T_\Diamond$ (Figure 6).[8] The systems MCP and KP are not listed above as they are respectively equivalent to MCD and KD ($P_\Box$ and D are interderivable given $mon_\Box$ and $C_\Box$). The resulting classical modal logics and their inclusion relations are displayed in Figure 7. In the following, we use L or MΣ, without specifying the set Σ, to denote any of the above classical logics.

In the following subsections, we show that the two methods define, for each classical modal logic L, the following minimal counterpart M.L.

Definition 5.1 (Minimal modal logics). Minimal modal logics *are axiomatically defined in the language $\mathcal{L}$ extending* MPL *with the following modal axioms and rules from Figure* 6:

M.M := $mon_\Box$, $mon_\Diamond$	M.MP := M.M + $P_\Diamond$
M.MN := M.M + $N_\Box$	M.MNP := M.MN + $P_\Diamond$
M.MC := M.M + $C_\Box$, $K_\Diamond$	
M.K := M.MC + $N_\Box$	
M.MD := M.M + D, $P_\Diamond$	M.MT := M.M + $T_\Box$, $T_\Diamond$
M.MND := M.MN + D	M.MNT := M.MN + $T_\Box$, $T_\Diamond$
M.MCD := M.MC + D, $P_\Diamond$	M.MCT := M.MC + $T_\Box$, $T_\Diamond$
M.KD := M.K + D	M.KT := M.K + $T_\Box$, $T_\Diamond$.

It is worth observing immediately that not all logics contain both the $\Box$- and the $\Diamond$-versions of the characteristic modal axioms of their classical counterparts. In particular, as we will show later in this section, $P_\Box$, $N_\Diamond$ and $C_\Diamond$ are not valid in the corresponding minimal systems, with the latter axiom that needs to be replaced with $K_\Diamond$ in order to obtain a complete axiomatisation. On the other hand, by means of routine axiomatic derivations one can show that the above definition of M.K is equivalent to the one of Definition 2.5 based on $K_\Box$ and *nec*, in particular $K_\Box$ is derivable from $mon_\Box$, $C_\Box$ and the axioms of MPL.

5.1. Minimal modal logics via bimodal companions. We prove that each minimal logic M.L above is the minimal counterpart of the classical logic L as defined in

8 A $\Box$- and a $\Diamond$-formulation of the axiom D are also possible, namely $\neg(\Box A \wedge \Box\neg A)$ and $\Diamond A \vee \Diamond\neg A$. We prefer to consider the more standard version $\Box A \supset \Diamond A$, which is adequate for both formulations of the logics and is commonly adopted in the definition of intuitionistic modal logics.

Definition 2.4 (that is, $\mathsf{M.L} \vdash A$ if and only if $\mathsf{S4} \oplus \mathsf{L} \vdash A^t$). As before, in order to prove this result, we first provide a semantics for minimal modal logics.

We start recalling the neighbourhood semantics for classical modal logics (cf. [9, 47]). A *classical neighbourhood model* is a tuple $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$, where $\mathcal{W}$ is a non-empty set of worlds, $\mathcal{V} : Atm \longrightarrow \mathcal{P}(\mathcal{W})$ is a valuation function for propositional variables, and $\mathcal{N}$ is a function $\mathcal{W} \longrightarrow \mathcal{P}(\mathcal{P}(\mathcal{W}))$, called neighbourhood function. Modal formulas are interpreted in classical neighbourhood models as $w \Vdash \Box B$ iff there is $\alpha \in \mathcal{N}(w)$ such that for all $v \in \alpha$, $v \Vdash B$; and $w \Vdash \Diamond B$ iff for all $\alpha \in \mathcal{N}(w)$, there is $v \in \alpha$ such that $v \Vdash B$. Each classical modal logic L considered in this work is characterised by the class of all classical neighbourhood models satisfying the following condition (C), (N), (P), (D), or (T), for all $\alpha, \beta \subseteq \mathcal{W}$, if L contains the axiom $C_\Box$, $N_\Box$, $P_\Box$, D, or $T_\Box$, respectively:

(C) If $\alpha, \beta \in \mathcal{N}(w)$, then $\alpha \cap \beta \in \mathcal{N}(w)$.
(N) $\mathcal{N}(w) \neq \emptyset$.
(D) If $\alpha, \beta \in \mathcal{N}(w)$, then $\alpha \cap \beta \neq \emptyset$.
(P) $\emptyset \notin \mathcal{N}(w)$.
(T) If $\alpha \in \mathcal{N}(w)$, then $w \in \alpha$.

We also remark that for each considered classical modal logic L, the fusion $\mathsf{S4} \oplus \mathsf{L}$ is characterised by the class of models $\langle \mathcal{W}, \mathcal{R}, \mathcal{N}, \mathcal{V} \rangle$, where $\mathcal{R}$ is a reflexive and transitive binary relation on $\mathcal{W}$, and $\mathcal{N}$ is a neighbourhood function satisfying the conditions among (C), (N), (D), (P), (T) satisfied by the models for L. This characterisation of fusions $\mathsf{S4} \oplus \mathsf{L}$ can be easily proved by combining the completeness proofs by canonical models for $\mathsf{S4}$ and for L (see e.g., [9]).

By combining relational models for MPL and classical neighbourhood models, we now define minimal neighbourhood models for minimal modal logics as follows.

Definition 5.2 (Minimal neighbourhood semantics). *A* minimal neighbourhood model *is a tuple $\mathcal{M} = \langle \mathcal{W}, \leq, \mathbb{F}, \mathcal{N}, \mathcal{V} \rangle$, where $\langle \mathcal{W}, \leq, \mathbb{F}, \mathcal{V} \rangle$ is a minimal relational model, and $\mathcal{N}$ is a neighbourhood function $\mathcal{W} \longrightarrow \mathcal{P}(\mathcal{P}(\mathcal{W}))$. The forcing relation $\mathcal{M}, w \Vdash A$ is inductively defined extending the clauses for $p, \bot, \wedge, \vee, \supset$ in Section 1.2 with the following clauses for the modalities*:

$$\begin{array}{lll} \mathcal{M}, w \Vdash \Box B & \textit{iff} & \textit{for all } v \geq w, \textit{there is } \alpha \in \mathcal{N}(v) \textit{ such that } \alpha \Vdash^{\forall} B; \\ \mathcal{M}, w \Vdash \Diamond B & \textit{iff} & \textit{for all } v \geq w, \textit{for all } \alpha \in \mathcal{N}(v), \alpha \Vdash^{\exists} B; \end{array}$$

where $\alpha \Vdash^{\forall} B$ and $\alpha \Vdash^{\exists} B$ are abbreviations for, respectively, 'for all $u \in \alpha$, $\mathcal{M}, u \Vdash B$', and 'there is $u \in \alpha$ such that $\mathcal{M}, u \Vdash B$'.

For each minimal modal logic $\mathsf{M.M\Sigma}$, we say that a minimal neighbourhood model $\mathcal{M}$ is a model for $\mathsf{M.M\Sigma}$ (or it is a $\mathsf{M.M\Sigma}$-model) if it satisfies the condition (X) above for all $\mathsf{X} \in \Sigma$. Note that $\mathsf{M.K}$ amounts to $\mathsf{M.MCN}$, hence the corresponding models must satisfy both (C) and (N).

By an easy induction on the construction of formulas one can prove the following.

Proposition 5.1 (Hereditary property). *For every $A \in \mathcal{L}$, every minimal neighbourhood model $\mathcal{M}$, and every world w of $\mathcal{M}$, if $w \Vdash A$ and $w \leq v$, then $v \Vdash A$.*

Now we prove that the logics $\mathsf{M.L}$ are sound and complete with respect to the corresponding classes of models.

THEOREM 5.2. *For all $A \in \mathcal{L}$ and all minimal modal logic* M.L, *if A is derivable in* M.L, *then A is valid in all minimal neighbourhood models for* M.L.

Proof. We show that all modal axioms and rules of M.L are valid, respectively validity preserving, in every minimal neighbourhood model $\mathcal{M}$ for M.L.

($mon_\Box$) Suppose that $\mathcal{M} \models A \supset B$ and $w \Vdash \Box A$. Then for all $v \geq w$, there is $\alpha \in \mathcal{N}(v)$ such that for all $z \in \alpha$, $z \Vdash A$, thus $z \Vdash B$, hence $w \Vdash \Box B$. Therefore $\mathcal{M} \models \Box A \supset \Box B$.

($mon_\Diamond$) Suppose that $\mathcal{M} \models A \supset B$ and $w \Vdash \Diamond A$. Then for all $v \geq w$, for all $\alpha \in \mathcal{N}(v)$, there is $z \in \alpha$ such that $z \Vdash A$, thus $z \Vdash B$, hence $w \Vdash \Diamond B$. Therefore $\mathcal{M} \models \Diamond A \supset \Diamond B$.

($N_\Box$) For all w and all $v \geq w$, by (N), there is $\alpha \in \mathcal{N}(v)$. Since $z \Vdash \top$ for all $z \in \alpha$, we have $w \Vdash \Box\top$. Thus $\mathcal{M} \models \Box\top$.

($C_\Box$) Suppose that $w \Vdash \Box A \wedge \Box B$. Then for all $v \geq w$, there are $\alpha, \beta \in \mathcal{N}(v)$ such that $\alpha \Vdash^\forall A$ and $\beta \Vdash^\forall B$. By (C), $\alpha \cap \beta \in \mathcal{N}(v)$, moreover $\alpha \cap \beta \Vdash^\forall A \wedge B$. Hence $\mathcal{M} \models \Box A \wedge \Box B \supset \Box(A \wedge B)$.

($K_\Diamond$) Suppose that $w \Vdash \Box(A \supset B)$ and $w \Vdash \Diamond A$. Then for all $v \geq w$, there is $\alpha \in \mathcal{N}(v)$ such that $\alpha \Vdash^\forall A \supset B$. Now, suppose that $\beta \in \mathcal{N}(v)$. By (C), $\alpha \cap \beta \in \mathcal{N}(v)$. Since $\alpha \cap \beta \subseteq \alpha$, $\alpha \cap \beta \Vdash^\forall A \supset B$. Moreover, by $w \Vdash \Diamond B$, $\alpha \cap \beta \Vdash^\exists A$. Thus $\alpha \cap \beta \Vdash^\exists B$, which implies $\beta \Vdash^\exists B$. Since this holds for every $\beta \in \mathcal{N}(v)$, $w \Vdash \Diamond B$. Therefore $\mathcal{M} \models \Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$.

($P_\Diamond$) For all w and all $v \geq w$, by (P), $\emptyset \notin \mathcal{N}(v)$. Hence, for all $\alpha \in \mathcal{N}(v)$, $\alpha \neq \emptyset$, thus $\alpha \Vdash^\exists \top$. Then we have $w \Vdash \Diamond\top$. Thus $\mathcal{M} \models \Diamond\top$.

(D) Suppose that $w \Vdash \Box A$. Then for all $v \geq w$, there is $\alpha \in \mathcal{N}(v)$ such that $\alpha \Vdash^\forall A$. Now, suppose that $\beta \in \mathcal{N}(v)$. By (D), there is $z \in \alpha \cap \beta$. Then $z \Vdash A$, hence $\beta \Vdash^\exists A$, therefore $w \Vdash \Diamond A$. Hence $\mathcal{M} \models \Box A \supset \Diamond A$.

($T_\Box$) Suppose that $w \Vdash \Box A$. Then for all $v \geq w$, there is $\alpha \in \mathcal{N}(v)$ such that $\alpha \Vdash^\forall A$. Hence in particular there is $\alpha \in \mathcal{N}(w)$ such that $\alpha \Vdash^\forall A$. By (T), $w \in \alpha$, then $w \Vdash A$. Therefore $\mathcal{M} \models \Box A \supset A$.

($T_\Diamond$) Suppose that $w \Vdash A$. By the hereditary property of minimal neighbourhood models, for all $v \geq w$, $v \Vdash A$. Moreover, by (T), for all $\alpha \in \mathcal{N}(v)$, $v \in \alpha$, hence $\alpha \Vdash^\exists A$. Thus $w \Vdash \Diamond A$, therefore $\mathcal{M} \models \Diamond A$. □

The proof of completeness proceeds essentially as the one in Section 2. First, we observe that Lemma 2.3 also holds for all logics M.L. We consider the following definition of neighbourhood segment.

DEFINITION 5.3. *For every logic* L *in $\mathcal{L}$, an* L-neighbourhood segment, *or just* segment, *is a pair $(\Phi, \mathscr{C})$, where Φ is an* L*-full set, and $\mathscr{C}$ is a class of sets of* L*-full sets such that*:

- *if $\Box A \in \Phi$, then there is $\mathscr{U} \in \mathscr{C}$ such that for all $\Psi \in \mathscr{U}$, $A \in \Psi$; and*
- *if $\Diamond A \in \Phi$, then for all $\mathscr{U} \in \mathscr{C}$, there is $\Psi \in \mathscr{U}$ such that $A \in \Psi$.*

Moreover, if L *contains the axiom $C_\Box$, or the axiom D, or the axiom $T_\Box$, then the* L*-segments must satisfy the following corresponding condition*:

(C-s) *If* $\mathscr{U}, \mathscr{V} \in \mathscr{C}$, *then* $\mathscr{U} \cap \mathscr{V} \in \mathscr{C}$. (T-s) *For all* $\mathscr{U} \in \mathscr{C}$, $\Phi \in \mathscr{U}$.
(D-s) *If* $\mathscr{U}, \mathscr{V} \in \mathscr{C}$, *then* $\mathscr{U} \cap \mathscr{V} \neq \emptyset$.

LEMMA 5.3. *For every minimal modal logic* M.L *and every* M.L*-full set* Φ,

(i) *there exists an* M.L*-neighbourhood segment* $(\Phi, \mathscr{C})$;
(ii) *if* $\Box A \notin \Phi$, *then there exists an* M.L*-neighbourhood segment* $(\Phi, \mathscr{C})$ *such that for all* $\mathscr{U} \in \mathscr{C}$, *there is* $\Psi \in \mathscr{U}$ *such that* $A \notin \Psi$;
(iii) *if* $\Diamond A \notin \Phi$, *then there exists an* M.L*-neighbourhood segment* $(\Phi, \mathscr{C})$ *such that there is* $\mathscr{U} \in \mathscr{C}$ *such that for all* $\Psi \in \mathscr{U}$, $A \notin \Psi$.

Proof.

(i) Given an M.L-full set Φ, we construct an M.L-segment $(\Phi, \mathscr{C})$ as follows. For all $\Box A \in \Phi$, we define $\mathscr{U}_A^- = \{\Psi \text{ M.L-full} \mid A \in \Psi \text{ and there is } \Diamond B \in \Phi \text{ such that } B \in \Psi\}$; and $\mathscr{U}_A = \mathscr{U}_A^-$ if M.L does not contain $T_\Box$, and $\mathscr{U}_A = \mathscr{U}_A^- \cup \{\Phi\}$ if M.L contains $T_\Box$. Moreover, we define $\mathscr{C} = \{\mathscr{U}_A \mid \Box A \in \Phi\}$. We show that $(\Phi, \mathscr{C})$ is an M.L-segment.

- If $\Box A \in \Phi$, then by definition $\mathscr{U}_A \in \mathscr{C}$. Moreover, if M.L does not contain $T_\Box$, then $A \in \Psi$ for all $\Psi \in \mathscr{U}_A$. If instead M.L contains $T_\Box$, then for all $\Psi \in \mathscr{U}_A$ we have $A \in \Psi$ or $\Psi = \Phi$, where, by $T_\Box$ and closure under derivation of M.L-full sets, $A \in \Phi$.
- If $\Diamond A \in \Phi$, then assume $\mathscr{U} \in \mathscr{C}$. Then, by definition, $\mathscr{U} = \mathscr{U}_B$ for some $\Box B \in \Phi$. By Lemma 2.3, there is an M.L-full set Ψ such that $A, B \in \Psi$, hence $\Psi \in \mathscr{U}_B = \mathscr{U}$ and $A \in \Psi$.

Moreover, the conditions (C-s), (D-s) and (T-s) are satisfied if M.L contains the axioms $C_\Box$, D, or $T_\Box$, respectively:

(C-s) Suppose $\mathscr{U}, \mathscr{V} \in \mathscr{C}$. Then $\mathscr{U} = \mathscr{U}_A$ and $\mathscr{V} = \mathscr{U}_B$ for some $\Box A, \Box B \in \Phi$. Hence, given that M.L contains $C_\Box$, by closure under derivation of M.L-full sets, we have $\Box(A \wedge B) \in \Phi$, thus $\mathscr{U}_{A \wedge B} \in \mathscr{C}$. Note also that for all M.L-full sets Ψ it holds $A, B \in \Psi$ if and only if $A \wedge B \in \Psi$. One can easily verify that this implies $\mathscr{U}_{A \wedge B} = \mathscr{U}_A \cap \mathscr{U}_B$, therefore $\mathscr{U} \cap \mathscr{V} = \mathscr{U}_A \cap \mathscr{U}_B \in \mathscr{C}$.

(D-s) Suppose $\mathscr{U}, \mathscr{V} \in \mathscr{C}$. Then $\mathscr{U} = \mathscr{U}_A$ and $\mathscr{V} = \mathscr{U}_B$ for some $\Box A, \Box B \in \Phi$. Given that M.L contains D, by closure under derivation of M.L-full sets, we have $\Diamond A, \Diamond B \in \Phi$. By Lemma 2.3, there is an M.L-full set Ψ such that $A, B \in \Psi$. Then by definition, $\Psi \in \mathscr{U}_A$ and $\Psi \in \mathscr{U}_B$, hence $\Psi \in \mathscr{U}_A \cap \mathscr{U}_B$, therefore $\mathscr{U} \cap \mathscr{V} = \mathscr{U}_A \cap \mathscr{U}_B \neq \emptyset$.

(T-s) By definition, for all $\mathscr{U} \in \mathscr{C}$, $\Phi \in \mathscr{U}$.

(ii) For all $\Box B \in \Phi$, we define $\mathscr{U}_B^- = \{\Psi \text{ M.L-full} \mid B \in \Psi \text{ and there is } \Diamond C \in \Phi \text{ such that } C \in \Psi\} \cup \{\Psi \text{ M.L-full} \mid B \in \Psi \text{ and } A \notin \Psi\}$; and $\mathscr{U}_B = \mathscr{U}_B^-$ if M.L does not contain $T_\Box$, and $\mathscr{U}_B = \mathscr{U}_B^- \cup \{\Phi\}$ if M.L contains $T_\Box$. Moreover, we define $\mathscr{C} = \{\mathscr{U}_B \mid \Box B \in \Phi\}$. We can show that $(\Phi, \mathscr{C})$ is an M.L-segment as in item (i). Now, suppose that $\mathscr{U} \in \mathscr{C}$. Then $\mathscr{U} = \mathscr{U}_B$ for some $\Box B \in \Phi$. Thus, since $\Box A \notin \Phi$, $\{B\} \nvdash A$ (otherwise $\vdash B \supset A$, and by $mon_\Box$, $\vdash \Box B \supset \Box A$, hence by closure under derivation, $\Box A \in \Phi$). Then by Lemma 2.3, there is an M.L-full set Ψ such that $B \in \Psi$ and $A \notin \Psi$, and by definition, $\Psi \in \mathscr{U}_B = \mathscr{U}$.

(iii) (iii.i) M.L does not contain $C_\Box$, $K_\Diamond$. We define $\mathscr{U}^- = \{\Psi \text{ M.L-full} \mid A \notin \Psi$ and there is $\Diamond B \in \Phi$ such that $B \in \Psi\}$, and for all $\Box C \in \Phi$, we define $\mathscr{U}_C^- = \{\Psi \text{ M.L-full} \mid C \in \Psi$ and there is $\Diamond B \in \Phi$ such that $B \in \Psi\}$. Moreover, we define $\mathscr{U} = \mathscr{U}^-$, $\mathscr{U}_C = \mathscr{U}_C^-$ if M.L does not contain $T_\Box$, and $\mathscr{U} = \mathscr{U}^- \cup \{\Phi\}$, $\mathscr{U}_C = \mathscr{U}_C^- \cup \{\Phi\}$ if M.L contains $T_\Box$. Finally, we define $\mathscr{C} = \{\mathscr{U}\} \cup \{\mathscr{U}_C \mid \Box C \in \Phi\}$. Note that $\mathscr{U}$ satisfies the condition of the lemma, in particular if $T_\Box$ belongs to M.L, then $A \notin \Phi$, since if $A \in \Phi$, then by $T_\Diamond$, $\Diamond A \in \Phi$, against the assumption. We can show that $(\Phi, \mathscr{C})$ is an M.L-segment. First, the conditions of M.L-segments for any $\Box B \in \Phi$ and for any $\Diamond B \in \Phi$ can be shown to be satisfied similarly to item (i). Moreover, the property (T-s) of M.L-segments for M.L containing $T_\Box$ follows immediately from the definition. We show that (D-s) is satisfied if M.L contains the axiom D: Suppose that $\mathscr{V}, \mathscr{Z} \in \mathscr{C}$. If $\mathscr{V} = \mathscr{U}_C, \mathscr{Z} = \mathscr{U}_D$ for some $\Box C, \Box D \in \Phi$, the proof is analogous to the one of (D-s) in item (i). Now suppose $\mathscr{V} = \mathscr{U}_C$ for some some $\Box C \in \Phi$ and $\mathscr{Z} = \mathscr{U}$. Then by axiom D, $\Diamond C \in \Phi$. Thus we have $\{C\} \nvdash A$, otherwise we would have $\vdash C \supset A$, and by $mon_\Diamond$, $\vdash \Diamond C \supset \Diamond A$, hence $\Diamond A \in \Phi$, against the assumption. By Lemma 2.3, there is an M.L-full set Ψ such that $C \in \Psi$ and $A \notin \Psi$. By definition, $\Psi \in \mathscr{U}$, moreover $\Psi \in \mathscr{U}_C$ (since $\Diamond C \in \Phi$), hence $\Psi \in \mathscr{U} \cap \mathscr{U}_C = \mathscr{V} \cap \mathscr{Z}$, therefore $\mathscr{V} \cap \mathscr{Z} \neq \emptyset$.(iii.ii) M.L contains $C_\Box$, $K_\Diamond$. We define $\mathscr{U}^- = \{\Psi \text{ M.L-full} \mid A \notin \Psi$, and $\Box^-\Phi \subseteq \Psi$, and $B \in \Psi$ for some $\Diamond B \in \Phi\}$, and $\mathscr{U} = \mathscr{U}^-$ if M.L does not contain $T_\Box$, and $\mathscr{U} = \mathscr{U}^- \cup \{\Phi\}$ M.L it contains $T_\Box$. Moreover, we define $\mathscr{C} = \{\mathscr{U}\}$. Clearly, $\mathscr{U}$ satisfies the claim of the lemma (in particular, if M.L contains $T_\Box$, then $A \notin \Phi$). We show that $(\Phi, \mathscr{C})$ is an M.L-segment. First, observe that for any $\Diamond B \in \Phi$, $\Box^-\Phi \cup \{B\} \nvdash A$. Indeed, if $\Box^-\Phi \cup \{B\} \vdash A$, then there are $C_1, \dots, C_n \in \Box^-\Phi$ such that $\vdash C_1 \wedge \dots \wedge C_n \wedge B \supset A$, hence $\vdash C_1 \wedge \dots \wedge C_n \supset (B \supset A)$, then by $mon_\Box$, $\vdash \Box(C_1 \wedge \dots \wedge C_n) \supset \Box(B \supset A)$, thus by $C_\Box$ (n times) and $K_\Diamond$, $\vdash \Box C_1 \wedge \dots \wedge \Box C_n \supset (\Diamond B \supset \Diamond A)$, which gives $\vdash \Box C_1 \wedge \dots \wedge \Box C_n \wedge \Diamond B \supset \Diamond A$, therefore $\Box C_1, \dots, \Box C_n, \Diamond B \vdash \Diamond A$; since $\Box C_1, \dots, \Box C_n, \Diamond B \in \Phi$, this entails $\Diamond A \in \Phi$, against the assumption. We then have: the condition for any $\Box B \in \Phi$ follows immediately from the definition. If $\Diamond B \in \Phi$, then $\Box^-\Phi \cup \{B\} \nvdash A$, thus by Lemma 2.3, there is an M.L-full set Ψ such that $\Box^-\Phi \subseteq \Psi$, $B \in \Psi$ and $A \notin \Psi$, hence $\Psi \in \mathscr{U}$. By the same argument, the property (D-s) is satisfied for M.L containing the axiom D given that $\Diamond\top \in \Phi$ entails the existence of such an M.L-full set Ψ, hence $\mathscr{U} \neq \emptyset$. Moreover, (C-s) is trivial, and (T-s) for M.L containing $T_\Box$ follows immediately from the definition.

□

DEFINITION 5.4. *For every logic* L *in* $\mathcal{L}$, *the* canonical neighbourhood model *for* L *is the tuple* $\mathcal{M} = \langle \mathcal{W}, \leq, \mathbb{F}, \mathcal{N}, \mathcal{V} \rangle$, *where*:

- $\mathcal{W}$ *is the set of all* L*-neighbourhood segments*;
- *for all* $(\Phi, \mathscr{C}), (\Psi, \mathscr{D}) \in \mathcal{W}$, $(\Phi, \mathscr{C}) \leq (\Psi, \mathscr{D})$ *if and only if* $\Phi \subseteq \Psi$;
- *for all* $(\Phi, \mathscr{C}) \in \mathcal{W}$, $(\Phi, \mathscr{C}) \in \mathbb{F}$ *if and only if* $\bot \in \Phi$;
- *for all sets* $\mathscr{U}$ *of* M.L*-full sets*, $\alpha_{\mathscr{U}} = \{(\Phi, \mathscr{C}) \mid \Phi \in \mathscr{U}\}$;
- *for all* $(\Phi, \mathscr{C}) \in \mathcal{W}$, $\alpha_{\mathscr{U}} \in \mathcal{N}((\Phi, \mathscr{C}))$ *if and only if* $\mathscr{U} \in \mathscr{C}$;
- *for all* $(\Phi, \mathscr{C}) \in \mathcal{W}$, $(\Phi, \mathscr{C}) \in \mathcal{V}(p)$ *if and only if* $p \in \Phi$.

LEMMA 5.4. *For every minimal modal logic* M.L, *the canonical neighbourhood model* $\mathcal{M}$ *for* M.L *is a minimal neighbourhood model for* M.L.

Proof. It is easy to very that $\mathcal{M}$ is a minimal neighbourhood model. We show that $\mathcal{M}$ satisfies the conditions among (C), (N), (P), (D), (T) associated with the axioms of M.L.

- (C) Suppose that $\alpha, \beta \in \mathcal{N}((\Phi, \mathscr{C}))$. Then, by definition, $\alpha = \alpha_{\mathscr{U}}$ and $\beta = \alpha_{\mathscr{V}}$ for some $\mathscr{U}, \mathscr{V} \in \mathscr{C}$. By the property (C-s) of M.L-segments, $\mathscr{U} \cap \mathscr{V} \in \mathscr{C}$, thus $\alpha_{\mathscr{U} \cap \mathscr{V}} \in \mathcal{N}((\Phi, \mathscr{C}))$, where $\alpha_{\mathscr{U} \cap \mathscr{V}} = \{(\Phi, \mathscr{C}) \mid \Phi \in \mathscr{U} \cap \mathscr{V}\} = \{(\Phi, \mathscr{C}) \mid \Phi \in \mathscr{U}\} \cap \{(\Phi, \mathscr{C}) \mid \Phi \in \mathscr{V}\} = \alpha_{\mathscr{U}} \cap \alpha_{\mathscr{V}} = \alpha \cap \beta$.
- (N) For all M.L-full sets Φ, $\Box\top \in \Phi$, then for all M.L-segments $(\Phi, \mathscr{C})$, $\mathscr{C} \neq \emptyset$, thus $\mathcal{N}((\Phi, \mathscr{C})) \neq \emptyset$.
- (P) For all M.L-full sets Φ, $\Diamond\top \in \Phi$, then for all M.L-segments $(\Phi, \mathscr{C})$ and all $\mathscr{U} \in \mathscr{C}$, $\mathscr{U} \neq \emptyset$, thus for all $\alpha_{\mathscr{U}} \in \mathcal{N}((\Phi, \mathscr{C}))$, $\alpha_{\mathscr{U}} \neq \emptyset$, that is, $\emptyset \notin \mathcal{N}((\Phi, \mathscr{C}))$.
- (D) Suppose that $\alpha, \beta \in \mathcal{N}((\Phi, \mathscr{C}))$. Then $\alpha = \alpha_{\mathscr{U}}$ and $\beta = \alpha_{\mathscr{V}}$ for some $\mathscr{U}, \mathscr{V} \in \mathscr{C}$. By (D-s), $\mathscr{U} \cap \mathscr{V} \neq \emptyset$, which implies $\alpha_{\mathscr{U}} \cap \alpha_{\mathscr{V}} = \alpha \cap \beta \neq \emptyset$.
- (T) Suppose that $\alpha \in \mathcal{N}((\Phi, \mathscr{C}))$. Then $\alpha = \alpha_{\mathscr{U}}$ for an $\mathscr{U} \in \mathscr{C}$. By (T-s), $\Phi \in \mathscr{U}$, thus $(\Phi, \mathscr{C}) \in \alpha_{\mathscr{U}} = \alpha$. □

LEMMA 5.5. *Let* M.L *be a minimal modal logic and* $\mathcal{M} = \langle \mathcal{W}, \leq, \mathbb{F}, \mathcal{N}, \mathcal{V} \rangle$ *be the canonical neighbourhood model for* M.L. *Then for all* $(\Phi, \mathscr{C}) \in \mathcal{W}$ *and all* $A \in \mathcal{L}$, $(\Phi, \mathscr{C}) \Vdash A$ *if and only if* $A \in \Phi$.

Proof. By induction on the construction of A. For the cases $A = p, \bot, B \wedge C, B \vee C, B \supset C$ the proof is exactly as the proof of Lemma 2.5. We consider the inductive cases $A = \Box B, \Diamond B$.

($A = \Box B$) Suppose that $\Box B \in \Phi$. Then for all $(\Psi, \mathscr{D}) \geq (\Phi, \mathscr{C})$, $\Box B \in \Psi$. By definition of segment, there is $\mathscr{U} \in \mathscr{D}$ such that for all $\Theta \in \mathscr{U}$, $B \in \Theta$. Then, by definition of canonical model, $\alpha_{\mathscr{U}} \in \mathcal{N}((\Psi, \mathscr{D}))$, and by i.h., $(\Theta, \mathscr{E}) \Vdash B$ for all $(\Theta, \mathscr{E}) \in \alpha_{\mathscr{U}}$. Therefore $(\Phi, \mathscr{C}) \Vdash \Box B$. Now suppose that $\Box B \notin \Phi$. By Lemma 5.3 (ii), there is an M.L-segment $(\Phi, \mathscr{D})$ such that for all $\mathscr{U} \in \mathscr{D}$, there is $\Psi \in \mathscr{U}$ such that $B \notin \Psi$. By definition, $(\Phi, \mathscr{D}) \in \mathcal{W}$ and $(\Phi, \mathscr{C}) \leq (\Phi, \mathscr{D})$. Moreover, assume $\alpha \in \mathcal{N}((\Phi, \mathscr{D}))$. Then $\alpha = \alpha_{\mathscr{U}}$ for some $\mathscr{U} \in \mathscr{D}$. Thus, there is $\Psi \in \mathscr{U}$ such that $B \notin \Psi$. By Lemma 5.3 (i), there is an M.L-segment $(\Psi, \mathscr{E})$, thus by definition, $(\Psi, \mathscr{E}) \in \alpha_{\mathscr{U}}$, and by i.h., $(\Psi, \mathscr{E}) \nVdash B$. Hence $\alpha = \alpha_{\mathscr{U}} \nVdash^{\forall} B$, therefore $(\Phi, \mathscr{C}) \nVdash \Box B$.

($A = \Diamond B$) Suppose that $\Diamond B \in \Phi$. Then for all $(\Psi, \mathscr{D}) \geq (\Phi, \mathscr{C})$, $\Diamond B \in \Psi$. By definition of segment, for all $\mathscr{U} \in \mathscr{D}$, there is $\Psi \in \mathscr{U}$ such that $B \in \Psi$. Now, assume $\alpha \in \mathcal{N}((\Psi, \mathscr{D}))$. By definition, $\alpha = \alpha_{\mathscr{U}}$ for some $\mathscr{U} \in \mathscr{D}$. Then there is $\Psi \in \mathscr{U}$ such that $B \in \Psi$. By Lemma 5.3 (i), there is an M.L-segment $(\Psi, \mathscr{E})$, thus by definition, $(\Psi, \mathscr{E}) \in \alpha_{\mathscr{U}}$, and by i.h., $(\Psi, \mathscr{E}) \Vdash B$, which implies $\alpha = \alpha_{\mathscr{U}} \Vdash^{\exists} B$. Since this holds for every $\alpha \in \mathcal{N}((\Psi, \mathscr{D}))$, we have $(\Phi, \mathscr{C}) \Vdash \Diamond B$. Now suppose that $\Diamond B \notin \Phi$. By Lemma 5.3 (iii), there is an M.L-segment $(\Phi, \mathscr{D})$ and a $\mathscr{U} \in \mathscr{D}$ such that for all $\Psi \in \mathscr{U}$, $B \notin \Psi$. By definition, $(\Phi, \mathscr{D}) \in \mathcal{W}$, $(\Phi, \mathscr{C}) \leq (\Phi, \mathscr{D})$, and $\alpha_{\mathscr{U}} \in \mathcal{N}((\Phi, \mathscr{D}))$. Moreover, for all $(\Psi, \mathscr{E}) \in \alpha_{\mathscr{U}}$, $B \notin \Psi$, then by i.h., $(\Psi, \mathscr{E}) \nVdash B$. Hence, $\alpha_{\mathscr{U}} \nVdash^{\exists} B$, therefore $(\Phi, \mathscr{C}) \nVdash \Diamond B$.

□

As a consequence of these lemmas, we obtain the following completeness result (cf. proof of Theorem 2.6).

THEOREM 5.6 (Completeness). *For all $A \in \mathcal{L}$ and all minimal modal logics* M.L, *if A is valid in every minimal neighbourhood model for* M.L, *then A is derivable in* M.L.

Finally, on the basis of this semantic characterisation of logics M.L, we can show that, for each classical logic L, the fusion $\mathsf{S4} \oplus \mathsf{L}$ is the bimodal companion of the corresponding minimal logic M.L.

THEOREM 5.7. *For all $A \in \mathcal{L}$ and all classical modal logics* L, *A is derivable in* M.L *if and only if A^t is derivable in* $\mathsf{S4} \oplus \mathsf{L}$.

Proof. ($\Rightarrow$) Suppose that $\mathsf{S4} \oplus \mathsf{L} \nvdash A^t$. Then there are a model $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{N}, \mathcal{V} \rangle$ for $\mathsf{S4} \oplus \mathsf{L}$ and a world w such that $\mathcal{M}, w \nVdash A^t$. We define $\mathcal{M}' = \langle \mathcal{W}, \leq, \mathbb{F}, \mathcal{N}, \mathcal{V}' \rangle$ over the same $\mathcal{W}$ and $\mathcal{N}$, where $\leq\, = \mathcal{R}$, for all $p \in Atm$, $\mathcal{V}'(p) = \{v \mid \text{for all } u, v\mathcal{R}u \text{ implies } u \in \mathcal{V}(p)\}$, and $\mathbb{F} = \{v \mid \text{for all } u, v\mathcal{R}u \text{ implies } u \in \mathcal{V}(f)\}$. By the properties of $\mathcal{N}$ in $\mathcal{M}$, it immediately follows that $\mathcal{M}'$ is a minimal neighbourhood model for M.L. We show that for all $v \in \mathcal{W}$ and all $B \in \mathcal{L}$, $\mathcal{M}', v \Vdash B$ if and only if $\mathcal{M}, v \Vdash B^t$, which implies that $\mathcal{M}', w \nVdash A$, therefore $\mathsf{M.L} \nvdash A$. The proof is by induction on the construction of B. The cases $B = p, \bot, C \wedge D, C \vee D, C \supset D$ are as in the proof of Theorem 2.7, case ($\Rightarrow$). We show the cases $B = \Box C, \Diamond C$.

($B = \Box C$) $\mathcal{M}', v \Vdash \Box C$ iff for all $u \geq v$, there is $\alpha \in \mathcal{N}(u)$ such that for all $z \in \alpha$, $\mathcal{M}', z \Vdash C$; iff (by definition of $\leq$ and i.h.) for all u, if $v\mathcal{R}u$, then there is $\alpha \in \mathcal{N}(u)$ such that for all $z \in \alpha$, $\mathcal{M}, z \Vdash C^t$; iff for all u, if $v\mathcal{R}u$, then $\mathcal{M}, u \Vdash \Box_2 C^t$; iff $\mathcal{M}, v \Vdash \Box_1 \Box_2 C^t$.

($B = \Diamond C$) $\mathcal{M}', v \Vdash \Diamond C$ iff for all $u \geq v$, for all $\alpha \in \mathcal{N}(u)$, there is $z \in \alpha$ such that $\mathcal{M}', z \Vdash C$; iff (by definition of $\leq$ and i.h.) for all u, if $v\mathcal{R}u$, then for all $\alpha \in \mathcal{N}(u)$, there is $z \in \alpha$ such that $\mathcal{M}, z \Vdash C^t$; iff for all u, if $v\mathcal{R}u$, then $\mathcal{M}, u \Vdash \Diamond_2 C^t$; iff $\mathcal{M}, v \Vdash \Box_1 \Diamond_2 C^t$.

($\Leftarrow$) Suppose that $\mathsf{M.L} \nvdash A$. Then there are a minimal neighbourhood model $\mathcal{M} = \langle \mathcal{W}, \leq, \mathbb{F}, \mathcal{N}, \mathcal{V} \rangle$ for M.L and a world w such that $\mathcal{M}, w \nVdash A$. We define $\mathcal{M}'' = \langle \mathcal{W}, \mathcal{R}, \mathcal{N}, \mathcal{V}'' \rangle$ over the same $\mathcal{W}$ and $\mathcal{N}$, where $\mathcal{R} = \leq$, for all $p \in Atm$, $\mathcal{V}''(p) = \mathcal{V}(p)$, and $\mathcal{V}''(f) = \mathbb{F}$. Then $\mathcal{M}''$ is a model for $\mathsf{S4} \oplus \mathsf{L}$. We show that for all $v \in \mathcal{W}$ and all $B \in \mathcal{L}$, $\mathcal{M}, v \Vdash B$ if and only if $\mathcal{M}'', v \Vdash B^t$, which implies that $\mathcal{M}'', w \nVdash A$, therefore $\mathsf{S4} \oplus \mathsf{L} \nvdash A$. The proof is by induction on the construction of B. The cases $B = p, \bot, C \wedge D, C \vee D, C \supset D$ are as in the proof of Theorem 2.7, case ($\Leftarrow$). We show the cases $B = \Box C, \Diamond C$.

($B = \Box C$) $\mathcal{M}, v \Vdash \Box C$ iff for all $u \geq v$, there is $\alpha \in \mathcal{N}(u)$ such that for all $z \in \alpha$, $\mathcal{M}, z \Vdash C$; iff (by definition of $\mathcal{R}$ and i.h.) for all u, if $v\mathcal{R}u$, then there is $\alpha \in \mathcal{N}(u)$ such that for all $z \in \alpha$, $\mathcal{M}'', z \Vdash C^t$; iff for all u, if $v\mathcal{R}u$, then $\mathcal{M}'', u \Vdash \Box_2 C^t$; iff $\mathcal{M}'', v \Vdash \Box_1 \Box_2 C^t$.

($B = \Diamond C$) $\mathcal{M}, v \Vdash \Diamond C$ iff for all $u \geq v$, for all $\alpha \in \mathcal{N}(u)$, there is $z \in \alpha$ such that $\mathcal{M}, z \Vdash C$; iff (by definition of $\mathcal{R}$ and i.h.) for all u, if $v\mathcal{R}u$, then for all $\alpha \in \mathcal{N}(u)$, there is $z \in \alpha$ such that $\mathcal{M}'', z \Vdash C^t$; iff for all u, if $v\mathcal{R}u$, then $\mathcal{M}'', u \Vdash \Diamond_2 C^t$; iff $\mathcal{M}'', v \Vdash \Box_1 \Diamond_2 C^t$. □

As an additional remark, based on this semantics we can also show that the axioms $P_\Box$, $N_\Diamond$ and $C_\Diamond$ are not valid in M.MP, M.MN and M.MC, respectively. For the first two axioms, consider a model $\mathcal{M} = \langle \mathcal{W}, \leq, \mathbb{F}, \mathcal{N}, \mathcal{V} \rangle$ where $\mathcal{W} = \{w, v\}$, $\leq\, = \{(w, w), (v, v)\}$, $\mathbb{F} = \{v\}$ and $\mathcal{N}(w) = \mathcal{N}(v) = \{\mathbb{F}\}$. $\mathcal{M}$ satisfies both conditions

$$\mathsf{M}^{c\ell}_{\Box}\ \frac{A \Rightarrow B}{\Box A \Rightarrow \Box B} \qquad \mathsf{M}^{c\ell}_{\Diamond}\ \frac{A \Rightarrow B}{\Diamond A \Rightarrow \Diamond B} \qquad \mathsf{mnc}^{c\ell}_{\mathsf{M}}\ \frac{A, B \Rightarrow}{\Box A, \Diamond B \Rightarrow} \qquad \mathsf{mem}^{c\ell}_{\mathsf{M}}\ \frac{\Rightarrow A, B}{\Rightarrow \Box A, \Diamond B}$$

$$\mathsf{C}^{c\ell}_{\Box}\ \frac{\Sigma, A \Rightarrow B, \Pi}{\Box\Sigma, \Box A \Rightarrow \Box B, \Diamond\Pi} \qquad \mathsf{C}^{c\ell}_{\Diamond}\ \frac{\Sigma, A \Rightarrow B, \Pi}{\Box\Sigma, \Diamond A \Rightarrow \Diamond B, \Diamond\Pi} \qquad \mathsf{mnc}^{c\ell}_{\mathsf{C}}\ \frac{\Sigma, A, B \Rightarrow}{\Box\Sigma, \Box A, \Diamond B \Rightarrow}$$

$$\mathsf{mem}^{c\ell}_{\mathsf{C}}\ \frac{\Rightarrow A, B, \Pi}{\Rightarrow \Box A, \Diamond B, \Diamond\Pi} \qquad \mathsf{N}^{c\ell}_{\Box}\ \frac{\Rightarrow A}{\Rightarrow \Box A} \qquad \mathsf{N}^{c\ell}_{\Diamond}\ \frac{A \Rightarrow}{\Diamond A \Rightarrow} \qquad \mathsf{K}^{c\ell}_{\Box}\ \frac{\Sigma \Rightarrow A, \Pi}{\Box\Sigma \Rightarrow \Box A, \Diamond\Pi}$$

$$\mathsf{K}^{c\ell}_{\Diamond}\ \frac{\Sigma, A \Rightarrow \Pi}{\Box\Sigma, \Diamond A \Rightarrow \Diamond\Pi} \qquad \mathsf{P}^{c\ell}_{\Box}\ \frac{A \Rightarrow}{\Box A \Rightarrow} \qquad \mathsf{P}^{c\ell}_{\Diamond}\ \frac{\Rightarrow A}{\Rightarrow \Diamond A} \qquad \mathsf{D}^{c\ell}\ \frac{A \Rightarrow B}{\Box A \Rightarrow \Diamond B} \qquad \mathsf{D}^{c\ell}_{\Box}\ \frac{A, B \Rightarrow}{\Box A, \Box B \Rightarrow}$$

$$\mathsf{D}^{c\ell}_{\Diamond}\ \frac{\Rightarrow A, B}{\Rightarrow \Diamond A, \Diamond B} \qquad \mathsf{CD}^{c\ell}\ \frac{\Sigma \Rightarrow \Pi}{\Box\Sigma \Rightarrow \Diamond\Pi} \qquad \mathsf{T}^{c\ell}_{\Box}\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta} \qquad \mathsf{T}^{c\ell}_{\Diamond}\ \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \Diamond A, \Delta}$$

Figure 8. Modal rules for classical sequent calculi G1-L.

(P) and (N). Moreover, since $\mathbb{F} \Vdash^{\forall} \bot$ and $\mathbb{F} \Vdash^{\exists} \bot$, we have $w \Vdash \Box\bot$ and $w \Vdash \Diamond\bot$. However, $w \not\Vdash \bot$, therefore $\mathcal{M} \not\models \neg\Box\bot$ and $\mathcal{M} \not\models \neg\Diamond\bot$. For $C_{\Diamond}$, consider a model $\mathcal{M}$ where $\mathcal{W} = \{w, v, u\}$, $\leq = \{(w, v), (w, u), (w, w), (v, v), (u, u)\}$, $\mathcal{V}(p) = \{v\}$, $\mathcal{V}(q) = \{u\}$, $\mathcal{N}(w) = \mathcal{N}(v) = \{\{v\}\}$ and $\mathcal{N}(u) = \{\{u\}\}$, that trivially satisfies (C). One can easily verify that $w \Vdash \Diamond(p \vee q)$ but $w \not\Vdash \Diamond p$ and $w \not\Vdash \Diamond q$, therefore $\mathcal{M} \not\models \Diamond(p \vee q) \supset \Diamond p \vee \Diamond q$.

5.2. Minimal modal logics via sequent calculi. G1-style sequent calculi for the considered classical modal logics are defined extending G1-CPL (Figure 1) with the following modal rules from Figure 8:

G1-M := $\mathsf{M}^{c\ell}_{\Box}$, $\mathsf{M}^{c\ell}_{\Diamond}$, $\mathsf{mnc}^{c\ell}_{\mathsf{M}}$, $\mathsf{mem}^{c\ell}_{\mathsf{M}}$
G1-MN := G1-M + $\mathsf{N}^{c\ell}_{\Box}$, $\mathsf{N}^{c\ell}_{\Diamond}$
G1-MC := $\mathsf{C}^{c\ell}_{\Box}$, $\mathsf{C}^{c\ell}_{\Diamond}$, $\mathsf{mnc}^{c\ell}_{\mathsf{C}}$, $\mathsf{mem}^{c\ell}_{\mathsf{C}}$
G1-K := $\mathsf{K}^{c\ell}_{\Box}$, $\mathsf{K}^{c\ell}_{\Diamond}$

G1-MP := G1-M + $\mathsf{P}^{c\ell}_{\Box}$, $\mathsf{P}^{c\ell}_{\Diamond}$
G1-MNP := G1-MN + $\mathsf{P}^{c\ell}_{\Box}$, $\mathsf{P}^{c\ell}_{\Diamond}$

G1-MD := G1-M + $\mathsf{D}^{c\ell}$, $\mathsf{D}^{c\ell}_{\Box}$, $\mathsf{D}^{c\ell}_{\Diamond}$
G1-MND := G1-MN + $\mathsf{D}^{c\ell}$, $\mathsf{D}^{c\ell}_{\Box}$, $\mathsf{D}^{c\ell}_{\Diamond}$
G1-MCD := G1-MC + $\mathsf{CD}^{c\ell}$
G1-KD := G1-K + $\mathsf{CD}^{c\ell}$

G1-MT := G1-M + $\mathsf{T}^{c\ell}_{\Box}$, $\mathsf{T}^{c\ell}_{\Diamond}$
G1-MNT := G1-MN + $\mathsf{T}^{c\ell}_{\Box}$, $\mathsf{T}^{c\ell}_{\Diamond}$
G1-MCT := G1-MC + $\mathsf{T}^{c\ell}_{\Box}$, $\mathsf{T}^{c\ell}_{\Diamond}$
G1-KT := G1-K + $\mathsf{T}^{c\ell}_{\Box}$, $\mathsf{T}^{c\ell}_{\Diamond}$.

These calculi are studied and shown to be cut-free complete in [27, 35, 38, 46]. By applying the single-succedent restriction to the classical calculi G1-L, we obtain the corresponding calculi G1-M.L which extend G1-MPL (Figure 1) with the following modal rules from Figure 9:

G1-M.M := $\mathsf{M}^{m}_{\Box}$, $\mathsf{M}^{m}_{\Diamond}$
G1-M.MN := G1-M.M + $\mathsf{N}^{m}_{\Box}$
G1-M.MC := $\mathsf{C}^{m}_{\Box}$, $\mathsf{K}^{m}_{\Diamond}$
G1-M.K := $\mathsf{K}^{m}_{\Box}$, $\mathsf{K}^{m}_{\Diamond}$

G1-M.MP := G1-M.M + $\mathsf{P}^{m}_{\Diamond}$
G1-M.MNP := G1-M.MN + $\mathsf{P}^{m}_{\Diamond}$

G1-M.MD := G1-M.M + D^{m}, $\mathsf{P}^{m}_{\Diamond}$
G1-M.MND := G1-M.MN + D^{m}, $\mathsf{P}^{m}_{\Diamond}$
G1-M.MCD := G1-M.MC + CD^{m}
G1-M.KD := G1-M.K + CD^{m}

G1-M.MT := G1-M.M + $\mathsf{T}^{m}_{\Box}$, $\mathsf{T}^{m}_{\Diamond}$
G1-M.MNT := G1-M.MN + $\mathsf{T}^{m}_{\Box}$, $\mathsf{T}^{m}_{\Diamond}$
G1-M.MCT := G1-M.MC + $\mathsf{T}^{m}_{\Box}$, $\mathsf{T}^{m}_{\Diamond}$
G1-M.KT := G1-M.K + $\mathsf{T}^{m}_{\Box}$, $\mathsf{T}^{m}_{\Diamond}$.

As before, the rules containing sequents with an empty succedent or with two active/principal formulas in the succedent are dropped (namely, $\mathsf{mnc}^{c\ell}_{\mathsf{M}}$, $\mathsf{mem}^{c\ell}_{\mathsf{M}}$, $\mathsf{mnc}^{c\ell}_{\mathsf{C}}$,

$$\mathsf{M}^m_\Box\ \frac{A \Rightarrow B}{\Box A \Rightarrow \Box B} \qquad \mathsf{M}^m_\Diamond\ \frac{A \Rightarrow B}{\Diamond A \Rightarrow \Diamond B} \qquad \mathsf{N}^m_\Box\ \frac{\Rightarrow A}{\Rightarrow \Box A} \qquad \mathsf{C}^m_\Box\ \frac{\Sigma, A \Rightarrow B}{\Box\Sigma, \Box A \Rightarrow \Box B}$$

$$\mathsf{K}^m_\Box\ \frac{\Sigma \Rightarrow A}{\Box\Sigma \Rightarrow \Box A} \qquad \mathsf{K}^m_\Diamond\ \frac{\Sigma, A \Rightarrow B}{\Box\Sigma, \Diamond A \Rightarrow \Diamond B} \qquad \mathsf{P}^m_\Diamond\ \frac{\Rightarrow A}{\Rightarrow \Diamond A} \qquad \mathsf{D}^m\ \frac{A \Rightarrow B}{\Box A \Rightarrow \Diamond B}$$

$$\mathsf{CD}^m\ \frac{\Sigma \Rightarrow A}{\Box\Sigma \Rightarrow \Diamond A} \qquad \mathsf{T}^m_\Box\ \frac{\Gamma, A \Rightarrow C}{\Gamma, \Box A \Rightarrow C} \qquad \mathsf{T}^m_\Diamond\ \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \Diamond A}$$

Figure 9. Modal rules for minimal sequent calculi G1-M.L.

$\mathsf{mem}^{c\ell}_{\mathsf{C}}$, $\mathsf{N}^{c\ell}_\Diamond$, $\mathsf{P}^{c\ell}_\Box$, $\mathsf{D}^{c\ell}_\Box$ and $\mathsf{D}^{c\ell}_\Diamond$), while the remaining rules preserve only one formula in the consequent of sequents (note in particular that the modal context $\Diamond\Pi$ is removed from $\mathsf{C}^m_\Box$, $\mathsf{K}^m_\Box$ and $\mathsf{K}^m_\Diamond$). Observe also that the single-succedent restriction applied to $\mathsf{C}^{c\ell}_\Diamond$ and $\mathsf{K}^{c\ell}_\Diamond$ produces the same rule $\mathsf{K}^m_\Diamond$. Finally, the calculi G1-M.MD and G1-M.MND contain the rule $\mathsf{P}^m_\Diamond$ that corresponds to the restriction of the rule $\mathsf{D}^{c\ell}_\Diamond$ in the particular case where $A = B$ ($\mathsf{P}^{c\ell}_\Diamond$ is derivable in G1-MD and G1-MND from $\mathsf{D}^{c\ell}_\Diamond$ and $\mathsf{ctr}^{cl}_{\mathsf{L}}$).

We now show that the rule cut is admissible in the calculi G1-M.L. As a consequence of this result, we prove that the calculi G1-M.L are equivalent to the corresponding axiomatic systems M.L.

THEOREM 5.8. *For every calculus* G1-M.L, *the rule* cut *is admissible in* G1-M.L.

Proof. The proof is in the appendix. □

THEOREM 5.9. *For every calculus* G1-M.L, *for all* $A \in \mathcal{L}$, *A is derivable in* G1-M.L *if and only if A is derivable in* M.L.

Proof. (⇒) For every modal rule $S_1, \dots, S_n/S$ of G1-M.L, we show that the rule $\iota(S_1), \dots, \iota(S_n)/\iota(S)$ is derivable in M.L. ($\mathsf{M}^m_\Box$) From $A \supset B$, by $mon_\Box$ we get $\Box A \supset \Box B$. ($\mathsf{M}^m_\Diamond$) From $A \supset B$, by $mon_\Diamond$ we get $\Diamond A \supset \Diamond B$. ($\mathsf{N}^m_\Box$) From A we get $\top \supset A$, then by $mon_\Box$, $\Box\top \supset \Box A$, hence with $\Box\top$ we obtain $\Box A$. ($\mathsf{P}^m_\Diamond$) From A we get $\top \supset A$, then by $mon_\Diamond$, $\Diamond\top \supset \Diamond A$, hence with $\Diamond\top$ we obtain $\Diamond A$. (D^m) From $A \supset B$, by $mon_\Box$, $\Box A \supset \Box B$, then with $\Box B \supset \Diamond B$ we get $\Box A \supset \Diamond B$. ($\mathsf{CD}^m$) Assume $\Sigma = A_1 \wedge \dots \wedge A_n$. Then from $A_1 \wedge \dots \wedge A_n \supset B$, by $mon_\Box$ we get $\Box(A_1 \wedge \dots \wedge A_n) \supset \Box B$. From $C_\Box$ we have $\Box A_1 \wedge \dots \wedge \Box A_n \supset \Box(A_1 \wedge \dots \wedge A_n)$, then with $\Box B \supset \Diamond B$ we obtain $\Box A_1 \wedge \dots \wedge \Box A_n \supset \Diamond B$. ($\mathsf{T}^m_\Box$) From $\bigwedge \Gamma \wedge A \supset B$, with $\Box A \supset A$ we get $\bigwedge \Gamma \wedge \Box A \supset B$. ($\mathsf{T}^m_\Diamond$) From $\bigwedge \Gamma \supset A$, with $A \supset \Diamond A$ we get $\bigwedge \Gamma \supset \Diamond A$. For $\mathsf{C}^m_\Box$, $\mathsf{K}^m_\Box$ and $\mathsf{K}^m_\Diamond$ see the derivations in Figure 3, replacing consecutive applications of *nec* and $K_\Box$ with one application of $mon_\Box$.

(⇐) For the other direction, it is easy to see that the modal axioms and rules of M.L are derivable, respectively admissible, in G1-M.L. We show as examples the derivations of $C_\Box$ and $mon_\Box$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A, B \Rightarrow A \qquad A, B \Rightarrow B}{A, B \Rightarrow A \wedge B}\ \wedge^m_{\mathsf{R}}}{\Box A, \Box B \Rightarrow \Box(A \wedge B)}\ \mathsf{C}^m_\Box}{\Box A \wedge \Box B, \Box B \Rightarrow \Box(A \wedge B)}\ \wedge^m_{\mathsf{L}}}{\Box A \wedge \Box B, \Box A \wedge \Box B \Rightarrow \Box(A \wedge B)}\ \wedge^m_{\mathsf{L}}}{\Box A \wedge \Box B \Rightarrow \Box(A \wedge B)}\ \mathsf{ctr}^m_{\mathsf{L}}}{\Rightarrow \Box A \wedge \Box B \supset \Box(A \wedge B)}\ \supset^m_{\mathsf{R}} \qquad \dfrac{\dfrac{\dfrac{\Rightarrow A \supset B \qquad A \supset B, A \Rightarrow B}{A \Rightarrow B}\ \mathsf{cut}}{\Box A \Rightarrow \Box B}\ \mathsf{M}_\Box}{\Rightarrow \Box A \supset \Box B}\ \supset^m_{\mathsf{R}}$$

□

§6. Constructive modal logics. On the basis of the relations between M.K and C.K observed in Section 4, we now define a constructive counterpart for each logic M.L. First, the constructive modal logics C.L are defined extending M.L with ex falso quodlibet $\bot \supset A$.

DEFINITION 6.1 (Constructive modal logics). *For every minimal modal logic* M.L, *the corresponding constructive modal logic* C.L *is defined as* M.L $+ \bot \supset A$.

We show that each logic C.L is semantically characterised by neighbourhood models obtained by suitably restricting the minimal neighbourhood models for the corresponding system M.L. The restriction is analogous to the one of Definition 4.1, with the difference that the neighbourhood function is now involved.

DEFINITION 6.2 (Constructive neighbourhood semantics). *For every constructive modal logic* C.L, *a* constructive neighbourhood model *for* C.L *is any minimal neighbourhood model* $\mathcal{M} = \langle \mathcal{W}, \leq, \mathbb{F}, \mathcal{N}, \mathcal{V} \rangle$ *for the corresponding minimal logic* M.L *such that the following hold for all* $w \in \mathbb{F}$:

(i) $w \in \mathcal{V}(p)$ *for all* $p \in Atm$;
(ii) *there is* $\alpha \in \mathcal{N}(w)$ *such that* $\alpha \subseteq \mathbb{F}$;
(iii) *for all* $\alpha \in \mathcal{N}(w)$, $\alpha \cap \mathbb{F} \neq \emptyset$.

THEOREM 6.1. *For all* $A \in \mathcal{L}$ *and all constructive modal logics* C.L, *A is valid in all constructive neighbourhood models for* C.L *if and only if A is derivable in* C.L.

Proof. ($\Rightarrow$) The proof extends the one of Theorem 5.2 by showing that $\bot \supset A$ is valid in every constructive neighbourhood model. Suppose that $w \Vdash \bot$. We show by construction on A that $w \Vdash A$, considering only the cases $A = \Box B, \Diamond B$ (see the proof of Theorem 4.1 for $A = p, \bot, B \wedge C, B \vee C, B \supset C$). ($A = \Box B$) Suppose $w \leq v$. Since $\mathbb{F}$ is $\leq$-upward closed, $v \in \mathbb{F}$. Then by Definition 6.2, item (ii), there is $\alpha \in \mathcal{N}(v)$ such that $\alpha \subseteq \mathbb{F}$. Hence $\alpha \Vdash^{\forall} \bot$, and by i.h., $\alpha \Vdash^{\forall} B$. Therefore $w \Vdash \Box B$. ($A = \Diamond B$) Suppose $w \leq v$. Since $\mathbb{F}$ is $\leq$-upward closed, $v \in \mathbb{F}$. Then by Definition 6.2, item (iii), for all $\alpha \in \mathcal{N}(v)$, $\alpha \cap \mathbb{F} \neq \emptyset$. Hence for all $\alpha \in \mathcal{N}(v)$, $\alpha \Vdash^{\exists} \bot$, then by i.h., $\alpha \Vdash^{\exists} B$. Therefore $w \Vdash \Diamond B$.

($\Leftarrow$) The proof extends the completeness proof of minimal modal logics by showing that the canonical neighbourhood model for C.L (Definition 5.4) satisfies the conditions (i), (ii), (iii) in Definition 6.2. Suppose that $(\Phi, \mathscr{C}) \in \mathbb{F}$. Then $\bot \in \Phi$. Since Φ is closed under derivation, by ex falso quodlibet we obtain $\Phi = \mathcal{L}$, which entails the following. (i) For all $p \in Atm$, $p \in \Phi$, hence by definition, $(\Phi, \mathscr{C}) \in \mathcal{V}(p)$. (ii) $\Box \bot \in \Phi$, hence by Definition 5.3, there is $\mathscr{U} \in \mathscr{C}$ such that for all $\Psi \in \mathscr{U}$, $\bot \in \Psi$. Then by Definition 5.4, there is $\alpha_{\mathscr{U}} \in \mathcal{N}((\Phi, \mathscr{C}))$ such that for all $(\Psi, \mathscr{D}) \in \alpha_{\mathscr{U}}$, $\bot \in \Psi$. Then for all $(\Psi, \mathscr{D}) \in \alpha_{\mathscr{U}}$, $(\Psi, \mathscr{D}) \in \mathbb{F}$, thus $\alpha_{\mathscr{U}} \subseteq \mathbb{F}$. (iii) $\Diamond \bot \in \Phi$, hence by Definition 5.3, for all $\mathscr{U} \in \mathscr{C}$, there is $\Psi \in \mathscr{U}$ such that $\bot \in \Psi$. Then by Definition 5.4 and Lemma 5.3 (which holds for C.L-full segments as well), for all $\alpha_{\mathscr{U}} \in \mathcal{N}((\Phi, \mathscr{C}))$, there is $(\Psi, \mathscr{D}) \in \alpha_{\mathscr{U}}$ such that $\bot \in \Psi$, hence $(\Psi, \mathscr{D}) \in \mathbb{F}$, thus $\alpha_{\mathscr{U}} \cap \mathbb{F} \neq \emptyset$. □

Now, we show that for each logic C.L, a sequent calculus G1-C.L can be obtained by extending G1-IPL with the modal rules of the corresponding calculus G1-M.L.

DEFINITION 6.3. *For every logic* C.L, *the sequent calculus* G1-C.L *contains the rules of* G1-IPL (*Figure* 5) *plus the modal rules of the corresponding minimal calculus* G1-M.L, *except for* $\mathsf{T}^m_\Box$ *which is replaced by its intuitionistic version* $\mathsf{T}^i_\Box$, *with* $0 \leq |\delta| \leq 1$ (*that is, with the succedent containing at most one formula*):

$$\mathsf{T}^i_\Box \ \frac{\Gamma, A \Rightarrow \delta}{\Gamma, \Box A \Rightarrow \delta}$$

Differently from the other modal rules, $\mathsf{T}^{c\ell}_\Box$ and $\mathsf{T}^{c\ell}_\Diamond$ are local and must therefore be treated like the propositional rules. Since $\mathsf{T}^{c\ell}_\Diamond$ has a principal formula in the succedent which is preserved by both kinds of sequent restrictions, this only impacts on $\mathsf{T}^i_\Box$ that requires an intuitionistic succedent containing zero or one formula.

THEOREM 6.2. *For every calculus* G1-C.L, *the rule* cut *is admissible in* G1-C.L.

Proof. The proof is in the appendix. □

THEOREM 6.3. *For every calculus* G1-C.L, *for all* $A \in \mathcal{L}$, A *is derivable in* G1-C.L *if and only if* A *is derivable in* C.L.

Proof. The derivations of the intuitionistic axioms and sequent rules are standard. For the derivations of the modal axioms and sequent rules we refer to the proof of Theorem 5.9. □

§7. Discussion and future work.

7.1. A framework of minimal and constructive modal logics. The aim of this paper was to provide a uniform characterisation of constructive modal logics. Our approach went through the definition of a family of minimal modal logics obtained from their classical counterparts (1) by means of a reduction into fusions of classical modal logics via the extended Gödel–Johansson translation, (2) by restricting G1 sequent calculi for classical modal logics to single-succedent sequents. We have seen that the resulting minimal counterpart of K is strictly connected with the constructive modal logic C.K studied in the literature, as the two systems validate the same modal principles. Moreover, we have seen that C.K can be obtained from M.K (1) axiomatically, by extending M.K with ex falso quodlibet $\bot \supset A$; (2) semantically, by adding suitable conditions on the set of fallible worlds; (3) based on the sequent calculi, by adding the minimal modal rules to an intuitionistic sequent calculus. By extending these relations to the other minimal systems, we have defined a constructive analog for each minimal system, obtaining a corresponding family of constructive modal logics. This family contains the logics C.K, C.KD and C.KT which are the constructive counterparts of K, KD and KT already studied in the literature. In particular, the same axiomatisations were defined in [2, 43] (C.KT also coincides with the propositional fragment of Fitch's first-order intuitionistic modal logic [18]), moreover our sequent calculi G1-C.K, G1-C.KD, G1-C.KT coincide with those of [6, 34, 36]. The remaining minimal and constructive logics are new. All in all, this work organises pre-existing constructive modal logics into a uniform framework and also extends this family with constructive counterparts of some non-normal modal logics, providing for each of them corresponding semantics and sequent calculi. At the same time, our approach offers an alternative view on constructive modal logics with respect to axiomatic systems; we observe in particular that a constructive modal logic does not necessarily contain both the □- and the ◇-version of the characteristic axioms of its classical counterpart, as one could expect when starting from the axiomatisation.

M.L	C.L	W.L
Definition 5.1	M.L + $\bot \supset A$	C.L + $\neg(\Box A \wedge \Diamond\neg A)$
Minimal modal models (Definition 5.2)	Minimal modal models with fallible worlds satisfying all formulas	Minimal modal models without fallible worlds
G1 propositional and modal rules with *exactly one* formula in the succedent (Figure 9)	G1 propositional rules with *at most one* formula in the succedent, G1 modal rules with *exactly one* formula in the succedent	G1 propositional and modal rules with *at most one* formula in the succedent

Figure 10. Axiomatic, semantical and proof-theoretical relations between minimal and constructive modal logics.

7.2. Comparison with Wijesekera-style constructive modal logics. Wijesekera's logic W.K, often presented as C.K + $N_\Diamond$, was defined (in a first-order formulation) and provided with a birelational semantics and a sequent calculus in [65]. Interestingly, the same models can be obtained from the birelational models for C.K (Definition 4.1) by dropping the fallible worlds, and the sequent calculus amounts to the restriction of G1-K (Figure 2) to sequents with *at most* one formula in the succedent (vs. restricting to *exactly* one formula in the succedent, that provides G1-C.K, cf. also [13]). We can now observe that analogous relations hold for all constructive modal logics C.L studied in this paper, considering the corresponding Wijesekera-style logic W.L defined in [11]. Above all, we point out that each system W.L (thus also W.K) can be obtained extending the corresponding system C.L with the axiom $\neg(\Box A \wedge \Diamond\neg A)$, which expresses one direction of the duality principle. This shows that the difference between C.K and W.K does not rely that much on a stronger $\Diamond$ of the latter, but rather on a different interaction of $\Box$ and $\Diamond$ in the two systems. In particular, the modalities in C.L systems are barely connected. The relations between minimal, constructive and Wijesekera-style modal logics are summarised in Figure 10.

7.3. Simpson's requirements. Simpson [59] listed some requirements that are now a standard to evaluate whether an intuitionistic modal logic I.L can be understood as an intuitionistic counterpart of a classical modal logic L, among which we have that I.L should be a conservative extension of IPL, it should contain all axioms of IPL (over the whole language $\mathcal{L}$) and be closed under modus ponens, it should satisfy the disjunction property (if $A \vee B$ is derivable, then A is derivable or B is derivable), the modalities in I.L should be independent, the extension of I.L with $A \vee \neg A$ should coincide with L. It looks natural to adapt these requirements to pairs of minimal and constructive/intuitionistic modal logics. It is easy to verify that each logic M.L is a conservative extension of MPL, contains all axioms of MPL and modus ponens, satisfies the disjunction property and has independent modalities. Moreover, the extension of M.L with $\bot \supset A$ coincides with the corresponding logic C.L. In this sense, each pair of corresponding logics M.L and C.L constitutes a Simpsonian pair of modal logics.

7.4. Computational properties. In this work, we have not considered the computational properties of the logics M.L and C.L. However, we can observe that the equation $(*)$ is not only a definitorial property of the logics M.L, it is also a polynomial reduction of the derivability problem for M.L into the derivability problem for $\mathsf{S4} \oplus \mathsf{L}$. Considering that the derivability problems for $\mathsf{S4} \oplus \mathsf{K}$, $\mathsf{S4} \oplus \mathsf{KD}$ and $\mathsf{S4} \oplus \mathsf{KT}$ are known to be PSPACE-complete [20], and that M.K is a conservative extension of MPL (with respect to the fragment of the language without the modalities) which is also PSPACE-complete, we can conclude that the derivability problems for M.K, M.KD and M.KT are PSPACE-complete. We conjecture that the same complexity bound applies to all logics M.L and C.L. In future work, we would like to address this problem by studying terminating sequent calculi and construction of finite models in the style of [12]. Moreover, we conjecture that PSPACE-complexity can be proved for M.M by combining the translation t with the reduction of classical M into multi-modal K presented in [22, 32]. We would also like to study reductions for the constructive logics C.L along the lines of [16].

7.5. Scalability and limitations of our approach. We have restricted our analysis to modal logics characterised by *non-iterative* axioms, namely, axioms without modal operators occurring within the scope of other modal operators. The reason is technical: models generated by modal logic fusions do not contain any interaction between the different relations (or between relations and neighbourhood functions). On the other hand, some interaction is needed in order to validate basic iterative axioms such as $4_\Box$ $\Box A \supset \Box\Box A$, as it is witnessed by the birelational semantics for C.S4 of [1] based on a confluence property of the form $w\mathcal{R}v \;\&\; v \leq u \Rightarrow \exists z(w \leq z \;\&\; z\mathcal{R}u)$. Interestingly, it can be proved that this property is satisfied by the canonical birelational model for C.K (Definition 2.8), which implies that this property is admissible in its birelational semantics. This remark suggests the possible applicability of our canonical model construction to minimal and constructive logics with iterative axioms. However, it not obvious to establish whether this is actually the case, and it is left to future work.

Concerning instead our proof-theoretical approach, we have considered pure label-free, Gentzen-style sequent calculi because of their simplicity and the fact that constructive modal logics allow for this kind of calculi (against intuitionistic modal logics, for which they seem not possible). Similar single-succedent restrictions of labels and nested sequent calculi have generated proof systems for intuitionistic modal logics [59, 60]. It would be interesting to apply a similar strategy starting from alternative types of calculi, such as hypersequent or 2-sequent calculi, in order to obtain constructive correspondents of additional classical logics as well as to better understand the relations between the properties of a calculus and the logics resulting from its restrictions.

Appendix: Proofs of cut admissibility.

THEOREM 3.1. *The following rule* cut *is admissible in* G1-M.K:

$$\mathsf{cut}\ \frac{\Gamma \Rightarrow A \qquad \Sigma, A \Rightarrow C}{\Gamma, \Sigma \Rightarrow C}$$

Proof. The proof follows a standard strategy that goes back to Gentzen [27] (cf. [61] for more details) and consists in proving the admissibility of the following generalisation of cut

$$\mathsf{mix}\ \frac{\Gamma \Rightarrow A \qquad \Sigma, A^n \Rightarrow C}{\Gamma, \Sigma \Rightarrow C}$$

also known as *multicut*, where A^n denotes one or more occurrences of A. The proof shows that every derivation containing one or more applications of mix can be transformed into an equivalent derivation not containing applications of mix by removing step by step all topmost applications of mix. Let us call *mix formula* the formula which is deleted by the application of mix. The proof proceeds by induction on lexicographically ordered pairs (c, h), where c is the *complexity* of the mix formula, defined as usual as $c(p) = c(\bot) = 1$, $c(B \circ C) = c(B) + c(C) + 1$, $C(\heartsuit B) = c(B) + 1$, with $\circ \in \{\wedge, \vee, \supset\}$, $\heartsuit \in \{\Box, \Diamond\}$, and h is the *cut height*, defined as the sum of the heights of the mix-free derivations of the premisses of mix, where the height of a mix-free derivation is in turn defined as the length of the longest branch from the root to an initial sequent. The proof distinguishes among the following cases. In each case, the derivation on the left is converted into the derivation on the right, where the original application of mix is possibly replaced by one or more applications of mix, each of them having a mix formula with lower complexity or having a lower mix height. In the derivations, given a rule R, we denote R^* an arbitrary number of repeated applications of R.

[1] At least one premiss of mix is an initial sequent. There are two subcases.

[1.1] The left premiss of mix is an initial sequent.

$$\mathsf{mix}\ \frac{A \Rightarrow A \qquad \overset{\triangledown}{\Gamma, A^n \Rightarrow C}}{\Gamma, A \Rightarrow C} \quad \rightsquigarrow \quad \frac{\overset{\triangledown}{\Gamma, A^n \Rightarrow C}}{\Gamma, A \Rightarrow C}\ \mathsf{ctr}_{\mathsf{L}}^{m^*}.$$

[1.2] The right premiss of mix is an initial sequent.

$$\mathsf{mix}\ \frac{\overset{\triangledown}{\Gamma \Rightarrow A} \qquad A \Rightarrow A}{\Gamma \Rightarrow A} \quad \rightsquigarrow \quad \overset{\triangledown}{\Gamma \Rightarrow A}.$$

[2] Neither premiss of mix is an initial sequent. There are three subcases.

[2.1] The mix formula is not principal in the last rule applied in the derivation $\mathcal{D}$ of the left premiss of mix. We consider several cases depending on the last rule applied in $\mathcal{D}$.

$(\wedge^m_{\mathsf{L}})$

$$\mathsf{mix}\,\dfrac{\wedge^m_{\mathsf{L}}\,\dfrac{\overset{\triangledown}{\Gamma, B_i \Rightarrow A}}{\Gamma, B_1 \wedge B_2 \Rightarrow A} \qquad \overset{\triangledown}{\Sigma, A^n \Rightarrow C}}{\Gamma, \Sigma, B_1 \wedge B_2 \Rightarrow C} \;\rightsquigarrow\; \dfrac{\dfrac{\overset{\triangledown}{\Gamma, B_i \Rightarrow A} \qquad \overset{\triangledown}{\Sigma, A^n \Rightarrow C}}{\Gamma, \Sigma, B_i \Rightarrow C}\,\mathsf{mix}}{\Gamma, \Sigma, B_1 \wedge B_2 \Rightarrow C}\,\wedge^m_{\mathsf{L}}$$

$(\vee^m_{\mathsf{L}})$

$$\mathsf{mix}\,\dfrac{\vee^m_{\mathsf{L}}\,\dfrac{\overset{\triangledown}{\Gamma, B \Rightarrow A} \qquad \overset{\triangledown}{\Gamma, C \Rightarrow A}}{\Gamma, B \vee C \Rightarrow A} \qquad \overset{\triangledown}{\Sigma, A^n \Rightarrow D}}{\Gamma, \Sigma, B \vee C \Rightarrow D} \;\rightsquigarrow$$

$$\vee^m_{\mathsf{L}}\,\dfrac{\mathsf{mix}\,\dfrac{\overset{\triangledown}{\Gamma, B \Rightarrow A} \qquad \overset{\triangledown}{\Sigma, A^n \Rightarrow D}}{\Gamma, \Sigma, B \Rightarrow D} \qquad \dfrac{\overset{\triangledown}{\Gamma, C \Rightarrow A} \qquad \overset{\triangledown}{\Sigma, A^n \Rightarrow D}}{\Gamma, \Sigma, C \Rightarrow D}\,\mathsf{mix}}{\Gamma, \Sigma, B \vee C \Rightarrow C}$$

$(\supset^m_{\mathsf{L}})$

$$\mathsf{mix}\,\dfrac{\supset^m_{\mathsf{L}}\,\dfrac{\overset{\triangledown}{\Gamma \Rightarrow B} \qquad \overset{\triangledown}{\Gamma, C \Rightarrow A}}{\Gamma, B \supset C \Rightarrow A} \qquad \overset{\triangledown}{\Sigma, A^n \Rightarrow D}}{\Gamma, \Sigma, B \supset C \Rightarrow D} \;\rightsquigarrow$$

$$\supset^m_{\mathsf{L}}\,\dfrac{\mathsf{wk}^{m^*}_{\mathsf{L}}\,\dfrac{\overset{\triangledown}{\Gamma \Rightarrow B}}{\Gamma, \Sigma \Rightarrow B} \qquad \dfrac{\overset{\triangledown}{\Gamma, C \Rightarrow A} \qquad \overset{\triangledown}{\Sigma, A^n \Rightarrow D}}{\Gamma, \Sigma, C \Rightarrow D}\,\mathsf{mix}}{\Gamma, \Sigma, B \supset C \Rightarrow D}$$

$(\mathsf{wk}^m_{\mathsf{L}})$

$$\mathsf{mix}\,\dfrac{\mathsf{wk}^m_{\mathsf{L}}\,\dfrac{\overset{\triangledown}{\Gamma \Rightarrow A}}{\Gamma, B \Rightarrow A} \qquad \overset{\triangledown}{\Sigma, A^n \Rightarrow C}}{\Gamma, \Sigma, B \Rightarrow C} \;\rightsquigarrow\; \dfrac{\dfrac{\overset{\triangledown}{\Gamma \Rightarrow A} \qquad \overset{\triangledown}{\Sigma, A^n \Rightarrow C}}{\Gamma, \Sigma \Rightarrow C}\,\mathsf{mix}}{\Gamma, \Sigma, B \Rightarrow C}\,\mathsf{wk}^m_{\mathsf{L}}$$

$(\mathsf{ctr}^m_{\mathsf{L}})$

$$\mathsf{mix}\,\dfrac{\mathsf{ctr}^m_{\mathsf{L}}\,\dfrac{\overset{\triangledown}{\Gamma, B, B \Rightarrow A}}{\Gamma, B \Rightarrow A} \qquad \overset{\triangledown}{\Sigma, A^n \Rightarrow C}}{\Gamma, \Sigma, B \Rightarrow C} \;\rightsquigarrow\; \dfrac{\dfrac{\overset{\triangledown}{\Gamma, B, B \Rightarrow A} \qquad \overset{\triangledown}{\Sigma, A^n \Rightarrow C}}{\Gamma, \Sigma, B, B \Rightarrow C}\,\mathsf{mix}}{\Gamma, \Sigma, B \Rightarrow C}\,\mathsf{ctr}^m_{\mathsf{L}}$$

Right rules and $\mathsf{K}^m_{\Box}$, $\mathsf{K}^m_{\Diamond}$ are not possible.

2.2 The mix formula is not principal in the last rule applied in the derivation $\mathcal{D}$ of the right premiss of mix. We consider several cases depending on the last rule applied in $\mathcal{D}$.

$(\wedge^m_{\mathsf{L}})$

$$\mathsf{mix}\,\dfrac{\overset{\triangledown}{\Gamma \Rightarrow A} \qquad \dfrac{\overset{\triangledown}{\Sigma, A^n, B_i \Rightarrow C}}{\Sigma, A^n, B_1 \wedge B_2 \Rightarrow C}\,\wedge^m_{\mathsf{L}}}{\Gamma, \Sigma, B_1 \wedge B_2 \Rightarrow C} \;\rightsquigarrow\; \dfrac{\dfrac{\overset{\triangledown}{\Gamma \Rightarrow A} \qquad \overset{\triangledown}{\Sigma, A^n, B_i \Rightarrow C}}{\Gamma, \Sigma, B_i \Rightarrow C}\,\mathsf{mix}}{\Gamma, \Sigma, B_1 \wedge B_2 \Rightarrow C}\,\wedge^m_{\mathsf{L}}$$

$(\wedge^m_{\mathsf{R}})$

$$\mathsf{mix}\,\dfrac{\overset{\triangledown}{\Gamma \Rightarrow A} \qquad \dfrac{\overset{\triangledown}{\Sigma, A^n \Rightarrow B} \qquad \overset{\triangledown}{\Sigma, A^n \Rightarrow C}}{\Sigma, A^n \Rightarrow B \wedge C}\,\wedge^m_{\mathsf{R}}}{\Gamma, \Sigma \Rightarrow B \wedge C} \;\rightsquigarrow$$

$$\dfrac{\mathsf{mix}\,\dfrac{\overset{\triangledown}{\Gamma \Rightarrow A} \qquad \overset{\triangledown}{\Sigma, A^n \Rightarrow B}}{\Gamma, \Sigma \Rightarrow B} \qquad \dfrac{\overset{\triangledown}{\Gamma \Rightarrow A} \qquad \overset{\triangledown}{\Sigma, A^n \Rightarrow C}}{\Gamma, \Sigma \Rightarrow C}\,\mathsf{mix}}{\Gamma, \Sigma \Rightarrow B \wedge C}\,\wedge^m_{\mathsf{R}}$$

 https://doi.org/10.1017/S1755020325000097

$(\vee_{\mathsf{L}}^{m})$

$$\mathsf{mix}\,\dfrac{\overset{\nabla}{\Gamma \Rightarrow A} \qquad \dfrac{\overset{\nabla}{\Sigma, A^n, B \Rightarrow D} \qquad \overset{\nabla}{\Sigma, A^n, C \Rightarrow D}}{\Sigma, A^n, B \vee C \Rightarrow D}\,\vee_{\mathsf{L}}^{m}}{\Gamma, \Sigma, B \vee C \Rightarrow D} \rightsquigarrow$$

$$\dfrac{\mathsf{mix}\,\dfrac{\overset{\nabla}{\Gamma \Rightarrow A} \qquad \overset{\nabla}{\Sigma, A^n, B \Rightarrow D}}{\Gamma, \Sigma, B \Rightarrow D} \qquad \dfrac{\overset{\nabla}{\Gamma \Rightarrow A} \qquad \overset{\nabla}{\Sigma, A^n, C \Rightarrow D}}{\Gamma, \Sigma, C \Rightarrow D}\,\mathsf{mix}}{\Gamma, \Sigma, B \vee C \Rightarrow D}\,\vee_{\mathsf{L}}^{m}$$

$(\vee_{\mathsf{R}}^{m})$

$$\mathsf{mix}\,\dfrac{\overset{\nabla}{\Gamma \Rightarrow A} \qquad \dfrac{\overset{\nabla}{\Sigma, A^n \Rightarrow B_i}}{\Sigma, A^n \Rightarrow B_1 \vee B_2}\,\vee_{\mathsf{R}}^{m}}{\Gamma, \Sigma \Rightarrow B_1 \vee B_2} \rightsquigarrow \dfrac{\dfrac{\overset{\nabla}{\Gamma \Rightarrow A} \qquad \overset{\nabla}{\Sigma, A^n \Rightarrow B_i}}{\Gamma, \Sigma \Rightarrow B_i}\,\mathsf{mix}}{\Gamma, \Sigma \Rightarrow B_1 \vee B_2}\,\vee_{\mathsf{R}}^{m}$$

$(\supset_{\mathsf{L}}^{m})$

$$\mathsf{mix}\,\dfrac{\overset{\nabla}{\Gamma \Rightarrow A} \qquad \dfrac{\overset{\nabla}{\Sigma, A^n \Rightarrow B} \qquad \overset{\nabla}{\Sigma, A^n, C \Rightarrow D}}{\Sigma, A^n, B \supset C \Rightarrow D}\,\supset_{\mathsf{L}}^{m}}{\Gamma, \Sigma, B \supset C \Rightarrow D} \rightsquigarrow$$

$$\dfrac{\mathsf{mix}\,\dfrac{\overset{\nabla}{\Gamma \Rightarrow A} \qquad \overset{\nabla}{\Sigma, A^n \Rightarrow B}}{\Gamma, \Sigma \Rightarrow B} \qquad \dfrac{\overset{\nabla}{\Gamma \Rightarrow A} \qquad \overset{\nabla}{\Sigma, A^n, C \Rightarrow D}}{\Gamma, \Sigma, C \Rightarrow D}\,\mathsf{mix}}{\Gamma, \Sigma, B \supset C \Rightarrow D}\,\supset_{\mathsf{L}}^{m}$$

$(\supset_{\mathsf{R}}^{m})$

$$\mathsf{mix}\,\dfrac{\overset{\nabla}{\Gamma \Rightarrow A} \qquad \dfrac{\overset{\nabla}{\Sigma, A^n, B \Rightarrow C}}{\Sigma, A^n \Rightarrow B \supset C}\,\supset_{\mathsf{R}}^{m}}{\Gamma, \Sigma \Rightarrow B \supset C} \rightsquigarrow \dfrac{\dfrac{\overset{\nabla}{\Gamma \Rightarrow A} \qquad \overset{\nabla}{\Sigma, A^n, B \Rightarrow C}}{\Gamma, \Sigma, B \Rightarrow C}\,\mathsf{mix}}{\Gamma, \Sigma \Rightarrow B \supset C}\,\supset_{\mathsf{R}}^{m}$$

$(\mathsf{wk}_{\mathsf{L}}^{m})$

$$\mathsf{mix}\,\dfrac{\overset{\nabla}{\Gamma \Rightarrow A} \qquad \dfrac{\overset{\nabla}{\Sigma, A^n \Rightarrow C}}{\Sigma, A^n, B \Rightarrow C}\,\mathsf{wk}_{\mathsf{L}}^{m}}{\Gamma, \Sigma, B \Rightarrow C} \rightsquigarrow \dfrac{\dfrac{\overset{\nabla}{\Gamma \Rightarrow A} \qquad \overset{\nabla}{\Sigma, A^n \Rightarrow C}}{\Gamma, \Sigma \Rightarrow C}\,\mathsf{mix}}{\Gamma, \Sigma, B \Rightarrow C}\,\mathsf{wk}_{\mathsf{L}}^{m}$$

$(\mathsf{ctr}_{\mathsf{L}}^{m})$

$$\mathsf{mix}\,\dfrac{\overset{\nabla}{\Gamma \Rightarrow A} \qquad \dfrac{\overset{\nabla}{\Sigma, A^n, B, B \Rightarrow C}}{\Sigma, A^n, B \Rightarrow C}\,\mathsf{ctr}_{\mathsf{L}}^{m}}{\Gamma, \Sigma, B \Rightarrow C} \rightsquigarrow \dfrac{\dfrac{\overset{\nabla}{\Gamma \Rightarrow A} \qquad \overset{\nabla}{\Sigma, A^n, B, B \Rightarrow C}}{\Gamma, \Sigma, B, B \Rightarrow C}\,\mathsf{mix}}{\Gamma, \Sigma, B \Rightarrow C}\,\mathsf{ctr}_{\mathsf{L}}^{m}$$

$\mathsf{K}_{\Box}^{m}$, $\mathsf{K}_{\Diamond}^{m}$ are not possible.

2.3 The mix formula is principal in the last rule applied in the derivations $\mathcal{D}_1$, $\mathcal{D}_2$ of both premisses of mix. We consider several cases depending on the last rule applied in $\mathcal{D}_1$, $\mathcal{D}_2$.

$(\wedge_{\mathsf{R}}^{m}$ - $\wedge_{\mathsf{L}}^{m})$ The mix formula A has the form $B \wedge C$. We consider the following case, the other case where the premiss of $\wedge_{\mathsf{L}}^{m}$ is $\Sigma, (B \wedge C)^{n-1}, C \Rightarrow D$ is analogous.

$(\vee_{\mathsf{R}}^{m}$ - $\vee_{\mathsf{L}}^{m})$ The mix formula A has the form $B \vee C$. We consider the following case, the other case where the premiss of $\vee_{\mathsf{R}}^{m}$ is $\Gamma \Rightarrow C$ is analogous.

$$\text{mix}\ \dfrac{\wedge^m_{\mathsf{R}}\ \dfrac{\overset{\triangledown}{\Gamma \Rightarrow B} \qquad \overset{\triangledown}{\Gamma \Rightarrow C}}{\Gamma \Rightarrow B \wedge C} \qquad \dfrac{\overset{\triangledown}{\Sigma, (B \wedge C)^{n-1}, B \Rightarrow D}}{\Sigma, (B \wedge C)^{n} \Rightarrow D}\ \wedge^m_{\mathsf{L}}}{\Gamma, \Sigma \Rightarrow D} \rightsquigarrow$$

$$\dfrac{\dfrac{\overset{\triangledown}{\Gamma \Rightarrow B} \qquad \dfrac{\wedge^m_{\mathsf{R}}\ \dfrac{\overset{\triangledown}{\Gamma \Rightarrow B} \qquad \overset{\triangledown}{\Gamma \Rightarrow C}}{\Gamma \Rightarrow B \wedge C} \qquad \overset{\triangledown}{\Sigma, (B \wedge C)^{n-1}, B \Rightarrow D}}{\Gamma, \Sigma, B \Rightarrow D}\ \text{mix}}{\Gamma, \Gamma, \Sigma \Rightarrow D}\ \text{mix}}{\Gamma, \Sigma \Rightarrow D}\ \text{ctr}^{m^*}_{\mathsf{L}}$$

$$\text{mix}\ \dfrac{\vee^m_{\mathsf{R}}\ \dfrac{\overset{\triangledown}{\Gamma \Rightarrow B}}{\Gamma \Rightarrow B \vee C} \qquad \dfrac{\overset{\triangledown}{\Sigma, (B \vee C)^{n-1}, B \Rightarrow D} \qquad \overset{\triangledown}{\Sigma, (B \vee C)^{n-1}, C \Rightarrow D}}{\Sigma, (B \vee C)^{n} \Rightarrow D}\ \vee^m_{\mathsf{L}}}{\Gamma, \Sigma \Rightarrow D} \rightsquigarrow$$

$$\dfrac{\dfrac{\overset{\triangledown}{\Gamma \Rightarrow B} \qquad \dfrac{\vee^m_{\mathsf{R}}\ \dfrac{\overset{\triangledown}{\Gamma \Rightarrow B}}{\Gamma \Rightarrow B \vee C} \qquad \overset{\triangledown}{\Sigma, (B \vee C)^{n-1}, B \Rightarrow D}}{\Gamma, \Sigma, B \Rightarrow D}\ \text{mix}}{\Gamma, \Gamma, \Sigma \Rightarrow D}\ \text{mix}}{\Gamma, \Sigma \Rightarrow D}\ \text{ctr}^{m^*}_{\mathsf{L}}$$

($\supset^m_{\mathsf{R}}$ - $\supset^m_{\mathsf{L}}$) The mix formula A has the form $B \supset C$.

$$\text{mix}\ \dfrac{\supset^m_{\mathsf{R}}\ \dfrac{\overset{\triangledown}{\Gamma, B \Rightarrow C}}{\Gamma \Rightarrow B \supset C} \qquad \dfrac{\overset{\triangledown}{\Sigma, (B \supset C)^{n-1} \Rightarrow B} \qquad \overset{\triangledown}{\Sigma, (B \supset C)^{n-1}, C \Rightarrow D}}{\Sigma, (B \supset C)^{n} \Rightarrow D}\ \supset^m_{\mathsf{L}}}{\Gamma, \Sigma \Rightarrow D} \rightsquigarrow$$

$$\dfrac{\text{mix}\ \dfrac{\supset^m_{\mathsf{R}}\ \dfrac{\overset{\triangledown}{\Gamma, B \Rightarrow C}}{\Gamma \Rightarrow B \supset C} \qquad \overset{\triangledown}{\Sigma, (B \supset C)^{n-1} \Rightarrow B}}{\Gamma, \Sigma \Rightarrow B} \qquad \overset{\triangledown}{\Gamma, B \Rightarrow C}}{\Gamma, \Gamma, \Sigma \Rightarrow C \quad (*)}\ \text{mix}$$

$$\dfrac{\dfrac{\Gamma, \Gamma, \Sigma \Rightarrow C \quad (*) \qquad \dfrac{\supset^m_{\mathsf{R}}\ \dfrac{\overset{\triangledown}{\Gamma, B \Rightarrow C}}{\Gamma \Rightarrow B \supset C} \qquad \overset{\triangledown}{\Sigma, (B \supset C)^{n-1}, C \Rightarrow D}}{\Gamma, \Sigma, C \Rightarrow D}\ \text{mix}}{\Gamma, \Gamma, \Gamma, \Sigma \Rightarrow D}\ \text{mix}}{\Gamma, \Sigma \Rightarrow D}\ \text{ctr}^{m^*}_{\mathsf{L}}$$

(R-$\text{ctr}^m_{\mathsf{L}}$) The transformation below applies for any last rule R in the derivation of the left premiss of mix.

$$\text{mix}\ \dfrac{\overset{\triangledown}{\Gamma \Rightarrow A} \qquad \dfrac{\overset{\triangledown}{\Sigma, A^n, A \Rightarrow C}}{\Sigma, A^n \Rightarrow C}\ \text{ctr}^m_{\mathsf{L}}}{\Gamma, \Sigma \Rightarrow C} \rightsquigarrow \dfrac{\overset{\triangledown}{\Gamma \Rightarrow A} \qquad \overset{\triangledown}{\Sigma, A^n, A \Rightarrow C}}{\Gamma, \Sigma \Rightarrow C}\ \text{mix}$$

(R-wk_L^m) The transformation below applies for any last rule R in the derivation of the left premiss of mix (note that if $n = 1$, then the conclusion of mix $\Gamma, \Sigma \Rightarrow C$ can be obtained from $\Sigma \Rightarrow C$ by wk_L^m).

$$\mathsf{mix}\ \dfrac{\overset{\triangledown}{\Gamma \Rightarrow A} \qquad \dfrac{\overset{\triangledown}{\Sigma, A^{n-1} \Rightarrow C}}{\Sigma, A^n \Rightarrow C}\ \mathsf{wk}_\mathsf{L}^m}{\Gamma, \Sigma \Rightarrow C} \quad \rightsquigarrow \quad \dfrac{\overset{\triangledown}{\Gamma \Rightarrow A} \qquad \overset{\triangledown}{\Sigma, A^{n-1} \Rightarrow C}}{\Gamma, \Sigma \Rightarrow C}\ \mathsf{mix}$$

($\mathsf{K}_\Box^m$ - $\mathsf{K}_\Box^m$) The mix formula A has the form $\Box B$.

$$\mathsf{mix}\ \dfrac{\mathsf{K}_\Box^m\ \dfrac{\overset{\triangledown}{\Sigma \Rightarrow B}}{\Box\Sigma \Rightarrow \Box B} \qquad \dfrac{\overset{\triangledown}{B^n, \Pi \Rightarrow C}}{(\Box B)^n, \Box\Pi \Rightarrow \Box C}\ \mathsf{K}_\Box^m}{\Box\Sigma, \Box\Pi \Rightarrow \Box C} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\triangledown}{\Sigma \Rightarrow B} \qquad \overset{\triangledown}{B^n, \Pi \Rightarrow C}}{\Sigma, \Pi \Rightarrow C}\ \mathsf{mix}}{\Box\Sigma, \Box\Pi \Rightarrow \Box C}\ \mathsf{K}_\Box^m$$

($\mathsf{K}_\Box^m$ - $\mathsf{K}_\Diamond^m$) The mix formula A has the form $\Box B$.

$$\mathsf{mix}\ \dfrac{\mathsf{K}_\Box^m\ \dfrac{\overset{\triangledown}{\Sigma \Rightarrow B}}{\Box\Sigma \Rightarrow \Box B} \qquad \dfrac{\overset{\triangledown}{B^n, \Pi, C \Rightarrow D}}{(\Box B)^n, \Box\Pi, \Diamond C \Rightarrow \Diamond D}\ \mathsf{K}_\Diamond^m}{\Box\Sigma, \Box\Pi, \Diamond C \Rightarrow \Diamond D} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\triangledown}{\Sigma \Rightarrow B} \qquad \overset{\triangledown}{B^n, \Pi, C \Rightarrow D}}{\Sigma, \Pi, C \Rightarrow D}\ \mathsf{mix}}{\Box\Sigma, \Box\Pi, \Diamond C \Rightarrow \Diamond D}\ \mathsf{K}_\Diamond^m$$

($\mathsf{K}_\Diamond^m$ - $\mathsf{K}_\Diamond^m$) The mix formula A has the form $\Diamond C$.

$$\mathsf{mix}\ \dfrac{\mathsf{K}_\Diamond^m\ \dfrac{\overset{\triangledown}{\Sigma, B \Rightarrow C}}{\Box\Sigma, \Diamond B \Rightarrow \Diamond C} \qquad \dfrac{\overset{\triangledown}{\Pi, C \Rightarrow D}}{\Box\Pi, \Diamond C \Rightarrow \Diamond D}\ \mathsf{K}_\Diamond^m}{\Box\Sigma, \Box\Pi, \Diamond B \Rightarrow \Diamond D} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\triangledown}{\Sigma, B \Rightarrow C} \qquad \overset{\triangledown}{\Pi, C \Rightarrow D}}{\Sigma, \Pi, B \Rightarrow D}\ \mathsf{mix}}{\Box\Sigma, \Box\Pi, \Diamond B \Rightarrow \Diamond D}\ \mathsf{K}_\Diamond^m$$

□

THEOREM 5.8. *For every calculus* G1-M.L, *the rule* cut *is admissible in* G1-M.L.

Proof. We extend the cases in the proof of Theorem 3.1 with the analysis of the new modal rules. The cases 1.1 and 1.2 are as before.

2.1 The mix formula is not principal in the last rule applied in the derivation $\mathcal{D}$ of the left premiss of mix.

($\mathsf{T}_\Box^m$)

$$\mathsf{mix}\ \dfrac{\mathsf{T}_\Box^m\ \dfrac{\overset{\triangledown}{\Gamma, B \Rightarrow A}}{\Gamma, \Box B \Rightarrow A} \qquad \overset{\triangledown}{\Sigma, A^n \Rightarrow C}}{\Gamma, \Sigma, \Box B \Rightarrow C} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\triangledown}{\Gamma, B \Rightarrow A} \qquad \overset{\triangledown}{\Sigma, A^n \Rightarrow C}}{\Gamma, \Sigma, B \Rightarrow C}\ \mathsf{mix}}{\Gamma, \Sigma, \Box B \Rightarrow C}\ \mathsf{T}_\Box^m$$

2.2 The mix formula is not principal in the last rule applied in the derivation $\mathcal{D}$ of the right premiss of mix.

$$(\mathsf{T}^m_\Box)\qquad \mathsf{mix}\,\dfrac{\overset{\nabla}{\Gamma\Rightarrow A}\qquad \dfrac{\overset{\nabla}{\Sigma, A^n, B\Rightarrow C}}{\Sigma, A^n, \Box B\Rightarrow C}\,\mathsf{T}^m_\Box}{\Gamma,\Sigma,\Box B\Rightarrow C} \rightsquigarrow \dfrac{\dfrac{\overset{\nabla}{\Gamma\Rightarrow A}\qquad \overset{\nabla}{\Sigma,A^n,B\Rightarrow C}}{\Gamma,\Sigma,B\Rightarrow C}\,\mathsf{mix}}{\Gamma,\Sigma,\Box B\Rightarrow C}\,\mathsf{T}^m_\Box$$

$$(\mathsf{T}^m_\Diamond)\qquad \mathsf{mix}\,\dfrac{\overset{\nabla}{\Gamma\Rightarrow A}\qquad \dfrac{\overset{\nabla}{\Sigma, A^n\Rightarrow B}}{\Sigma, A^n\Rightarrow \Diamond B}\,\mathsf{T}^m_\Diamond}{\Gamma,\Sigma\Rightarrow \Diamond B} \rightsquigarrow \dfrac{\dfrac{\overset{\nabla}{\Gamma\Rightarrow A}\qquad \overset{\nabla}{\Sigma,A^n\Rightarrow B}}{\Gamma,\Sigma\Rightarrow B}\,\mathsf{mix}}{\Gamma,\Sigma\Rightarrow \Diamond B}\,\mathsf{T}^m_\Diamond$$

2.3 The mix formula is principal in the last rule applied in the derivations $\mathcal{D}_1$, $\mathcal{D}_2$ of both premisses of mix. For the cases where the last rule applied in $\mathcal{D}_1$, $\mathcal{D}_2$ is propositional see the proof of Theorem 3.1. We show the other cases.

($\mathsf{N}^m_\Box$ - $\mathsf{M}^m_\Box$) The mix formula A has the form $\Box B$.

$$\mathsf{mix}\,\dfrac{\mathsf{N}^m_\Box\,\dfrac{\overset{\nabla}{\Rightarrow B}}{\Rightarrow \Box B}\qquad \dfrac{\overset{\nabla}{B\Rightarrow C}}{\Box B\Rightarrow \Box C}\,\mathsf{M}^m_\Box}{\Rightarrow \Box C} \rightsquigarrow \dfrac{\dfrac{\overset{\nabla}{\Rightarrow B}\qquad \overset{\nabla}{B\Rightarrow C}}{\Rightarrow C}\,\mathsf{mix}}{\Rightarrow \Box C}\,\mathsf{N}^m_\Box$$

($\mathsf{P}^m_\Diamond$ - $\mathsf{M}^m_\Diamond$) The mix formula A has the form $\Diamond B$.

$$\mathsf{mix}\,\dfrac{\mathsf{P}^m_\Diamond\,\dfrac{\overset{\nabla}{\Rightarrow B}}{\Rightarrow \Diamond B}\qquad \dfrac{\overset{\nabla}{B\Rightarrow C}}{\Diamond B\Rightarrow \Diamond C}\,\mathsf{M}^m_\Diamond}{\Rightarrow \Diamond C} \rightsquigarrow \dfrac{\dfrac{\overset{\nabla}{\Rightarrow B}\qquad \overset{\nabla}{B\Rightarrow C}}{\Rightarrow C}\,\mathsf{mix}}{\Rightarrow \Diamond C}\,\mathsf{P}^m_\Diamond$$

($\mathsf{N}^m_\Box$ - D^m) The mix formula A has the form $\Box B$ (note that by definition $\mathsf{P}^m_\Diamond$ belongs to the calculus).

$$\mathsf{mix}\,\dfrac{\mathsf{N}^m_\Box\,\dfrac{\overset{\nabla}{\Rightarrow B}}{\Rightarrow \Box B}\qquad \dfrac{\overset{\nabla}{B\Rightarrow C}}{\Box B\Rightarrow \Diamond C}\,\mathsf{D}^m}{\Rightarrow \Diamond C} \rightsquigarrow \dfrac{\dfrac{\overset{\nabla}{\Rightarrow B}\qquad \overset{\nabla}{B\Rightarrow C}}{\Rightarrow C}\,\mathsf{mix}}{\Rightarrow \Diamond C}\,\mathsf{P}^m_\Diamond$$

($\mathsf{N}^m_\Box$ - $\mathsf{T}^m_\Box$) The mix formula A has the form $\Box B$.

$$\mathsf{mix}\,\dfrac{\mathsf{N}^m_\Box\,\dfrac{\overset{\nabla}{\Rightarrow B}}{\Rightarrow \Box B}\qquad \dfrac{\overset{\nabla}{\Gamma,(\Box B)^{n-1},B\Rightarrow C}}{\Gamma,(\Box B)^n\Rightarrow C}\,\mathsf{T}^m_\Box}{\Gamma\Rightarrow C} \rightsquigarrow$$

$$\dfrac{\overset{\nabla}{\Rightarrow B}\qquad \dfrac{\mathsf{N}^m_\Box\,\dfrac{\overset{\nabla}{\Rightarrow B}}{\Rightarrow \Box B}\qquad \overset{\nabla}{\Gamma,(\Box B)^{n-1},B\Rightarrow C}}{\Gamma,B\Rightarrow C}\,\mathsf{mix}}{\Gamma\Rightarrow C}\,\mathsf{mix}$$

($\mathsf{C}^m_\Box$ - $\mathsf{C}^m_\Box$) The mix formula A has the form $\Box C$.

$$\dfrac{\dfrac{\overset{\triangledown}{\Sigma, B \Rightarrow C}}{\Box\Sigma, \Box B \Rightarrow \Box C}\,\mathsf{C}^m_\Box \qquad \dfrac{\overset{\triangledown}{C^n, \Pi \Rightarrow D}}{(\Box C)^n, \Box\Pi \Rightarrow \Box D}\,\mathsf{C}^m_\Box}{\Box\Sigma, \Box\Pi, \Box B \Rightarrow \Box D}\,\mathsf{mix} \rightsquigarrow \dfrac{\dfrac{\overset{\triangledown}{\Sigma, B \Rightarrow C} \qquad \overset{\triangledown}{C^n, \Pi \Rightarrow D}}{\Sigma, \Pi, B \Rightarrow D}\,\mathsf{mix}}{\Box\Sigma, \Box\Pi, \Box B \Rightarrow \Box D}\,\mathsf{C}^m_\Box$$

($\mathsf{C}^m_\Box$ - $\mathsf{K}^m_\Diamond$) The mix formula A has the form $\Box C$.

$$\dfrac{\dfrac{\overset{\triangledown}{\Sigma, B \Rightarrow C}}{\Box\Sigma, \Box B \Rightarrow \Box C}\,\mathsf{C}^m_\Box \qquad \dfrac{\overset{\triangledown}{C^n, \Pi, D \Rightarrow E}}{(\Box C)^n, \Box\Pi, \Diamond D \Rightarrow \Diamond E}\,\mathsf{K}^m_\Diamond}{\Box\Sigma, \Box\Pi, \Box B, \Diamond D \Rightarrow \Diamond E}\,\mathsf{mix} \rightsquigarrow$$

$$\dfrac{\dfrac{\overset{\triangledown}{\Sigma, B \Rightarrow C} \qquad \overset{\triangledown}{C^n, \Pi, D \Rightarrow E}}{\Sigma, \Pi, B, D \Rightarrow E}\,\mathsf{mix}}{\Box\Sigma, \Box\Pi, \Box B, \Diamond D \Rightarrow \Diamond E}\,\mathsf{K}^m_\Diamond$$

($\mathsf{C}^m_\Box$ - CD^m) The mix formula A has the form $\Box C$.

$$\dfrac{\dfrac{\overset{\triangledown}{\Sigma, B \Rightarrow C}}{\Box\Sigma, \Box B \Rightarrow \Box C}\,\mathsf{C}^m_\Box \qquad \dfrac{\overset{\triangledown}{C^n, \Pi \Rightarrow D}}{(\Box C)^n, \Box\Pi \Rightarrow \Diamond D}\,\mathsf{CD}^m}{\Box\Sigma, \Box B, \Box\Pi \Rightarrow \Diamond D}\,\mathsf{mix} \rightsquigarrow \dfrac{\dfrac{\overset{\triangledown}{\Sigma, B \Rightarrow C} \qquad \overset{\triangledown}{C^n, \Pi \Rightarrow D}}{\Sigma, B, \Pi \Rightarrow D}\,\mathsf{mix}}{\Box\Sigma, \Box B, \Box\Pi \Rightarrow \Diamond D}\,\mathsf{CD}^m$$

(CD^m - $\mathsf{K}^m_\Diamond$) The mix formula A has the form $\Diamond B$.

$$\dfrac{\dfrac{\overset{\triangledown}{\Sigma \Rightarrow B}}{\Box\Sigma \Rightarrow \Diamond B}\,\mathsf{CD}^m \qquad \dfrac{\overset{\triangledown}{\Pi, B \Rightarrow C}}{\Box\Pi, \Diamond B \Rightarrow \Diamond C}\,\mathsf{K}^m_\Diamond}{\Box\Sigma, \Box\Pi \Rightarrow \Diamond C}\,\mathsf{mix} \rightsquigarrow \dfrac{\dfrac{\overset{\triangledown}{\Sigma \Rightarrow B} \qquad \overset{\triangledown}{\Pi, B \Rightarrow C}}{\Sigma, \Pi \Rightarrow C}\,\mathsf{mix}}{\Box\Sigma, \Box\Pi \Rightarrow \Diamond C}\,\mathsf{CD}^m$$

($\mathsf{C}^m_\Box$ - $\mathsf{T}^m_\Box$) The mix formula A has the form $\Box C$.

$$\dfrac{\dfrac{\overset{\triangledown}{\Sigma, B \Rightarrow C}}{\Box\Sigma, \Box B \Rightarrow \Box C}\,\mathsf{C}^m_\Box \qquad \dfrac{\overset{\triangledown}{\Gamma, (\Box C)^{n-1}, C \Rightarrow D}}{\Gamma, (\Box C)^n \Rightarrow D}\,\mathsf{T}^m_\Box}{\Gamma, \Box\Sigma, \Box B \Rightarrow D}\,\mathsf{mix} \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\overset{\triangledown}{\Sigma, B \Rightarrow C} \qquad \dfrac{\dfrac{\overset{\triangledown}{\Sigma, B \Rightarrow C}}{\Box\Sigma, \Box B \Rightarrow \Box C}\,\mathsf{C}^m_\Box \qquad \overset{\triangledown}{\Gamma, (\Box C)^{n-1}, C \Rightarrow D}}{\Gamma, \Box\Sigma, \Box B, C \Rightarrow D}\,\mathsf{mix}}{\Gamma, \Box\Sigma, \Box B, \Sigma, B \Rightarrow D}\,\mathsf{mix}}{\Gamma, \Box\Sigma, \Box B, \Box\Sigma, \Box B \Rightarrow D}\,\mathsf{T}^{m^*}_\Box}{\Gamma, \Box\Sigma, \Box B \Rightarrow D}\,\mathsf{ctr}^{m^*}_{\mathsf{L}}$$

($\mathsf{T}^m_\Diamond$ - $\mathsf{K}^m_\Diamond$) The mix formula A has the form $\Diamond B$.

$$\dfrac{\mathsf{T}^m_\Diamond\dfrac{\overset{\triangledown}{\Gamma \Rightarrow B}}{\Gamma \Rightarrow \Diamond B} \qquad \dfrac{\overset{\triangledown}{\Sigma, B \Rightarrow C}}{\Box\Sigma, \Diamond B \Rightarrow \Diamond C}\mathsf{K}^m_\Diamond}{\Gamma, \Box\Sigma \Rightarrow \Diamond C}\mathsf{mix} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\overset{\triangledown}{\Gamma \Rightarrow B} \qquad \overset{\triangledown}{\Sigma, B \Rightarrow C}}{\Gamma, \Sigma \Rightarrow C}\mathsf{mix}}{\Gamma, \Box\Sigma \Rightarrow C}\mathsf{T}^{m*}_\Box}{\Gamma, \Box\Sigma \Rightarrow \Diamond C}\mathsf{T}^m_\Diamond$$

For the remaining combinations, ($\mathsf{M}^m_\Box$ - $\mathsf{M}^m_\Box$) is analogous to ($\mathsf{C}^m_\Box$ - $\mathsf{C}^m_\Box$) with $|\Sigma| = |\Pi| = 0$ and $n = 1$; ($\mathsf{M}^m_\Diamond$ - $\mathsf{M}^m_\Diamond$) is analogous to ($\mathsf{K}^m_\Diamond$ - $\mathsf{K}^m_\Diamond$) with $|\Sigma| = |\Pi| = 0$; ($\mathsf{M}^m_\Box$ - D^m) is analogous to ($\mathsf{C}^m_\Box$ - CD^m) with $|\Sigma| = |\Pi| = 0$ and $n = 1$; (D^m - $\mathsf{M}^m_\Diamond$) is analogous to (CD^m - $\mathsf{K}^m_\Diamond$) with $|\Sigma| = |\Pi| = 0$; ($\mathsf{M}^m_\Box$ - $\mathsf{T}^m_\Box$) is analogous to ($\mathsf{C}^m_\Box$ - $\mathsf{T}^m_\Box$) with $|\Sigma| = 0$; and ($\mathsf{T}^m_\Diamond$ - $\mathsf{M}^m_\Diamond$) is analogous to ($\mathsf{T}^m_\Diamond$ - $\mathsf{K}^m_\Diamond$) with $|\Sigma| = 0$. □

THEOREM 6.2. *For every calculus* G1-C.L, *the rule* cut *is admissible in* G1-C.L.

Proof. We extend the cases in the proof of Theorem 5.8 with the combinations involving $\bot^i_\mathsf{L}$ and wk^i_R. The cases 1.1 and 2.1 are as in the proof of Theorem 5.8.

1.2 The right premiss of mix is the initial sequent $\bot^i_\mathsf{L}$:

$$\dfrac{\overset{\triangledown}{\Gamma \Rightarrow \bot} \qquad \bot \Rightarrow}{\Gamma \Rightarrow}\mathsf{mix}$$

We need to consider the last rule applied in the derivation of the left premiss of mix $\Gamma \Rightarrow \bot$, which is a left propositional rule or $\mathsf{T}^i_\Box$. We show as an example the latter possibility.

$$\dfrac{\mathsf{T}^i_\Box\dfrac{\overset{\triangledown}{\Gamma, B \Rightarrow \bot}}{\Gamma, \Box B \Rightarrow \bot} \qquad \bot \Rightarrow}{\Gamma, \Box B \Rightarrow}\mathsf{mix} \quad \rightsquigarrow \quad \mathsf{T}^i_\Box\dfrac{\dfrac{\overset{\triangledown}{\Gamma, B \Rightarrow \bot} \qquad \bot \Rightarrow}{\Gamma, B \Rightarrow}\mathsf{mix}}{\Gamma, \Box B \Rightarrow}$$

2.2 The mix formula is not principal in the last rule applied in the derivation $\mathcal{D}$ of the right premiss of mix.

(wk^i_R)

$$\dfrac{\overset{\triangledown}{\Gamma \Rightarrow A} \qquad \dfrac{\overset{\triangledown}{\Sigma, A^n \Rightarrow}}{\Sigma, A^n \Rightarrow B}\mathsf{wk}^i_\mathsf{R}}{\Gamma, \Sigma \Rightarrow B}\mathsf{mix} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\triangledown}{\Gamma \Rightarrow A} \qquad \overset{\triangledown}{\Sigma, A^n \Rightarrow}}{\Gamma, \Sigma \Rightarrow}\mathsf{mix}}{\Gamma, \Sigma \Rightarrow B}\mathsf{wk}^i_\mathsf{R}$$

2.3 The mix formula is principal in the last rule applied in the derivations $\mathcal{D}_1$, $\mathcal{D}_2$ of both premisses of mix.

(wk^i_R - R) The transformation below applies for any last rule R in the derivation of the left premiss of mix.

$$\dfrac{\mathsf{wk}^i_\mathsf{R}\dfrac{\overset{\triangledown}{\Gamma \Rightarrow}}{\Gamma \Rightarrow A} \qquad \overset{\triangledown}{\Sigma, A^n \Rightarrow \delta}}{\Gamma, \Sigma \Rightarrow \delta}\,\mathsf{mix} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\triangledown}{\Gamma \Rightarrow}}{\Gamma, \Sigma \Rightarrow}\,\mathsf{wk}^{i^*}_\mathsf{L}\ (\text{if } |\Sigma| \geq 0)}{\Gamma, \Sigma \Rightarrow \delta}\,\mathsf{wk}^{i^*}_\mathsf{R}\ (\text{if } |\delta| = 1)$$

□

Acknowledgments. I thank the anonymous reviewers for their very careful reading and useful comments that helped me to significantly improve the paper.

Funding. I acknowledge the financial support through the 'Abstractron' project funded by the Autonome Provinz Bozen - Südtirol (Autonomous Province of Bolzano-Bozen) through the Research Südtirol/Alto Adige 2022 Call.

BIBLIOGRAPHY

[1] Alechina, N., Mendler, M., de Paiva, V., & Ritter, E. (2001). Categorical and Kripke semantics for constructive S4 modal logic. In Fribourg, L., editor. *Computer Science Logic, 15th International Workshop, CSL 2001. 10th Annual Conference of the EACSL, Paris, France, September 10-13, 2001, Proceedings*, volume 2142 of Lecture Notes in Computer Science, Berlin, Heidelberg: Springer, pp. 292–307.

[2] Arisaka, R., Das, A., & Straßburger, L. (2015). On nested sequents for constructive modal logics. *Logical Methods in Computer Science*, **11**(3), 1–33.

[3] Avron, A. (1996). The method of hypersequents in the proof theory of propositional nonclassical logics. In Hodges, W., Hyland, M., Steinhorn, C., Truss, J., editors. *Logic: From Foundations to Applications*. Oxford: Clarendon Press, pp. 1–32.

[4] Balbiani, P., Gao, H., Gencer, Ç., & Olivetti, N. (2024). Local intuitionistic modal logics and their calculi. In Benzmüller, C., Heule, M. J. H., & Schmidt, R. A., editors. *Automated Reasoning—12th International Joint Conference, IJCAR 2024, Nancy, France, July 3-6, 2024, Proceedings, Part II*, volume 14740 of Lecture Notes in Computer Science, Cham: Springer, pp. 78–96.

[5] ———. (2024). A natural intuitionistic modal logic: Axiomatization and bi-nested calculus. In Murano, A., & Silva, A., editors. *32nd EACSL Annual Conference on Computer Science Logic, CSL 2024, February 19-23, 2024, Naples, Italy*, volume 288 of LIPIcs, 21. Schloss Dagstuhl—Leibniz-Zentrum für Informatik, pp. 1–13.

[6] Bellin, G., de Paiva, V., & Ritter, E. (2001). Extended Curry-Howard correspondence for a basic constructive modal logic. In Araces, C. and de Rijke, M., editors. *Proceedings of M4M-2*.

[7] Bílková, M., Majer, O., Pelis, M., & Restall, G. (2010). Relevant agents. In Beklemishev, L. D., Goranko, V., & Shehtman, V. B., editors. *Advances in Modal Logic*, Vol. 8. College Publications, pp. 22–38.

[8] Brünnler, K. (2009). Deep sequent systems for modal logic. *Archive for Mathematical Logic*, **48**(6), 551–577.

[9] Chellas, B. F. (1980). *Modal Logic—An Introduction*. New York: Cambridge University Press.

[10] D'Agostino, M., Gabbay, D. M., & Russo, A. Grafting modalities onto substructural implication systems. *Studia Logica*, **59**(1), 65–102, 1997.

[11] Dalmonte, T. (2022). Wijesekera-style constructive modal logics. In Fernández-Duque, D., Palmigiano, A., & Pinchinat, S., editors. *Advances in Modal Logic*, Vol. 14. College Publications, pp. 281–304.

[12] Dalmonte, T., Grellois, C., & Olivetti, N. (2021). Terminating calculi and countermodels for constructive modal logics. In Das, A., & Negri, S., editors. *Automated Reasoning with Analytic Tableaux and Related Methods—30th International Conference, TABLEAUX 2021, Birmingham, UK, September 6-9, 2021, Proceedings*, volume 12842 of Lecture Notes in Computer Science, Cham: Springer, pp. 391–408.

[13] Das, A., & Marin, S. (2023). On intuitionistic diamonds (and lack thereof). In Ramanayake, R., & Urban, J., editors. *Automated Reasoning with Analytic Tableaux and Related Methods - 32nd International Conference, TABLEAUX 2023, Prague, Czech Republic, September 18-21, 2023, Proceedings*, volume 14278 of Lecture Notes in Computer Science, Cham: Springer, pp. 283–301.

[14] Davies, R., & Pfenning, F. (2001). A modal analysis of staged computation. *Journal of the ACM*, **48**(3), 555–604.

[15] Ewald, W. B. (1986). Intuitionistic tense and modal logic. *The Journal of Symbolic Logic*, **51**(1), 166–179.

[16] Fairtlough, M., & Mendler, M. (1997). Propositional lax logic. *Information & Computation*, **137**(1), 1–33.

[17] Ferenz, N., & Tedder, A. (2023). Neighbourhood semantics for modal relevant logics. *Journal of Philosophical Logic*, **52**(1), 145–181.

[18] Fitch, F. B. (1948). Intuitionistic modal logic with quantifiers. *Portugaliae Mathematica*, **7**(2), 113–118.

[19] Fuhrmann, A. (1990). Models for relevant modal logics. *Studia Logica*, **49**(4), 501–514.

[20] Gabbay, D. M., Kurucz, A., Wolter, F., & Zakharyaschev, M. (2003). *Many-Dimensional Modal Logics: Theory and Applications*. Amsterdam: Elsevier Science B.V.

[21] Garg, D., Bauer, L., Bowers, K. D., Pfenning, F., & Reiter, M. K. (2006). A linear logic of authorization and knowledge. In Gollmann, D., Meier, J., & Sabelfeld, A., editors. *Computer Security—ESORICS 2006, 11th European Symposium on Research in Computer Security, Hamburg, Germany, September 18-20, 2006, Proceedings*, volume 4189 of Lecture Notes in Computer Science, Berlin, Heidelberg: Springer, pp. 297–312.

[22] Gasquet, O., & Herzig, A. (1996). From classical to normal modal logics. In Wansing, H., editor. *Proof Theory of Modal Logic*. Dordrecht: Springer, pp. 293–311.

[23] Gentzen, G. (1935). Untersuchungen über das logische Schliessen I. *Matematische Zeitschrift*, **39**(1), 176–210.

[24] Gödel, K. (1933). Eine Interpretation des intuitionistischen Aussagenkalküls. *Ergebnisse eines mathematischen Kolloquiums*, **4**, 39–40.

[25] de Groot, J., Shillito, I., & Clouston, R. (2024). Semantical analysis of intuitionistic modal logics between CK and IK. Preprint. CoRR, abs/2408.00262.

[26] Hansen, H. H. Monotonic modal logic, Master's thesis, Universiteit van Amsterdam, 2003 (ILLC technical report: PP-2003-24).

[27] Indrzejczak, A. (2005). Sequent calculi for monotonic modal logics. *Bulletin of the Section of Logic*, **34**(3), 151–164.

[28] ———. (2021). *Sequents and Trees*. Cham: Birkhäuser.

[29] Johansson, I. (1937). Der Minimalkalkül, ein reduzierter intuitionistischer Formalismus. *Compositio Mathematica*, **4**, 119–136.

[30] Kamide, N. (2002). Kripke semantics for modal substructural logics. *Journal of Logic, Language and Information*, **11**(4), 453–470.

[31] Kojima, K. (2012). Relational and neighborhood semantics for intuitionistic modal logic. *Reports on Mathematical Logic*, **47**, 87–113.

[32] Kracht, M., & Wolter, F. (1999). Normal monomodal logics can simulate all others. *The Journal of Symbolic Logic*, **64**(1), 99–138.

[33] Kripke, S. A . (1965). *Semantical Analysis of Intuitionistic Logic I*, Studies in Logic and the Foundations of Mathematics, 40, Amsterdam: Elsevier, pp. 92–130.

[34] Kuznets, R., Marin, S., & Straßburger, L. (2021). Justification logic for constructive modal logic. *Journal of Applied Logics*, **8**(8), 2313–2332.

[35] Lavendhomme, R., & Lucas, T. (2000). Sequent calculi and decision procedures for weak modal systems. *Studia Logica*, **66**(1), 121–145.

[36] Lellmann, B. (2013). *Sequent Calculi with Context Restrictions and Applications to Conditional Logic*. PhD thesis, Imperial College London.

[37] ———. (2016). Hypersequent rules with restricted contexts for propositional modal logics. *Theoretical Computer Science*, **656**, 76–105.

[38] Lellmann, B., & Pimentel, E. (2019). Modularisation of sequent calculi for normal and non-normal modalities. *ACM Transactions on Computational Logic*, **20**(2), 7. 1–7.46.

[39] Mares, E. D. (1993). Classically complete modal relevant logics. *Mathematical Logic Quarterly*, **39**, 165–177.

[40] Marion, M., & Sadrzadeh, M. (2004). Reasoning about knowledge in linear logic: Modalities and complexity. In Rahman, S., Symons, J., Gabbay, D. M., & Van Bendegem, J. P., editors. *Logic, Epistemology, and the Unity of Science*, volume 1 of Logic, Epistemology, and the Unity of Science, Dordrecht: Springer, pp. 327–350.

[41] Mendler, M., & de Paiva, V. (2005). Constructive CK for contexts. In Serafini, L. and Bouquet, P., editors. *Proceedings of CRR'05*. CEUR-WR.org/Vol-136.

[42] Mendler, M., & Scheele, S. (2010). Towards constructive DL for abstraction and refinement. *Journal of Automated Reasoning*, **44**(3), 207–243.

[43] ———. (2011). Cut-free Gentzen calculus for multimodal CK. *Information & Computation*, **209**(12), 1465–1490.

[44] ———. (2014). On the computational interpretation of CKn for contextual information processing. *Fundamenta Informaticae*, **130**(1), 125–162.

[45] Negri, S. (2005). Proof analysis in modal logic. *Journal of Philosophical Logic*, **34**(5–6), 507–544.

[46] Orlandelli, E. (2020). Sequent calculi and interpolation for non-normal modal and deontic logics. *Logic and Logical Philosophy*, **30**(1), 139–183.

[47] Pacuit, E. (2017). *Neighborhood Semantics for Modal Logic*. Cham: Springer.

[48] de Paiva, V., & Ritter, E. (2011). Basic constructive modality. In Beziau, J.-Y., & Coniglio, M. E., editors. *Logic without Frontiers. Festschrift for Walter Alexandre Carnielli on the Occasion of his 60th Birthday*. College Publications, pp. 411–428.

[49] Plotkin, G. D., & Stirling, C. (1986). A framework for intuitionistic modal logics. In Halpern, J. Y., editor. *Proceedings of the 1st Conference on Theoretical Aspects of Reasoning about Knowledge, Monterey, CA, USA, March 1986*. Burlington, Massachusetts: Morgan Kaufmann, pp. 399–406.

[50] Poggiolesi, F. (2008). A cut-free simple sequent calculus for modal logic S5. *Review of Symbolic Logic*, **1**(1), 3–15.

[51] Porello, D., & Troquard, N. (2015). Non-normal modalities in variants of linear logic. *Journal of Applied Non-Classical Logics*, **25**(3), 229–255.

[52] Restall, G. (2007). Proofnets for S5: Sequents and circuits for modal logic. In Dimitracopoulos, C., Newelski, L., Normann, D., and Steel, J. R., editors. *Logic Colloquium 2005*, Vol. 28. Cambridge: Cambridge University Press, pp. 151–172.

[53] Sedlár, I. (2016). Epistemic extensions of modal distributive substructural logics. *Journal of Logic and Computation*, **26**(6), 1787–1813.

[54] Segerberg, K. (1968). Propositional logics related to Heyting's and Johansson's. *Theoria*, **34**(1), 26–61.

[55] Seki, T. (2003). General frames for relevant modal logics. *Notre Dame Journal of Formal Logic*, **44**(2), 93–109.

[56] ———. (2003). A Sahlqvist theorem for relevant modal logics. *Studia Logica*, **73**(3), 383–411.

[57] Servi, G. F. (1980). Semantics for a class of intuitionistic modal calculi. In Luisa, M., and Chiara, D., editors. *Italian Studies in the Philosophy of Science*. Dordrecht: Springer, pp. 59–72.

[58] ———. (1984). Axiomatizations for some intuitionistic modal logics. *Rendiconti del Seminario Matematico—PoliTO*, **42**(3), 179–194.

[59] Simpson, A. K. (1994). *The Proof Theory and Semantics of Intuitionistic Modal Logic*. PhD thesis, University of Edinburgh.

[60] Straßburger, L. (2013). Cut elimination in nested sequents for intuitionistic modal logics. In Pfenning, F., editor. *Foundations of Software Science and Computation Structures - 16th International Conference, FOSSACS 2013, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2013, Rome, Italy, March 16-24, 2013. Proceedings*, volume 7794 of Lecture Notes in Computer Science, Berlin, Heidelberg: Springer, pp. 209–224.

[61] Troelstra, A. S., & Schwichtenberg, H. (2000). *Basic Proof Theory* (second edition), volume 43 of Cambridge Tracts in Theoretical Computer Science, Cambridge: Cambridge University Press.

[62] Viganò, L. (2000). *Labelled Non-Classical Logics*. New York: Springer.

[63] Wansing, H. (1995). Strong cut-elimination for constant domain first-order *S*5. *Logic Journal of the IGPL*, **3**(5), 797–810.

[64] ———. (2002). Sequent systems for modal logics. In Guenther, F., & Gabbay, D., editors. *Handbook of Philosophical Logic*, Vol. 8, Dordrecht: Springer, pp. 61–145.

[65] Wijesekera, D. (1990). Constructive modal logics I. *Annals of Pure and Applied Logic*, **50**(3), 271–301.

[66] Wijesekera, D., & Nerode, A. (2005). Tableaux for constructive concurrent dynamic logic. *Annals of Pure and Applied Logic*, **135**(1–3), 1–72.

[67] Wolter, F., & Zakharyaschev, M. (1999). Intuitionistic modal logics as fragments of classical bimodal logics. In Orlowska, E., editor. *Logic at Work*. Heidelberg: Physica-Verlag, pp. 168–186.

FREE UNIVERSITY OF BOZEN-BOLZANO
FACULTY OF ENGINEERING
NOI TECHPARK - BRUNO-BUOZZI-STRAßE
1 - VIA BRUNO BUOZZI, 1, 39100
BOZEN-BOLZANO
ITALY
E-mail: tiziano.dalmonte@unibz.it