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Nontrivial invariant subspaces of linear operator pencils

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Abstract. In this paper, we introduce the spherical polar decomposition of the linear pencil of an ordered pair $\mathbf{T} = (T_1, T_2)$ and investigate nontrivial invariant subspaces between the generalized spherical Aluthge transform of the linear pencil of $\mathbf{T}$ and the linear pencil of the original pair $\mathbf{T}$ of bounded operators with dense ranges.

1 Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and let $\mathcal{B}(\mathcal{H}, \mathcal{K})$ be the algebra of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. If $\mathcal{H} = \mathcal{K}$, we write $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H}, \mathcal{H})$. For an operator $S \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, the kernel of S is denoted by $\ker(S)$ and the range of S is denoted by $\operatorname{ran}(S)$. The *linear pencil* of an ordered pair $\mathbf{T} = (T_1, T_2)$ of operators T_1 and T_2 in $\mathcal{B}(\mathcal{H})$ is defined by $\{T_1 - \lambda T_2 : \lambda \in \mathbb{C}\}$, and the *linear pencil* of $\mathbf{T} = (T_1, T_2)$ at $\lambda \in \mathbb{C}$ is defined by $\mathbb{T}_\lambda := T_1 - \lambda T_2$. (For a detailed discussion of the linear pencil of $\mathbf{T}$, the reader may refer to [3, 7].) A subspace $\mathcal{M} \subseteq \mathcal{H}$ is called a nontrivial invariant subspace (NIS) of the pencil of $\mathbf{T}$ if $\mathcal{M} \neq \{0\}, \mathcal{H}$ and $\mathbb{T}_\lambda \mathcal{M} \subseteq \mathcal{M}$ for any $\lambda \in \mathbb{C}$.

In this paper, we introduce the spherical polar decomposition of the linear pencil of an ordered pair $\mathbf{T} = (T_1, T_2)$ and investigate nontrivial invariant subspaces between the generalized spherical Aluthge transform of the linear pencil of $\mathbf{T}$ and the linear pencil of the original pair $\mathbf{T}$ of bounded operators with dense ranges. We briefly state our main results. In Theorem 1.5, we show that for a pair $\mathbf{T}$ of operators with dense range and for $0 \le t \le 1$, if the linear pencil of $\mathbf{T}$ has an NIS, then the generalized spherical Aluthge transform of the linear pencil of $\mathbf{T}$ has also an NIS. In Theorem 1.7, we show that the converse of the result in Theorem 1.5 is true when $T = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$ is bounded below. Next, in Theorem 1.8, we show that for $0 \le t \le 1$, $\{T_1 - \lambda T_2 : \lambda \in \mathbb{C}\}$ has an NIS if and only if the generalized spherical Aluthge transform $\{\widehat{T_1^t} - \lambda \widehat{T_2^t} : \lambda \in \mathbb{C}\}$ has an NIS, when $\mathbf{T} = (T_1, T_2)$ is a commuting pair of operators with dense ranges. Next, in Theorem 1.9, we can show that for $0 \le t \le 1$, $\widehat{\mathbf{T}}^t$ is a commuting pair and $\widehat{\mathbf{T}}^t$ has a nontrivial joint invariant subspace (NJIS) if and only if $\mathbf{T}$ does, when $\mathbf{T} = (T_1, T_2)$ is a

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commuting pair of operators with dense ranges. In Theorem 1.9, we first employ a tool and technique which connects a relation between an NJIS and NIS in multivariable and single operator theories to obtain a new proof of the known results in [8, 12, 13]. Finally, in Example 1.11, we give a partial answer to Conjecture 1.10 in [13, Conjecture 2.29].

We now introduce some definitions and terminology for our paper. The polar decomposition of a bounded linear operator S in $\mathcal{B}(\mathcal{H},\mathcal{K})$ is a canonical factorization $S = U|S|$, where $|S| = \sqrt{S^*S}$ is a positive operator, U is a partial isometry with $(\ker S)^\perp$ as its initial space and $\overline{\operatorname{ran} S}$, the closure of $\operatorname{ran} S$, as its final space, and $\ker S = \ker U = \ker|S|$. It is known that if $S = WP$, where P is positive and W is a partial isometry with $\ker W = \ker P$, then $P = |S|$ and $U = W$. The polar decomposition for the linear pencil of an ordered pair $\mathbf{T} = (T_1, T_2)$ needs to be defined uniformly for all operators in $\{T_1 - \lambda T_2 : \lambda \in \mathbb{C}\}$. For this, we observe the polar decomposition for the operator T such that

$$T = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = VP : \mathcal{H} \to \mathcal{H} \oplus \mathcal{H},$$

where $T = VP$ is the polar decomposition of T, $V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ is a partial isometry from $\mathcal{H}$ to $\mathcal{H} \oplus \mathcal{H}$, and $P = |T| = (T^*T)^{\frac{1}{2}} = (T_1^*T_1 + T_2^*T_2)^{\frac{1}{2}}$ is the positive operator on $\mathcal{H}$ (see [5, 6, 12, 13]). Now we define the polar decomposition of the linear pencil of an ordered pair $\mathbf{T} = (T_1, T_2)$ by using the polar decomposition of T in $\mathcal{B}(\mathcal{H}, \mathcal{H} \oplus \mathcal{H})$.

Definition 1.1 The polar decomposition of the linear pencil of an ordered pair $\mathbf{T} = (T_1, T_2)$ is the linear pencil of the ordered pair (V_1P, V_2P), i.e.,

$$\{V_1P - \lambda V_2P = (V_1 - \lambda V_2)P : \lambda \in \mathbb{C}\}.$$

Recall the *Aluthge transform* $\widetilde{S} := |S|^{\frac{1}{2}}U|S|^{\frac{1}{2}}$, the *generalized Aluthge transform* $\widetilde{S}^t := |S|^tU|S|^{1-t}$ $(0 \le t \le 1)$, and the *Duggal transform* $\widetilde{S}^D := |S|U$ of $S = U|S| \in \mathcal{B}(\mathcal{H})$. These transformations have received considerable attention in recent years. For more details, the reader is referred to [1, 2, 4, 10, 11, 13].

Let $\mathbf{T} = (T_1, T_2) \equiv (V_1P, V_2P)$ be a pair of operators, where P, V_1, and V_2 are given above. Naturally, we can get the *spherical polar decomposition* of a pair of operators $\mathbf{T} = (T_1, T_2)$ as follows [12, 13]:

$$\mathbf{T} = (T_1, T_2) \equiv (V_1P, V_2P).$$

Then, for $0 \le t \le 1$, the *generalized spherical Aluthge transform* $\widehat{\mathbf{T}}^t$ is defined by

$$\widehat{\mathbf{T}}^t = (\widehat{T}_1^t, \widehat{T}_2^t) = (P^tV_1P^{1-t}, P^tV_2P^{1-t}).$$

In particular, when $t = \frac{1}{2}$ (resp. $t = 1$), we call the *spherical Aluthge (resp. Duggal) transform* $\widehat{\mathbf{T}}$ *(resp.* $\widehat{\mathbf{T}}^D$*) of* $\mathbf{T}$, that is,

$$\widehat{\mathbf{T}} := (\widehat{T}_1, \widehat{T}_2) \equiv (P^{\frac{1}{2}}V_1P^{\frac{1}{2}}, P^{\frac{1}{2}}V_2P^{\frac{1}{2}}) \text{ (resp. } \widehat{\mathbf{T}}^D := (\widehat{T}_1^D, \widehat{T}_2^D) \equiv (PV_1, PV_2)).$$

Now we define the generalized spherical Aluthge transform of the linear pencil of $\mathbf{T} = (T_1, T_2)$.

Definition 1.2 For $0 \le t \le 1$, the generalized spherical Aluthge transform of the linear pencil of $\mathbf{T} = (T_1, T_2)$ is defined as the linear pencil of $\widehat{\mathbf{T}}^t = (\widehat{T}_1^t, \widehat{T}_2^t)$, that is,

$$\{\widehat{T}_1^t - \lambda \widehat{T}_2^t : \lambda \in \mathbb{C}\} = \left\{P^t(V_1 - \lambda V_2)P^{1-t} : \lambda \in \mathbb{C}\right\}.$$

Also, the generalized spherical Aluthge transform $\widehat{\mathbb{T}}_\lambda^t$ at $\lambda \in \mathbb{C}$ of the linear pencil of $\mathbf{T} = (T_1, T_2)$ is defined by

$$\widehat{\mathbb{T}}_\lambda^t = \widehat{T}_1^t - \lambda \widehat{T}_2^t = P^t(V_1 - \lambda V_2)P^{1-t}.$$

When $t = \frac{1}{2}$ (resp. $t = 1$), we will denote $\widehat{\mathbb{T}}_\lambda \equiv \widehat{\mathbb{T}}_\lambda^{\frac{1}{2}}$ (resp. $\widehat{\mathbb{T}}_\lambda^D \equiv \widehat{\mathbb{T}}_\lambda^1$) and call the *spherical Aluthge transform* (resp. the *spherical Duggal transform*) of the linear pencil of $\mathbf{T} = (T_1, T_2)$ at λ, i.e.,

$$\widehat{\mathbb{T}}_\lambda = P^{\frac{1}{2}}(V_1 - \lambda V_2)P^{\frac{1}{2}} \ (\text{resp. } \widehat{\mathbb{T}}_\lambda^D = P(V_1 - \lambda V_2)).$$

It is known that the Aluthge transform has a natural connection with the invariant subspace problem, because every normal operator has nontrivial invariant subspaces and Aluthge transform is to convert an operator into another operator which shares with the first one many spectral properties, but which is closer to being a normal operator. However, for an infinite-dimensional Hilbert space $\mathcal{H}$, one needs to remember the classical example of Exner in [9], who proved that, even for weighted shifts, subnormality is not preserved by the Aluthge transform; in this case, the transformed shift is farther from normal than the original one. In recent years, Jung, Ko, and Pearcy proved in [11] that an operator $S \in \mathcal{B}(\mathcal{H})$ with dense range has a nontrivial invariant subspace if and only if $\widetilde{S}$ does. In [8, 12, 13], the authors extended the above result to the generalized Aluthge transform and the generalized spherical Aluthge transform for commuting pairs of operators. We now consider a relation between the invariant subspaces for the linear pencil of $\mathbf{T} = (T_1, T_2)$ and those for the linear pencil of its generalized Aluthge transform.

We let $\operatorname{Lat}(S)$ be the set of common invariant subspaces for S. First, we have the following.

Proposition 1.3 *Let* $\mathbf{T} = (T_1, T_2)$ *be a pair of operators. Then* $\operatorname{Lat}\{T_1 - \lambda T_2 : \lambda \in \mathbb{C}\} = \operatorname{Lat} T_1 \cap \operatorname{Lat} T_2$.

Proof $(\subseteq)$: Let $\mathcal{M} \in \operatorname{Lat}\{T_1 - \lambda T_2 : \lambda \in \mathbb{C}\}$. Then $(T_1 - 0 \cdot T_2)(\mathcal{M}) \subseteq \mathcal{M}$, i.e., $T_1\mathcal{M} \subseteq \mathcal{M}$. Let $x \in \mathcal{M}$. Then $(T_1 - \lambda T_2)x \in \mathcal{M}$ for all $\lambda \in \mathbb{C}$, which implies that $\left(\frac{1}{\lambda}T_1 - T_2\right)x \in \mathcal{M}$ for all $\lambda \neq 0$. Since $\mathcal{M}$ is closed, letting $\lambda \to \infty$, $-T_2x \in \mathcal{M}$, i.e., $T_2x \in \mathcal{M}$. Thus, $T_2\mathcal{M} \subseteq \mathcal{M}$. Therefore, we have that $\mathcal{M} \in \operatorname{Lat} T_1 \cap \operatorname{Lat} T_2$.
$(\supseteq)$: Let $\mathcal{N} \in \operatorname{Lat} T_1 \cap \operatorname{Lat} T_2$. Then $T_1\mathcal{N} \subseteq \mathcal{N}$ and $T_2\mathcal{N} \subseteq \mathcal{N}$. Let $x \in \mathcal{N}$. Then $T_1x, T_2x \in \mathcal{N}$. So $(T_1 - \lambda T_2)x \in \mathcal{N}$, i.e., $(T_1 - \lambda T_2)(\mathcal{N}) \subseteq \mathcal{N}$ for all $\lambda \in \mathbb{C}$. Therefore, $\mathcal{N} \in \operatorname{Lat}\{T_1 - \lambda T_2 : \lambda \in \mathbb{C}\}$. ■

Proposition 1.4 *For a pair of operators* $\mathbf{T} = (T_1, T_2) = (V_1 P, V_2 P)$ *and* $0 \le t \le 1$*, we have:*

(i) $P^t \mathbb{T}_\lambda = \widehat{\mathbb{T}}_\lambda^t P^t$ *for* $\lambda \in \mathbb{C}$.

(ii) $(V_1 - \lambda V_2) P^{1-t} \widehat{\mathbb{T}}_\lambda^t = \mathbb{T}_\lambda (V_1 - \lambda V_2) P^{1-t}$ *for* $\lambda \in \mathbb{C}$.

Proof (i) and (ii) are clear from direct calculations. ∎

Next, we study nontrivial invariant subspaces between the linear pencil of $\mathbf{T}$ and the linear pencil of its generalized Aluthge transform.

Theorem 1.5 *Let* $\mathbf{T} = (T_1, T_2)$ *be a pair of operators with dense range. For* $0 \le t < 1$*, we have that if the linear pencil of* $\mathbf{T}$ *has an NIS, then the generalized spherical Aluthge transform of the linear pencil of* $\mathbf{T}$ *has an NIS.*

Proof For $t = 0$, the desired one is clear.

For $0 < t < 1$, suppose that $\ker P \ne \{0\}$. Then $\ker P \ne \mathcal{H}$, because $T_1 \ne 0$ and $T_2 \ne 0$. Since $\ker P = \ker T_1 \cap \ker T_2 = \bigcap_{\lambda \in \mathbb{C}} \ker(T_1 - \lambda T_2)$, $\ker P \in \operatorname{Lat} T_1 \cap \operatorname{Lat} T_2$, and by Proposition 1.3, $\ker P \in \operatorname{Lat}\{T_1 - \lambda T_2 : \lambda \in \mathbb{C}\}$, i.e., the linear pencil of $\mathbf{T}$ has a nontrivial invariant subspace. On the other hand, since $\ker P^{1-t} = \ker P \ne \{0\}, \mathcal{H}$ and $\ker P^{1-t} \in \operatorname{Lat}\{\widehat{T_1^t} - \lambda \widehat{T_2^t} : \lambda \in \mathbb{C}\}$, the generalized spherical Aluthge transform of the linear pencil of $\mathbf{T}$ has a nontrivial invariant subspace. So, we may assume that $\ker P = \{0\}$. Let $\mathcal{M} \in \operatorname{Lat}\{T_1 - \lambda T_2 : \lambda \in \mathbb{C}\}$ be nontrivial. Consider $\mathcal{N} = \overline{P^t \mathcal{M}}$, where $\overline{P^t \mathcal{M}}$ means the smallest closed set containing $P^t \mathcal{M}$. Since $\ker P = \{0\}$, we have $\mathcal{N} \ne \{0\}$. Suppose that $\mathcal{N} = \mathcal{H}$, i.e., $\overline{P^t \mathcal{M}} = \mathcal{H}$. Since T_i $(i = 1, 2)$ has dense range, V_i $(i = 1, 2)$ also has dense range. Since $\ker P^{1-t} = \ker P = \{0\}$, P^{1-t} has dense range, and so $V_i P^{1-t}$ has dense range. Thus, we have that

$$\mathcal{H} = \overline{V_i P^{1-t} \mathcal{N}} = \overline{V_i P^{1-t} (P^t \mathcal{M})} = \overline{V_i P \mathcal{M}} = \overline{T_i \mathcal{M}} \subset \mathcal{M} \ne \mathcal{H}, \tag{1.1}$$

which is a contradiction. Hence, $\mathcal{N} \ne \mathcal{H}$, which means that $\mathcal{N}$ is nontrivial. Now, by Proposition 1.4(i), note that, for $\lambda \in \mathbb{C}$,

$$\widehat{\mathbb{T}}_\lambda^t P^t \mathcal{M} = P^t \mathbb{T}_\lambda \mathcal{M} \subset P^t \mathcal{M} \Longrightarrow \widehat{\mathbb{T}}_\lambda^t \mathcal{N} \subset \mathcal{N}.$$

Therefore, $\mathcal{N} \in \operatorname{Lat}\{\widehat{T_1^t} - \lambda \widehat{T_2^t} : \lambda \in \mathbb{C}\}$ and we have the desired one.

For $t = 1$, by the proof of the case for $0 < t < 1$, we can assume that $\ker P = \{0\}$. Next, for a nontrivial $\mathcal{M}$ in $\operatorname{Lat}\{T_1 - \lambda T_2 : \lambda \in \mathbb{C}\}$, we consider $\mathcal{L} = \overline{P\mathcal{M}}$. Then $\mathcal{L} \ne \{0\}$. Suppose that $\mathcal{L} = \mathcal{H}$, i.e., $\overline{P\mathcal{M}} = \mathcal{H}$. Since V_i $(i = 1, 2)$ also has dense range, by (1.1) when $t = 1$, we have that

$$\mathcal{H} = \overline{V_i \mathcal{L}} = \overline{V_i (P\mathcal{M})} = \overline{T_i \mathcal{M}} \subset \mathcal{M} \ne \mathcal{H},$$

which is also a contradiction. Hence, $\mathcal{L}$ is nontrivial. Now, by Proposition 1.4(i), note that, for $\lambda \in \mathbb{C}$,

$$\widehat{\mathbb{T}}_\lambda^D P\mathcal{M} = P \mathbb{T}_\lambda \mathcal{M} \subset P\mathcal{M} \Longrightarrow \widehat{\mathbb{T}}_\lambda^D \mathcal{N} \subset \mathcal{N}.$$

Therefore, $\mathcal{N} \in \operatorname{Lat}\{\widehat{T_1^t} - \lambda \widehat{T_2^t} : \lambda \in \mathbb{C}\}$ and our proof is now completed. ∎

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The converse of the result in Theorem 1.5 is true when $T = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = VP = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} P$is bounded below, i.e., if T is bounded below, then we have that the linear pencil of $\mathbf{T}$ has an NIS if and only if the generalized spherical Aluthge transform of the linear pencil of $\mathbf{T}$ has an NIS. For this, we first consider the following.

Proposition 1.6 *If T is bounded below, then P is invertible.*

Proof Since T is bounded below, there exists $c > 0$ such that $\|Tx\| = \|VPx\| \geq c\|x\|$ for all $x \in \mathcal{H}$. Thus, for all $x \in \mathcal{H}$,

$$\begin{aligned} \|VPx\| \geq c\|x\| &\Longrightarrow \|V\|\,\|Px\| \geq c\|x\| \\ &\overset{V \text{ is a partial isometry}}{\Longrightarrow} \|Px\| \geq c\|x\|. \end{aligned}$$

Therefore, P is bounded below. Since P is positive, P is invertible, as desired. ∎

Now we have the following.

Theorem 1.7 *Let T be bounded below. Then, for $0 \leq t \leq 1$, we have that the linear pencil $\{T_1 - \lambda T_2 : \lambda \in \mathbb{C}\}$ has an NIS if and only if the generalized spherical Aluthge transform*

$$\{\widehat{T_1^t} - \lambda \widehat{T_2^t} : \lambda \in \mathbb{C}\}$$

has an NIS.

Proof $(\Longrightarrow)$ It is clear from Theorem 1.5.
$(\Longleftarrow)$ For $t = 0$, the desired one is clear.

For $0 < t \leq 1$, let $\mathcal{N} \in \mathrm{Lat}\{\widehat{T_1^t} - \lambda \widehat{T_2^t} : \lambda \in \mathbb{C}\}$ be nontrivial. Since T is bounded below, by Proposition 1.6, P is invertible. So, $\mathcal{R} = P^{-t}\mathcal{N}$ is nontrivial and closed. Since $\widehat{\mathbb{T}_\lambda^t}\mathcal{N} \subset \mathcal{N}$, we have that $P^t(V_1 - \lambda V_2)P^{1-t}\mathcal{N} \subset \mathcal{N}$. Since P^t is invertible, $(V_1 - \lambda V_2)P^{1-t}\mathcal{N} \subset P^{-t}\mathcal{N}$, and so $(V_1 - \lambda V_2)P(P^{-t}\mathcal{N}) \subset P^{-t}\mathcal{N}$, i.e., $\mathbb{T}_\lambda\mathcal{R} \subset \mathcal{R}$. Therefore, the linear pencil $\{T_1 - \lambda T_2 : \lambda \in \mathbb{C}\}$ has an NIS, as desired. ∎

Theorem 1.8 *Let $\mathbf{T} = (T_1, T_2)$ be a commuting pair of operators with dense ranges. Then, for $0 \leq t \leq 1$, the linear pencil $\{T_1 - \lambda T_2 : \lambda \in \mathbb{C}\}$ has an NIS if and only if the generalized spherical Aluthge transform $\{\widehat{T_1^t} - \lambda \widehat{T_2^t} : \lambda \in \mathbb{C}\}$ has an NIS.*

Proof $(\Longrightarrow)$ It is clear from Theorem 1.5.
$(\Longleftarrow)$ Suppose that $\ker T_1 \neq \{0\}$. Since $\mathbf{T} = (T_1, T_2)$ is a commuting pair of operators, for all $x \in \ker T_1$, $T_1(T_2 x) = T_2(T_1 x) = 0$. So $T_2(\ker T_1) \subseteq \ker T_1$. Hence, $(T_1 - \lambda T_2)(\ker T_1) \subseteq \ker T_1$ for all $\lambda \in \mathbb{C}$ and $\ker T_1 \in \mathrm{Lat}\{T_1 - \lambda T_2 : \lambda \in \mathbb{C}\}$. Similarly, if $\ker T_2 \neq \{0\}$, then $\ker T_2 \in \mathrm{Lat}\{T_1 - \lambda T_2 : \lambda \in \mathbb{C}\}$. Thus, we may assume that T_1 and T_2 are both quasiaffinities.

For $t = 0$, the desired one is clear.

For $0 < t \leq 1$, let $\mathcal{M}$ be an NIS for the linear pencil of the generalized spherical Aluthge transform $\{\widehat{T_1^t} - \lambda \widehat{T_2^t} : \lambda \in \mathbb{C}\}$. Let $\mathcal{L} = \overline{V_1 P V_2 P^{1-t}\mathcal{M}}$. Then $\mathcal{L} \neq \mathcal{H}$. Indeed, if so, then $P^t\mathcal{L}$ is dense in $\mathcal{H}$, since

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$$\ker P^t = \ker P = \ker T_1 \cap \ker T_2 = \{0\}. \tag{1.2}$$

Let $x \in V_1PV_2P^{1-t}\mathcal{M}$ with $x \neq 0$. Then there is $z \in \mathcal{M}$ such that $x = V_1PV_2P^{1-t}z$. Since $z \in \mathcal{M}$, $(\widehat{T_1^t} - \lambda\widehat{T_2^t})z \in \mathcal{M}$, which means that $(\frac{1}{\lambda}\widehat{T_1^t} - \widehat{T_2^t})z \in \mathcal{M}$ for $\lambda \neq 0$. Letting $\lambda \to \infty$, we have that $\widehat{T_2^t}z \in \mathcal{M}$. Of course, $\widehat{T_1^t}z \in \mathcal{M}$ when $\lambda = 0$. Since $\widehat{T_2^t}z \in \mathcal{M}$, $(\widehat{T_1^t} - \lambda\widehat{T_2^t})(\widehat{T_2^t}z) \in \mathcal{M}$ for all $\lambda \in \mathbb{C}$. Thus, we have that

$$\widehat{T_1^t}\,\widehat{T_2^t}z \in \mathcal{M}, \text{ by letting } \lambda = 0. \tag{1.3}$$

Consider $P^t x$. Then, by (1.3), we have

$$P^t x = P^t\left(V_1PV_2P^{1-t}z\right) = P^tV_1P^{1-t}P^tV_2P^{1-t}z = \widehat{T_1^t}\,\widehat{T_2^t}z \in \mathcal{M}. \tag{1.4}$$

So, by (1.4), we have $P^t\mathcal{L} \subseteq \mathcal{M} \neq \mathcal{H}$, which contradicts to the fact that $\overline{P^t\mathcal{L}} = \mathcal{H}$.

Also, $\mathcal{L} \neq \{0\}$. Indeed, if so, $V_1PV_2P^{1-t}z = 0$ for a nonzero vector $z \in \mathcal{M}$. Since $V_1PV_2P^{1-t}z = T_1\left(V_2P^{1-t}z\right) = 0$, we have $V_2P^{1-t}z = 0$ because of the quasiaffinity of T_1. Thus, we have that $P^{1-t}z \in \ker V_2$. On the other hand, since T_1 and T_2 are commuting, $V_1PV_2 = V_2PV_1$. Hence, $V_1PV_2P^{1-t}z = V_2PV_1P^{1-t}z = T_2\left(V_1P^{1-t}z\right) = 0$, so $V_1P^{1-t}z = 0$, i.e., $P^{1-t}z \in \ker V_1$. Therefore, $P^{1-t}z \in \ker V_1 \cap \ker V_2 = \{0\}$. Since $\ker P^{1-t} = \{0\}$, $z = 0$, which is a contradiction to $z \neq 0$.

Let $x \in V_1PV_2P^{1-t}\mathcal{M} \subset \mathcal{L}$ with $x \neq 0$. Then there exists a nonzero vector $z \in \mathcal{M}$ such that $x = V_1PV_2P^{1-t}z$. Note that

$$\begin{aligned}
&(T_1 - \lambda T_2)x = T_1x - \lambda T_2x \\
&= V_1P\left(V_1PV_2P^{1-t}z\right) - \lambda V_2P\left(V_1PV_2P^{1-t}z\right) \\
&\overset{V_1PV_2=V_2PV_1}{=} V_1P\left(V_2PV_1P^{1-t}z\right) - \lambda V_1PV_2PV_2P^{1-t}z \\
&= V_1PV_2P^{1-t}(\widehat{T_1^t}z) - \lambda V_1PV_2P^{1-t}(\widehat{T_2^t}z) \\
&\in V_1PV_2P^{1-t}\mathcal{M} - \lambda V_1PV_2P^{1-t}\mathcal{M} \\
&\subset \overline{V_1PV_2P^{1-t}\mathcal{M}} = \mathcal{L}.
\end{aligned}$$

Thus, $(T_1 - \lambda T_2)\mathcal{L} \subset \mathcal{L}$ for all $\lambda \in \mathbb{C}$ and we have the desired one. Therefore, our proof is now completed. ■

As a corollary of Proposition 1.3 and Theorem 1.8, we obtain new proofs of the following results in [8, Theorem 2.6] and [13, Theorem 2.24].

Theorem 1.9 ([8, Theorem 2.6], [13, Theorem 2.24]) *Let* $\mathbf{T} = (T_1, T_2)$ *be a commuting pair of operators with dense ranges. Then, for* $0 \leq t \leq 1$, $\widehat{\mathbf{T}}^t$ *has an NJIS if and only if* $\mathbf{T}$ *does.*

Proof ($\Longrightarrow$) For $t = 0$, the desired one is clear.

For $0 < t \leq 1$, we assume that $\widehat{\mathbf{T}}^t$ has an NJIS. Let $\mathcal{M} \in \operatorname{Lat}\widehat{T_1^t} \cap \operatorname{Lat}\widehat{T_2^t}$ with $\mathcal{M} \neq \{0\}, \mathcal{H}$. Then, by Proposition 1.3, we have that $\mathcal{M} \in \operatorname{Lat}\{\widehat{T_1^t} - \lambda\widehat{T_2^t} : \lambda \in \mathbb{C}\}$. Thus, by the proof of Theorem 1.8, we have that $\mathcal{L} \in \operatorname{Lat}\{T_1 - \lambda T_2 : \lambda \in \mathbb{C}\}$, where $\mathcal{L} = \overline{V_1PV_2P^{1-t}\mathcal{M}}$. By Proposition 1.3 again, we have that $\mathcal{L} \in \operatorname{Lat}T_1 \cap \operatorname{Lat}T_2$. Therefore, $\mathbf{T}$ has an NIS, as desired.

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($\Longleftarrow$) It is clear from Proposition 1.3, Theorem 1.8, and the same argument given above. ∎

Next, we recall the following open problem.

Conjecture 1.10 [13, Conjecture 2.29] *For a commuting n-tuple $\mathbf{T}$, we have that* $\mathrm{Lat}\,(\mathbf{T})$ *and* $\mathrm{Lat}(\widehat{\mathbf{T}}^t)$ $(t=1)$ *are isomorphic.*

The following result gives a partial answer to Conjecture 1.10 and more.

Example 1.11 Let R and Q be positive operators in $\mathcal{B}(\mathcal{H})$ such that $[R, Q] = RQ - QR \neq 0$ and $R+Q$ is invertible. Consider the linear pencil $\{T_1 - \lambda T_2 : \lambda \in \mathbb{C}\}$, where $(T_1, T_2) = (\sqrt{R}, \sqrt{Q})$. Then we have that for $0 \le t \le 1$, $\mathrm{Lat}(\widehat{\mathbf{T}}^t) = \mathrm{Lat}\,(\mathbf{T})$. For this, note that $T_1^* T_1 + T_2^* T_2 = R + Q$ and $P = \sqrt{R+Q}$ is invertible. Thus, the polar decomposition of the pencil is

$$\{T_1 - \lambda T_2 : \lambda \in \mathbb{C}\} = \left\{(T_1 P^{-1} - \lambda T_1 P^{-1})P : \lambda \in \mathbb{C}\right\},$$

because for $i = 1, 2$, $V_i = T_i P^{-1}$. So, the generalized Aluthge transform is

$$\left\{P^t T_1 P^{-t} - \lambda P^t T_2 P^{-t} : \lambda \in \mathbb{C}\right\}.$$

Hence, by Proposition 1.3, we have that for $0 \le t \le 1$,

$$\begin{aligned}\mathrm{Lat}(\widehat{\mathbf{T}}^t) &= \mathrm{Lat}\,\widehat{T_1}^t \cap \mathrm{Lat}\,\widehat{T_2}^t = \mathrm{Lat}\{\widehat{T_1}^t - \lambda \widehat{T_2}^t : \lambda \in \mathbb{C}\}\\ &= \mathrm{Lat}\left\{P^t T_1 P^{-t} - \lambda P^t T_2 P^{-t} : \lambda \in \mathbb{C}\right\}\\ &= \mathrm{Lat}(P^t T_1 P^{-t}) \cap \mathrm{Lat}(P^t T_2 P^{-t})\\ &= \mathrm{Lat}\,T_1 \cap \mathrm{Lat}\,T_2 = \mathrm{Lat}\,(\mathbf{T}).\end{aligned}$$

References

[1] A. Aluthge, *On p-hyponormal operators for* $0 < p < 1$. Integr. Equ. Oper. Theory **13**(1990), 307–315.

[2] C. Benhida, R. Curto, S. H. Lee, and J. Yoon, *The spectral picture and joint spectral radius of the generalized spherical Aluthge transforms*. Adv. Math. **408**(2022), 108602.

[3] S. Bōgli and M. Marletta, *Essential numerical ranges for linear operator pencils*. IMA J. of Numer. Anal. **40**(2020), 2256–2308.

[4] M. Cho, I. Jung, and W. Y. Lee, *On the iterated Duggal transform*. Kyungpook Math. J. **49**(2009), 647–650.

[5] R. Curto and J. Yoon, *Aluthge transforms of* 2*-variable weighted shifts*. Integr. Equ. Oper. Theory **90**(2018), Article no. 52, 32 pp.

[6] R. Curto and J. Yoon, *Spherical Aluthge transforms and quasinormality for commuting pairs of operators*. In: Analysis of operators on function spaces (the Serguei Shimorin memorial volume), Trends in Mathematics, Birkhäuser, Cham, 2019, pp. 213–237.

[7] S. Djordjević, J. Kim, and J. Yoon, *Spectra of the spherical Aluthge transform, the linear pencil, and a commuting pair of operators*. Linear Multilinear Algebra **70**(2022), 2047–2064.

[8] S. Djordjević, J. Kim, and J. Yoon, *Generalized spherical Aluthge transforms and binormality for commuting pairs of operators*. Math. Nachr., to appear. (in press).

[9] G. Exner, *Aluthge transforms and n-contractivity of weighted shifts*. J. Operator Theory **61**(2009), 419–438.

[10] M. Ito, T. Yamazaki, and M. Yanagida, *On the polar decomposition of the Aluthge transformation and related results*. J. Operator Theory **51**(2004), 303–319.

[11] I. Jung, E. Ko, and C. Pearcy, *Aluthge transform of operators*. Integr. Equ. Oper. Theory **37**(2000), 437–448.

[12] J. Kim and J. Yoon, *Aluthge transforms and common invariant subspaces for a commuting n-tuple of operators*. Integr. Equ. Oper. Theory **87**(2017), 245–262.

[13] J. Kim and J. Yoon, *Taylor spectra and common invariant subspaces through the Duggal and generalized Aluthge transforms for commuting n-tuples of operators*. J. Operator Theory **81**(2019), 81–105.

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