THE BULLETIN OF SYMBOLIC LOGIC
Volume 30, Number 2, June 2024

REVIEWS

The Association for Symbolic Logic publishes analytical reviews of selected books and articles in the field of symbolic logic. The reviews were published in ***The Journal of Symbolic Logic*** from the founding of the JOURNAL in 1936 until the end of 1999. The Association moved the reviews to this BULLETIN, beginning in 2000.

The Reviews Section is edited by Graham Leach-Krouse (Managing Editor), Albert Atserias, Mark van Atten, Clinton Conley, Johanna Franklin, Dugald Macpherson, Antonio Montalbán, Valeria de Paiva, Christian Retoré, Marion Scheepers, and Nam Trang. Authors and publishers are requested to send, for review, copies of books to *ASL, Department of Mathematics, University of Connecticut, 341 Mansfield Road, U-1009, Storrs, CT 06269-1009, USA.*

COMPACTNESS OF ω_1 AND STRONG AXIOMS OF DETERMINACY

N. Trang, *Structure theory of* $L(\mathbb{R}, \mu)$ *and its applications*. ***Journal of Symbolic Logic***, vol. 80 (2015), no. 1, pp. 29–55.

N. Trang, *Supercompactness can be equiconsistent with measurability*. ***Notre Dame Journal of Formal Logic***, vol. 62 (2021), no. 4, pp. 593–618.

N. Trang and T. Wilson, *Determinacy from strong compactness of* ω_1. ***Annals of Pure and Applied Logic***, vol. 172 (2021), no. 6, Article no. 102944, 30pp.

D. Ikegami and N. Trang, *On supercompactness of* ω_1, ***Advances in Mathematical Logic*** (T. Arai, M. Kikuchi, S. Kuroda, M. Okada, T. Yorioka, editors), Springer, Proceedings Mathematics & Statistics, Singapore, 369, 2021, pp. 27–45.

For an uncountable cardinal κ and a set X, we say that κ is *X-supercompact* if there is a fine, normal, κ-complete measure on $\wp_\kappa(X)$ and say that κ is *supercompact* if it is X-supercompact for any set X. Although this is a rather strong large cardinal property, even ω_1 can be supercompact in the absence of the axiom of choice. Indeed, Takeuti showed that if κ is a supercompact cardinal and $g \subseteq \mathrm{Col}(\omega, {<}\kappa)$ is V-generic, then $V(\mathbb{R}^{V[g]}) \models \omega_1$ is supercompact (cf. G. Takeuti, *A relativization of axioms of strong infinity to* ω_1. ***Annals of the Japan Association for Philosophy of Science***, vol. 3 (1970), pp. 191–204). It turned out that the (partial) supercompactness of ω_1 is particularly interesting because of its connection to the axiom of determinacy (AD) and its strengthenings. Through this review, we give a brief summary of the progress made on this topic over the last 10 years.

Let us start with a classical result by Solovay: Under the axiom of determinacy for games on reals ($\mathsf{AD}_{\mathbb{R}}$), ω_1 is $\mathbb{R}$-supercompact witnessed by the club filter on $\wp_{\omega_1}(\mathbb{R})$ (cf. R. M. Solovay, *The independence of* DC *from* AD, ***Large Cardinals, Determinacy and Other Topics: The Cabal Seminar Volume IV*** (A. S. Kechris, B. Löwe, J. R. Steel, editors), Cambridge University Press, Cambridge, Lecture Notes in Logic, 49, 2021, pp. 66–95). As a corollary of this, one can easily show that $\mathsf{AD}_{\mathbb{R}}$ implies that ω_1 is ${<}\Theta$-supercompact, i.e., κ-supercompact for any $\kappa < \Theta := \sup\{\alpha \in \mathrm{Ord} \mid \text{There is a surjection } f : \mathbb{R} \to \alpha\}$. This corollary, however, can be greatly improved using inner model theory. The key result is due to Steel and Woodin, who showed that if AD holds in $L(\mathbb{R})$, $\mathsf{HOD}^{L(\mathbb{R})}$ can be characterized as a fine structural model (cf. J. R. Steel and W. H. Woodin, ***HOD*** *as a core model*, ***Ordinal Definability and***

1079-8986/24/3002-0005
DOI:10.1017/bsl.2024.5

Recursion Theory: The Cabal Seminar Volume III (A. S. Kechris, B. Löwe, J. R. Steel, editors), Cambridge University Press, Cambridge, Lecture Notes in Logic, 43, 2016, pp. 257–345). Based on this analysis of $\mathsf{HOD}^{L(\mathbb{R})}$, Woodin showed that under $\mathsf{AD} + V = L(\mathbb{R})$, ω_1 is $<\Theta$-supercompact. Also, using Woodin's method, Neeman showed the uniqueness of supercompact measures on $\wp_{\omega_1}(\alpha)$ for any $\alpha < \Theta$ under the same assumption (cf. I. Neeman, *Inner models and ultrafilters in* $L(\mathbb{R})$. ***Bulletin of Symbolic Logic***, vol. 13 (2007), no. 1, pp. 31–53). Woodin also observed that the hypothesis $\mathsf{AD} + V = L(\mathbb{R})$ of these results can be replaced with AD^+, which is a technical strengthening of AD. Note that it is conjectured that AD^+ is equivalent to AD. Although the full details of Woodin's observation have never been written down, one can find discussion about this in J. R. Steel, *Ordinal definability in models of determinacy: Introduction to part V,* ***Ordinal Definability and Recursion Theory: The Cabal Seminar Volume III*** (A. S. Kechris, B. Löwe, J. R. Steel, editors), Cambridge University Press, Cambridge, Lecture Notes in Logic, 43, 2016, pp. 3–48, and G. Sargsyan, $\mathsf{AD}_{\mathbb{R}}$ *implies that all sets of reals are* Θ *universally Baire.* ***Archive for Mathematical Logic***, vol. 60 (2021), pp. 1–15.

Now let us consider $\mathbb{R}$-supercompactness of ω_1. This is not very strong assumption in terms of consistency strength because if one starts with a measurable cardinal, then Takeuti's model would satisfy "$\mathsf{DC} + \omega_1$ is $\mathbb{R}$-supercompact." The theory "$\mathsf{AD} + \mathsf{DC} + \omega_1$ is $\mathbb{R}$-supercompact" is also much weaker than $\mathsf{AD}_{\mathbb{R}}$. Indeed, Woodin showed that the following theories are equiconsistent:

(1-1) $\mathsf{ZFC} +$ there are ω^2 many Woodin cardinals.
(1-2) $L(\mathbb{R}, \mu) \models$"$\mathsf{ZF} + \mathsf{DC} + \mathsf{AD} + \omega_1$ is $\mathbb{R}$-supercompact," where μ is the club filter on $\wp_{\omega_1}(\mathbb{R})$.

The model $L(\mathbb{R}, \mu)$ is called the Solovay model. In the first paper under review, *Structure theory of* $L(\mathbb{R}, \mu)$ *and its applications*, Trang starts with a proof of this equiconsistency. The forward direction is a variant of derived model construction in J. R. Steel, *The derived model theorem,* ***Logic Colloquium 2006*** (S. B. Cooper, H. Geuvers, A. Pillay, J. Väänänen, editors), Cambridge University Press, Cambridge, Lecture Notes in Logic, 32, 2009, p. 280–327. The reverse direction makes use of Prikry forcing associated with the measure μ on $\wp_{\omega_1}(\mathbb{R})$ as in P. Koellner and W. H. Woodin, *Large cardinals from determinacy,* ***Handbook of Set Theory (M. Foreman, A. Kanamori, editors), Springer Dordrecht, Dordrecht, vol. 3***, 2010, pp. 1951–2119. Using these techniques, Trang gave two kinds of applications. One is the HOD analysis in $L(\mathbb{R}, \mu)$, which is a natural adaptation of Steel–Woodin's result for $L(\mathbb{R})$. The other is the construction of a certain ideal on $\wp_{\omega_1}(\mathbb{R})$ in the $\mathbb{P}_{\max}$ extension of $L(\mathbb{R}, \mu)$. It is also shown that existence of such an ideal is equiconsistent with (1-2). One can find further results on the Solovay model in N. Trang, *Determinacy in* $L(\mathbb{R}, \mu)$. ***Journal of Mathematical Logic***, vol. 14 (2014), no. 1, Article no. 1450006, 23pp and D. Rodríguez and N. Trang, $L(\mathbb{R}, \mu)$ *is unique.* ***Advances in Mathematics***, vol. 324 (2018), pp. 355–393.

Next, we consider $\wp(\mathbb{R})$-supercompactness of ω_1. The easiest way to get such supercompactness together with AD is by assuming much stronger determinacy than $\mathsf{AD}_{\mathbb{R}}$. Indeed, it is a folklore result that "$\mathsf{DC} + \mathsf{AD}_{\mathbb{R}} +$ there is a normal $\mathbb{R}$-complete measure on Θ" implies that ω_1 is $\wp(\mathbb{R})$-supercompact. (cf. N. Trang, *Derived models and supercompact measures on* $\wp_{\omega_1}(\wp(\mathbb{R}))$. ***Mathematical Logic Quarterly***, vol. 61 (2015), no. 1–2, pp. 56–65.) In the second paper under review, *Supercompactness can be equiconsistent with measurability*, it is shown that the following theories are equiconsistent:

(2-1) $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}^+ + \mathsf{AD}_{\mathbb{R}} +$ there is a normal $\mathbb{R}$-complete measure on Θ.
(2-2) $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}^+ + \mathsf{AD}_{\mathbb{R}} + \Theta$ is regular $+ \omega_1$ is $\wp(\mathbb{R})$-supercompact.

In the paper, Trang first proved Woodin's theorem on Vopěnka forcing over HOD in a determinacy model, which is now considered as a standard tool. Assuming (2-2), a model of (2-1) is obtained as a symmetric extension via Vopěnka forcing of some ZFC model that has V_Θ^{HOD} as its rank initial segment and carries a normal measure on Θ. The argument to find such a ZFC model is based on the HOD analysis in G. Sargsyan, *Hod mice and the mouse set conjecture.* ***Memoirs of the American Mathematical Society***, vol. 236 (2015), no. 1111, viii+172 pp. It is worth noting that a similar argument can be found in R. Atmai and G. Sargsyan, *Hod up to* $\mathsf{AD}_{\mathbb{R}} + \Theta$ *is measurable.* ***Annals of Pure Applied Logic***, vol. 170 (2019), no. 1, pp. 95–108, where they analyze HOD in a minimal model of (2-1).

Unlike $\mathbb{R}$-compactness of ω_1, $\wp(\mathbb{R})$-supercompactness of ω_1 entails some strong form of determinacy. The third paper under review, *Determinacy from strong compactness of* ω_1, Trang and Wilson showed that "DC + ω_1 is $\wp(\mathbb{R})$-supercompact" implies the existence of a sharp for a transitive model of $\mathsf{AD}_{\mathbb{R}}$ + DC including all reals and ordinals. (The exact consistency strength of $\wp(\mathbb{R})$-supercompactness of ω_1 is still unknown.) This follows from one of their main results, which claims the following theories are equiconsistent:

(3-1) ZF + DC + $\mathsf{AD}_{\mathbb{R}}$.
(3-1) ZF + DC + ω_1 is $\wp(\mathbb{R})$-strongly compact.

Here, we say that ω_1 is *X-strongly compact* if there is a fine countably complete measure on $\wp_{\omega_1}(X)$. The forward direction is easy because by the proof of the aforementioned folklore result, in the minimal model of ZF + DC + $\mathsf{AD}_{\mathbb{R}}$, ω_1 is $\wp(\mathbb{R})$-strongly compact. Most of the paper is devoted to showing the other direction by a technique called the core model induction, which combines inner model theory with descriptive set theory to obtain a model of determinacy. The strong compactness of ω_1 is mainly used for the descriptive set theoretic part: Under ZF + DC, if Γ is an inductive-like scaled pointclass and ω_1 is $\mathrm{Env}(\Gamma)$-strongly compact, then there is a scale on a universal $\check{\Gamma}$ set, each of whose prewellorderings is in $\mathrm{Env}(\Gamma)$. This fact is essentially proved in Wilson's PhD thesis, *Contributions to descriptive inner model theory* (2012).

In the fourth paper under review, *On supercompactness of* ω_1, Ikegami and Trang obtained several structural consequences of full supercompactness of ω_1. They first show that supercompactness of ω_1 implies DC. They also show that supercompactness of ω_1 implies determinacy-like consequences such as non-existence of an ω_1-sequence of distinct reals, ∞-Borelness of sets of reals in the Chang model $\mathsf{CM} := \bigcup_{\alpha\in\mathrm{Ord}} L({}^{\omega}\alpha)$, and weak homogeneity of any tree on $\omega \times \mathrm{Ord}$. One interesting corollary of DC and the weak homogeneity of the Martin–Solovay trees is the equivalence of AD^+ and $\mathsf{AD}_{\mathbb{R}}$. Lastly, they show that under the inner model theoretic assumption called Hod Pair Capturing, supercompactness of ω_1 implies determinacy for all Suslin sets of reals. Note that one cannot expect full determinacy here because Takeuti's model does not satisfy AD. However, the consistency of "$\mathsf{AD}_{\mathbb{R}}$ + ω_1 is supercompact" is proved by Woodin in his unpublished work. He showed that assuming a proper class of Woodin limits of Woodin cardinals, the generalized Chang model $\mathsf{CM}^+ := \bigcup_{\alpha\in\mathrm{Ord}} L({}^{\omega}\alpha)[\mu_\alpha]$, where μ_α is the club filter on $\wp_{\omega_1}({}^{\omega}\alpha)$, satisfies "$\mathsf{AD}_{\mathbb{R}}$ + ω_1 is supercompact." Ikegami and Trang conjectured that:

(4-1) ZFC + there is a proper class of Woodin limits of Woodin cardinals,
(4-2) ZF + ω_1 is supercompact,
(4-3) ZF + $\mathsf{AD}_{\mathbb{R}}$ + ω_1 is supercompact,

are equiconsistent.

One thing that the articles under review do not discuss is the relation between supercompactness for ω_1 and long game determinacy. For example, in Trang's PhD thesis, *Generalized Solovay measures, the HOD analysis, and the core model induction* (2013), it is shown that the following theories are equivalent over ZFC:

https://doi.org/10.1017/bsl.2024.5 Published online by Cambridge University Press

(5-1) There is a sharp for an inner model with ω^2 many Woodin cardinals.
(5-2) All $<\omega^2$-$\mathbf{\Pi}^1_1$ games on natural numbers of length ω^3 are determined.
(5-3) There is a sharp for $L(\mathbb{R}, \mu)$ and $L(\mathbb{R}, \mu) \models$"$\mathsf{AD} + \omega_1$ is $\mathbb{R}$-supercompact," where μ is the club filter on $\wp_{\omega_1}(\mathbb{R})$.

Trang also obtained such equivalence for a sharp for an inner model with ω^α many Woodin cardinals for any $\alpha < \omega_1$ by introducing generalized Solovay models. Supercompactness of ω_1 seems important beyond determinacy of fixed countable length games too. Steel proved several results on the relation between the theory (2-2) and games ending at the first Σ_n-admissible relative to the play (cf. J. R. Steel, *Long games,* ***Games, Scales, and Suslin Cardinals: The Cabal Seminar Volume I*** (A. S. Kechris, B. Löwe, J. R. Steel, editors), Cambridge University Press, Cambridge, Lecture Notes in Logic, 31, 2008, pp. 223–259). Also, based on Neeman's consistency proof of long game determinacy (cf. I. Neeman, ***The Determinacy of Long Games*** De Gruyter, Berlin, De Gruyter Series in Logic and its Applications, 7, 2004, xii+317 pp.), Woodin showed that, assuming a sharp for an inner model with a Woodin limit of Woodin cardinals, it is consistent that ZFC + all games on natural numbers of length ω_1 with payoff sets that are definable from real and ordinal parameters are determined. He then used such determinacy to prove the aforementioned theorem on CM^+.

Many questions about the supercompactness or strong compactness of ω_1 are still open and seem crucial for a proper understanding of the connection between inner models and determinacy axioms. The four papers under review could be good starting points for anyone interested in tackling such questions.

Takehiko Gappo
Institut Für Diskrete Mathematik Und Geometrie, Tu Wien, Wiedner Hauptstrasse 8-10/104, 1040 Wien, Austria. E-mail: takehiko.gappo@tuwien.ac.at

Christopher Pincock. ***Mathematics and Explanation***. Elements in the Philosophy of Mathematics. Cambridge University Press, Cambridge, UK, 2023, 80 pp.

Can mathematics play an explanatory role in science? Can mathematics play an explanatory role 'internally', that is, within mathematics itself? Although these questions may be seen as referring to two independent areas of research, namely that concerning mathematical explanations in science (also called by Pincock 'genuine mathematical explanations') and that relative to mathematical explanations in pure mathematics, they both revolve around the idea that mathematics can disclose the reasons why something (an empirical phenomenon or a mathematical fact) is the way it is. Thus, philosophers of mathematics usually consider them as pertaining to a broader, unified field of research focused on the nature of mathematical explanation. In ***Mathematics and Explanation***, Christopher Pincock provides a comprehensive and critical overview of this field of research. Furthermore, he goes beyond existing studies on mathematical explanation by proposing novel ideas and questions.

The book is organised into five sections. The first section serves as a brief introduction in which Pincock outlines the general architecture of ***Mathematics and Explanation***. The introductory section also provides insight into a crucial aspect of Pincock's approach to the philosophy of mathematics, namely his attempt to 'attend carefully to mathematical and scientific practice in philosophical work' (p. 2). What does this attempt amount to? In line with his previous works (e.g., his 2012 book ***Mathematics and Scientific Representation***), Pincock pursues an epistemology of mathematics that is sensitive to actual mathematical and scientific practice. His work on mathematical explanation can therefore be situated within a broader trend in current philosophy of mathematics, known as the 'philosophy of mathematical practice', which opts for a bottom-up methodology that draws particular attention to the way(s) in which mathematics is actually practiced (see P. Mancosu (Ed.), ***The Philosophy of Mathematical Practice***. Oxford University Press, 2008).