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CHARACTERS OF NON-CONNECTED, REDUCTIVE p-ADIC GROUPS

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1. Introduction. In this paper, we extend to non-connected, reductive groups over a p-adic field of characteristic zero Harish-Chandra's theorem on the local integrability of characters.

Harish-Chandra's theorem states that the distribution character of an admissible, irreducible representation of a (connected) reductive p-adic group is locally integrable. We show that this extends to any reductive group; just as in the connected case, one even gets a very precise control over the singularities of the character along the singular elements.

As will be seen, the proof in the non-connected case is an easy extension of Harish-Chandra's. The reader may wonder why we have bothered to write its generalization completely. The reason is that the original article **[8]** does not contain proofs for the crucial lemmas, and this makes it impossible to explain why the theorem extends. Because this result is needed for work of Arthur and the author on base change, it has been thought necessary to give complete arguments. We have done so only to a degree. Harish-Chandra's paper contains three separate parts, two dealing with theorems on the Lie algebra (Parts I and II), one with the group (Part III). We have relied on the self-contained parts I and II, though some proofs are missing there also. On the basis of that, we supply complete proofs for the theorems on the group. To keep the length of this paper to a minimum, we have constantly referred to the results proved in **[8]**.

Of course Harish-Chandra had completely written proofs of all the missing lemmas. It is expected that those for Parts I and II will eventually appear. The proofs we supply for Part III rely completely on his unpublished notes.

We would like to remark that the study of representation theory of non-connected groups is not an idle generalization. In fact, some of the most important applications of the trace formula to the study of automorphic forms are likely to come from the consideration of outer automorphisms. (For the trace formula in that case see **[7]**). Facts of local harmonic analysis seem to be useful there. In particular they are used in

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[1] to prove the identities of orbital integrals for Base Change; I have also used them to prove the corresponding identities for the lifting between $SL(2)$ and $PGL(3)$, simplifying an earlier proof of Langlands; this is used by Flicker in [7]. Also, some interesting questions appear in the representation theory of non-connected groups. Here is one. Harish-Chandra has conjectured that orthogonality relations, generalizing those between discrete series characters, must hold between elliptic characters [5] of reductive groups. On the other hand, in the study of base change, certain elusive "twisted orthogonality relations" appear ([**13**, Chapter 7]). These twisted orthogonality relations can be construed as (ordinary) orthogonality relations between certain elliptic characters of a non-connected group arising as the semi-direct product of a connected group by a Galois automorphism. I do not know if this may lead to a local proof of the twisted orthogonality relations. Another question is to extend to non-connected groups the "Langlands classification", the theory of Jacquet modules, etc.

The organization of this paper is as follows: Section 2 contains some preliminaries on semi-simple elements, centralizers, and Harish-Chandra's D function for non-connected groups. In Section 3, we prove the main theorems and in particular the local integrability of characters. The results (Theorems 1, 2 and 3) are stated in Section 3.3.

Harish-Chandra's untimely death does not allow me to thank him here; I can only record my gratitude for his great generosity. Besides his proof of the local integrability theorem, I have also relied on his unpublished notes on characters of non-connected real groups.

Thanks are due to Howe for indications about his representation theory of small compact p-adic groups. I have also made use of a recent paper of Rodier, where he proves the local integrability of characters of p-adic $GL(n)$ in large enough characteristics.

2. Non-connected reductive groups.

2.1 We will denote by $\mathbf{G}$ a linear algebraic group defined over a non-Archimedian local field k of characteristic 0. We say that $\mathbf{G}$ is reductive if $\mathbf{G}^0$, its neutral component, is a reductive connected group.

We denote by R_k the ring of integers of k, by $\tilde{\omega}$ a uniformizing parameter, by q the cardinal of the residue field, by p the prime divisor of q.

Let $G = \mathbf{G}(k)$ be the group of k-points of $\mathbf{G}$: it is a locally compact, totally discontinuous group. We call G a reductive k-group; if $G = \mathbf{G}(k)$ with $\mathbf{G}$ connected, we say that G is connected. We write $G^0 = \mathbf{G}^0(k)$, and often call it the *neutral component* of G. We have $G = \coprod G^i$, a finite disjoint union, where each G^i is a G^0-coset in G, of the form $\mathbf{G}^i(k)$ where $\mathbf{G}^i$ is a connected component of $\mathbf{G}$. We call each G^i a connected component of G.

We write Z_G for the center of G, where G is any reductive group.

Since $\mathbf{G}$ is linear, semi-simple and unipotent elements are defined in $\mathbf{G}(k)$ ([**2**]). Assume γ is a semi-simple element of G: then $\mathrm{Ad}(\gamma)$ is a semi-simple automorphism of $\mathbf{G}^0$. Let $\mathbf{M}$ be the neutral component of the centralizer of γ in $\mathbf{G}$ (We will write $\mathbf{G}^\gamma$ for the centralizer of γ, $\mathfrak{g}^\gamma$ for its Lie algebra). We will call $M = \mathbf{M}(k)$ the *connected centralizer* of γ. Obviously $\mathbf{M} \subset \mathbf{G}^0$, and since $\mathrm{Ad}(\gamma)$ is semi-simple, $\mathbf{M}$ is a reductive group by a theorem of Steinberg [**15**, 2.10].

2.2 *Harish-Chandra's discriminant; regular elements.* Assume $G = \coprod G^i$, a union of connected components. For all i, let r_i be the first non-zero power of T in the polynomial

$$P_i(T) = \det((T - \mathrm{Ad}(g) + 1)|_{\mathfrak{g}}), \quad (g \in G^i).$$

Here $\mathfrak{g}$ is the Lie algebra of G. Then r_i is called the rank of G^i. Notice that r_i may vary when i ranges over the connected components.

Example. Let k'/k be a finite extension of local fields. Let $\mathbf{G}$ be a connected reductive k'-group. Set

$$\mathbf{H}^0 = \mathrm{Res}_{k'/k}(\mathbf{G}),$$

the k-group obtained by restriction of scalars. Then $\Sigma = \mathrm{Gal}(k'/k)$ acts by k-automorphisms over $\mathbf{H}^0$. Let $\mathbf{H} = \mathbf{H}^0 \rtimes \Sigma$, a non-connected k-group. Assume Σ is cyclic of order l, generated by σ. Then, if r is the rank of $\mathbf{G}$ over k':

$$\mathrm{rank}(H^0) = rl$$

$$\mathrm{rank}(H^0 \times \sigma) = r$$

as an easy computation shows.

We define the discriminant function D_G on G componentwise, by putting $D_G(g)$ equal to the coefficient of T^{r_i} in $P_i(T)$ if $g \in G^i$.

An element γ of G will be called *regular* if $D_G(\gamma) \neq 0$. We denote by G_{reg} the set of regular elements of G.

LEMMA 1. *Assume γ is regular. Then γ is semi-simple and the neutral component of its centralizer is a torus.*

Remark. For connected groups it is well known that the converse to Lemma 1 is true.

Proof of Lemma 1. By definition, γ is regular if the multiplicity of the eigenvalue 1 in $\mathrm{Ad}(g)$, g ranging over the connected component of γ, is minimal at γ. Let $\gamma = \sigma\nu$, σ semi-simple, ν unipotent, be the Jordan decomposition of γ. Then the multiplicity of 1 in $\mathrm{Ad}(\gamma)$ is at least equal to the dimension of $\mathfrak{g}^\sigma$. Thus, if γ is regular, σ is regular also. Now assume σ is regular semi-simple. Let M be the neutral component of G^σ. We must show that M is a torus. Indeed, if that was false, there would be a unipotent element $u \in M$ with $u \neq 1$. If $g \in G$ centralizes σu, it must centralize σ and u by unicity of the Jordan decomposition. Thus $G^{\sigma u}$ is the centralizer of u in G^{σ}, whose dimension is smaller than that of G^{σ}: this contradicts the regularity of σ. This shows that M must be a torus. Now if $\gamma = \sigma\nu$ is regular, we have seen that its semi-simple part σ is regular: then ν must belong to the torus M, which shows that γ is in fact semi-simple.

Assume $\gamma \in G$ is regular. Then, by Lemma 1, it is semi-simple and thus diagonalizable (on an algebraic closure). Let $\mathfrak{t} = \mathfrak{g}^{\gamma}$ be the space where $\mathrm{Ad}(\gamma)$ has eigenvalue 1. Then $1 - \gamma$ is bijective on $\mathfrak{g}/\mathfrak{t}$ and it is easy to prove the following result:

LEMMA 2. *Assume γ regular*, $\mathfrak{t} = \mathfrak{g}^{\gamma}$. *Then*

$$D_G(\gamma) = \det((1 - \gamma)|_{\mathfrak{g}/\mathfrak{t}}).$$

2.3 *Representation theory*. The group G is a locally compact, totally discontinuous group, and the elementary results of representation theory of such groups contained, for example, in [**3**] apply. In particular, we have the notion of admissible representation. We will use the habitual notations: for example, if (π, V) is an admissible representation of G, $K \subset G$ a compact-open subgroup, V^K denotes the vectors fixed by K, $\widetilde{V}$ the admissible dual.

3. Local integrability of characters.

3.1 *Representations of compact groups* ([**8**, Section 13]). If K is a compact group, we denote by $\mathscr{E}(K)$ the dual of K. If K_1, K_2 are two compact subgroups of a same group G, and F_i ($i = 1, 2$) is a finite subset of $\mathscr{E}(K_i)$, we say that F_1, F_2 *interact* if there is a common representation in their restrictions to $K_1 \cap K_2$. We say that $x \in G$ *intertwines* F_1 and F_2 if F_1 interacts with $\mathrm{Ad}(x)F_2$, defined in the obvious way as a subset of $\mathscr{E}(\mathrm{Ad}(x)K_2)$. We write $[F_1 : F_2] \neq 0$ or $[F_1 : F_2] = 0$ according as F_1 and F_2 do, or do not, interact.

Obvious properties of these notions are listed in [**8**, Section 13]. We record the following one. For K, F as above, let $\mathscr{A}(F)$ be the space of functions on K of type F.

LEMMA 3 ([**1**, Corollary to Lemma 31]). *Assume $x \in G$. Let f be a complex function on $K_1 x K_2$ such that the function*

$$(k_1, k_2) \mapsto f(k_1 x k_2)$$

lies in $\mathscr{A}(F_1) \otimes \mathscr{A}(F_2)$. Then, if $f \neq 0$, x intertwines F_2 and F_1.

If μ is an irreducible representation of K, we will denote by ξ_μ its character; we write

$$\xi_F = \sum_{\mu \in F} \xi_\delta$$

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for a finite $F \subset \mathscr{E}(K)$. We will use without comment the notation K to denote a compact-open subgroup of a topological group G.

3.2 *Admissible distributions.* We now assume that G satisfies the assumptions of Section 2. We recall from [**8**, Section 14] the definition of admissible distributions. If U is open in G, Θ a distribution on U, $K_0 \subset G$ a compact open subgroup, Θ is (G, K_0)-admissible at $\gamma \in U$ if

(1) $\gamma K_0 \subset U$.

(2) If K is an open-compact subgroup of K_0, and $\mu \in \mathscr{E}(K)$, $\Theta * \xi_\mu = 0$ on γK_0 unless γ intertwines μ and the trivial representation 1_{K_0} of K_0.

We say that Θ is *admissible* at γ if it is (G, K_0)-admissible at γ for some K_0. An admissible distribution is one that is admissible at each point of U.

LEMMA 4. *Let π be an irreducible admissible representation of G on a space V. Let K_0 be such that $V^{K_0} \neq \{0\}$. Then the character Θ_π is (G, K_0)-admissible at each point.*

Proof. Assume $K \subset K_0$. Assume that, for $\mu \in \mathscr{E}(K)$, $\Theta * \xi_\mu$ is not identically 0. This implies that μ occurs in the restriction of π to K. Let $v \in V$ be of type μ, $v \neq 0$. Since V is generated by V^{K_0}, we have

$$v = \sum \pi(g_i)v_i, \quad v_i \in V^{K_0}.$$

So for some i, we must have

$$\langle w, \pi(g_i)v_i \rangle \neq 0$$

with $w \in \widetilde{V}$ of type μ, such that $\langle w, v \rangle = 1$. But then, applying Lemma 3 to the function

$$f(k_1 g_i k_2) = \langle w, \pi(k_1 g_i k_2)v_i \rangle = \langle \widetilde{\pi}(k_1)w, \pi(g_i k_2)v_i \rangle$$

of $k_1 \in K$, $k_2 \in K_0$, we see that g_i intertwines μ and 1_{K_0}.

3.3 *Statement of results.*

THEOREM 1. *Let π be an admissible irreducible representation of G. Then the character Θ_π of π is a locally integrable function on G, locally constant on G_{reg}. Moreover, the function*

$$|D_G|^{1/2}\Theta_\pi$$

is locally bounded on G.

This will follow from the following two results:

THEOREM 2. *Let Θ be an admissible distribution on an open, G-invariant subset U of G. Let γ be a semi-simple element of U. Then, if Θ is admissible at γ, it coincides with a locally integrable function around γ.*

THEOREM 3. *Under the assumptions of Theorem* 2, *let M be the connected component of the centralizer of γ in G, $\mathfrak{m}$ its Lie algebra. Then there exist unique complex numbers $c_{\mathcal{O}}$ such that, for Y close to zero in $\mathfrak{m}$*

$$\Theta(\gamma \exp Y) = \sum c_{\mathcal{O}} \hat{\nu}_{\mathcal{O}}(Y).$$

Here $\mathcal{O}$ ranges over nilpotent M-orbits in $\mathfrak{m}$, $\nu_{\mathcal{O}}$ is the invariant measure on $\mathcal{O}$, and $\hat{\nu}_{\mathcal{O}}$ its Fourier transform.

3.4 *Reduction to Lemma* 6: *Descent.* From now on, let Θ, U, γ be as in Theorem 2. Let M be the neutral component of the centralizer of γ in G (or G^0): then M is a connected reductive group. Let $\mathfrak{m}$ be its Lie algebra,

$$\mathfrak{q} = (\mathrm{Ad}(\gamma) - 1)\mathfrak{g}:$$

then $\mathfrak{g} = \mathfrak{m} + \mathfrak{q}$, a direct sum. Define, for $m \in M$,

$$D_{G/M}(m) = \det(\mathrm{Ad}\, m - 1)\,|_{\mathfrak{g}/\mathfrak{m}}.$$

Let $M' \subset M$ be defined by

$$D_{G/M}(\gamma m) \neq 0, \quad U_M = M' \cap \gamma^{-1}U.$$

Then M' is an open, M-invariant neighborhood of 1 in M, and the map

$$(x, m) \mapsto x\gamma m x^{-1}$$

$$G \times U_M \to U$$

is submersive ([**14**, Proposition 1]). By integration along the fibers, Θ defines an M-invariant distribution, θ, on U_M.

Roughly speaking, the idea of the proof is to show that θ is close to being admissible; and, by descent to the Lie algebra, to show that an admissible character has the required properties near the identity. However, the route followed for that is quite tortuous.

Let $K_0 \subset G$ be such that Θ is (G, K_0)-admissible at γ. We fix a compact-open subgroup K_M of M such that $K_M \subset U_M \cap K_0$. As before, K is any open subgroup of K_0. Let F_i ($i \geqq 1$) be finite subsets of $\mathcal{E}(K)$, disjoint and exhausting $\mathcal{E}(K)$. We set

$$\Theta_\mu = \deg(\mu)\Theta * \xi_\mu,$$

where ξ_μ is the character of $\mu \in \mathcal{E}(K)$,

$$\Theta_{F_i} = \sum_{\mu \in F_i} \Theta_\mu.$$

The same notations apply to θ and $\delta \in \mathcal{E}(K_M)$. It is then easy to prove ([**8**, Lemma 36]):

LEMMA 5. *Let $\delta \in \mathcal{E}(K_M)$. Then $\theta(\overline{\xi}_\delta) = 0$ unless there is i such that*:
1) *δ interacts with F_i*
2) *$\Theta_{F_i}(\gamma m) \neq 0$ for some $m \in K_M$.*

Now let $\mathfrak{m}_0$ be an invariant, open, closed subset of $\mathfrak{m}$ satisfying the following conditions:

(1) $R_k\mathfrak{m}_0 = \mathfrak{m}_0$; the exponential mapping is an analytic isomorphism of $\mathfrak{m}_0$ onto an open subset M_0 of U_M.

(2) $M_0 \cap QZ_M$ is compact for any compact $Q \subset M$.

(3) $|D_M(\exp X)| = |\eta_M(X)|$ for $X \in \mathfrak{m}_0$ where η_M is the discriminant in the Lie algebra ([**8**, Section 1]). We may restrict the distribution θ to M_0, pull it back to $\mathfrak{m}_0$, and extend it by 0 to $\mathfrak{m}$: whence a distribution θ_0 on $\mathfrak{m}$. The core of the proof will be to deduce from Lemma 5 the following assertion. Let $|\ \ |$ be a p-adic norm on $\mathfrak{g}$: we denote also by $|\ \ |$ its restriction to $\mathfrak{m}$. Let $\mathcal{N}$ be the nilpotent cone in $\mathfrak{g}$, S the set $|X| = 1$. Let V be a neighborhood of $\mathcal{N} \cap S \cap \mathfrak{m}$ in $S \cap \mathfrak{m}$. Let Λ be a lattice in $\mathfrak{m}$ that is small (see Section 3.5) and well-adapted with respect to M ([**8**, Section 12]). If B is a non-degenerate, invariant bilinear form on $\mathfrak{m}$, χ a non-trivial character of k, they define a dual lattice $\Lambda^* \subset \mathfrak{m}$. If $Z \in \mathfrak{m}$, let f_Z be the characteristic function of $Z + \Lambda^*$.

LEMMA 6. *If the integer ν is large enough, the following property holds. Assume $Z \in \mathfrak{m}$ is such that $|Z| > q^\nu$, and $Z \notin kV$. Then, $\hat{\theta}_0$ being the Fourier transform of θ_0:*

$$\hat{\theta}_0(f_Z) = 0.$$

(The Fourier transform is defined by means of $\chi(B(X, Y))$, so it is a distribution on $\mathfrak{m}$.)

We now prove Theorems 1, 2, and 3, relying on Lemma 6. If $t > 0$, and V is as before, let $J(V, t, \Lambda^*)$ be the space of invariant distributions T on $\mathfrak{m}$ such that, if $Z \in \mathfrak{m}$ and $|Z| \geqq t$, then $T(f_Z) = 0$ unless $Z \in kV$. By Lemma 6,

$$\hat{\theta}_0 \in J(V, t, \Lambda^*)$$

for large enough t. If T is an invariant distribution, let $j_{\Lambda^*}T$ denote its restriction to the space of Λ^*-invariant functions. Let J_0 be the space of invariant distributions whose support is compact modulo conjugation. By a fundamental result from Howe's theory [**8**, Corollary to Theorem 17] we have that, for suitably chosen V:

$$j_{\Lambda^*}J(V, t, \Lambda^*) \subset j_{\Lambda^*}J_0$$

for any t. (That such a V exists results from Lemmas 28, 29 of [**8**].) Thus $\hat{\theta}_0 \in j_{\Lambda^*}J_0$. Taking Fourier transform, we see that θ_0 agrees on Λ with a distribution in the space of Fourier transforms of J_0. But now Theorem 3 of [**8**] implies that θ_0 is locally integrable in a neighborhood of 0. Transporting by the exponential, we see that θ is locally integrable in a neighborhood of 1. The usual theory of invariant distributions ([**9**, Theorem 11 and Corollary]) now shows that Θ is locally integrable around γ, whence Theorem 2. Moreover, Theorem 4 of **[8]** gives the more precise estimates of Theorem 3.

By Theorem 3 of **[8]**, $|\eta_M|^{1/2}\theta_0$ is bounded near 0 and θ_0 is locally constant on regular elements. Via the exponential isomorphism, this means that $|D_M|^{1/2}\theta$ is bounded, and θ is smooth on regular elements, near 1. To obtain the regularity assertions of Theorem 1 near γ, we need:

LEMMA 7. (a) *If* $\gamma x \in G_{\text{reg}} \cap \gamma M$, *then* $x \in M_{\text{reg}}$.

(b) $\dfrac{D_G(\gamma x)}{D_M(x)}$ *is bounded near* 1.

Proof. If $\gamma x \in G_{\text{reg}}$, the connected centralizer of γx in G is a torus. Since M centralizes γ, the connected centralizer of x in M must be a torus. This proves (a). In (b), we may assume γx regular, whence x regular in M. If $\mathfrak{t} \subset \mathfrak{g}$ is the centralizer of γx, we have

$$D_G(\gamma x) = \det(1 - \mathrm{Ad}(\gamma x)\,|_{\mathfrak{g}/\mathfrak{t}}).$$

The argument for (a), and counting ranks, show that $\mathfrak{t}$ is the centralizer of x in $\mathfrak{m}$. Thus

$$D_M(x) = \det(1 - \mathrm{Ad}\, x|_{\mathfrak{m}/\mathfrak{t}}) = \det(1 - \mathrm{Ad}(\gamma x)\,|_{\mathfrak{m}/\mathfrak{t}})$$

since M commutes with γ. But for x close to 1,

$$\det(1 - \mathrm{Ad}(\gamma x)\,|_{\mathfrak{g}/\mathfrak{m}}) \neq 0,$$

whence (b).

Now Theorem 1 follows from Harish-Chandra's usual argument and well-known lemma:

LEMMA 8. *Assume* $F \subset G$ *is non-empty, closed and* G*-invariant. Then* F *contains a semi-simple element.*

For this see, e.g., **[4**, Lemma 4.14]. Applied to the complement of the set where Θ_π is not locally integrable (resp. $|D|^{1/2}\Theta_\pi$ is not locally bounded) this shows that it must be empty, whence the theorem.

3.5 *Representations of small compact subgroups.* Let $\mathfrak{g}$ be the Lie algebra of a reductive group G over k. Let $\mathfrak{g}_0 \subset \mathfrak{g}$ be a domain satisfying the conditions (1)-(3) after Lemma 5. We recapitulate Howe's theory of representations of small, compact-open subgroups of G (**[10]**).

Assume k is an extension of $\mathbf{Q}_p$. Set

$$\beta = \frac{3}{2(p-1)}\mathrm{ord}_k(p) + 1 \quad \text{if } p \neq 3$$
$$\beta = \mathrm{ord}_k(3) + 1 \quad \text{if } p = 3.$$

Let L be a lattice in $\mathfrak{g}$, contained in $\mathfrak{g}_0$. Assume that $[L, L] \subset \tilde{\omega}^\beta L$. Then

$K = \exp L$ is an open compact subgroup of G ([**10**]). We say that L is *small* if

(a) $[L, L] \subset \tilde{\omega}^{\beta+1}L \quad (p \neq 2)$,

(b) $[L, L] \subset 2\tilde{\omega}^{\beta+1}L \quad (p = 2)$.

Condition (b) is equivalent to:

$$[L', L'] \subset \tilde{\omega}^{\beta+1}L', \quad \text{with } L' = \frac{1}{2}L.$$

We set $K = \exp L$, $K^{1/2} = \exp L'$. (Thus $K^{1/2} = K$ unless $p = 2$.) Then K is a normal subgroup of finite index of $K^{1/2}$. Let $\mathcal{E}^{1/2}(K)$ be the set of $K^{1/2}$ orbits in $\mathcal{E}(K)$. If $\mu \in \mathcal{E}^{1/2}(K)$, let μ_i be the representations belonging to μ. We set $d(\mu) = d(\mu_i)$, the dimension of any μ_i, and

$$\xi_\mu = \sum_i \xi_{\mu_i}.$$

We assume given an invariant bilinear form $B(X, Y)$ on $\mathfrak{g}$, and a non-trivial character χ on k. Let then L^* be the dual lattice of L with respect to the character $\chi(B(X, Y))$ of $\mathfrak{g} \times \mathfrak{g}$. Then $K^{1/2}$ operates on $\mathfrak{g}/L^*$.

THEOREM 4 (Howe). *Assume L is small. There is a bijection* $\mu \mapsto \mathcal{O}_\mu$ *from* $\mathcal{E}^{1/2}(K)$ *to the set of all* $K^{1/2}$*-orbits in* $\mathfrak{g}/L^*$ *such that*

(1) $d(\mu)\xi_\mu(\exp \lambda) = \sum_{X \in \mathcal{O}_\mu/L^*} \chi(B(X, \lambda))$.

(2) *For any* $X \in \mathcal{O}_\mu$,

$$d(\mu) = [K:K_X]^{1/2}$$

where

$$K_X = \{k \in K:\mathrm{Ad}(k)X \in X + L^*\}.$$

This is Theorem 1.1 of [**10**], at least for $p \neq 2$. The case $p = 2$ is not covered there, but the analogue is proved in [**11**] for discrete nilpotent groups: this is Theorem 1(b) of [**11**]. The reader will check that, for the arguments there to work, one needs that $\tilde{\omega}^{-\beta}L'$ be pronilpotent and "elementarily exponentiable". That is true under our assumptions.

PROPOSITION 1 (Howe). *Assume* L_1, L_2 *are small lattices,* $K_i = \exp(L_i)$. *We identify subsets of* $\mathfrak{g}/L_i^*$ *with* L_i^**-invariant subsets of* $\mathfrak{g}$. *Then, if* $x \in G$, x *intertwines* μ_2 *with* μ_1 *if and only if,* $\mathcal{O}_i$ *being the associated orbits*:

$$\mathcal{O}_1 \cap \mathrm{Ad}(x)\mathcal{O}_2 \neq \emptyset.$$

This is Proposition 1.4 of [**10**].

3.6 *Properties of the Harish-Chandra map for orbits.* In this section we study the relation between representations of small compact open subgroups of M and G. This is analogous to the Harish-Chandra map between centers of enveloping algebras in the real case. It is expressed, of course, in terms of Howe's orbit parametrization.

We assume given G-invariant and M-invariant domains $\mathfrak{g}_0 \subset \mathfrak{g}$ and $\mathfrak{m}_0 \subset \mathfrak{m}$ satisfying the conditions after Lemma 5. Let $L \subset \mathfrak{g}$, $\Lambda \subset \mathfrak{m}$ be small lattices. We set $K = \exp L$, $K_M = \exp \Lambda$. Let $\mathfrak{g} = \mathfrak{m} + \mathfrak{q}$, and $P_{\mathfrak{m}}$, $P_{\mathfrak{q}}$ be the corresponding projections.

LEMMA 9. *Assume $\mu \in \mathscr{E}^{1/2}(K)$, $\delta \in \mathscr{E}^{1/2}(K_M)$. Then μ and δ interact if and only if*

$$P_{\mathfrak{m}}\mathcal{O}_\mu \cap \mathcal{O}_\delta \neq \emptyset.$$

Proof. By [**8**, Lemma 30], μ and δ interact if and only if

$$\int_{K \cap K_M} \xi_\mu(x)\overline{\xi_\delta(x)}dx \neq 0.$$

Using Howe's character formula, this is equivalent to

$$\sum_{Y \in \mathcal{O}_\mu / L^*} \sum_{Z \in \mathcal{O}_\delta / \Lambda^*} \int_{L \cap \Lambda} \chi(B(Y - Z, u))du \neq 0.$$

(Note that exp sends Haar measure to Haar measure for small compact subgroups.)

But, by duality theory for Abelian groups,

$$\int_{L \cap \Lambda} \chi(B(Y - Z, u))du = \begin{cases} \mathrm{vol}(L \cap \Lambda) & \text{if } Y - Z \in (L \cap \Lambda)^* \\ 0 & \text{otherwise} \end{cases}$$

where $(L \cap \Lambda)^*$ is the orthogonal of $L \cap \Lambda$ in $\mathfrak{g}$ for $\chi \circ B$. Furthermore

$$(L \cap \Lambda)^* = L^* + \Lambda^* + \mathfrak{q}.$$

So the term associated with Y and Z is equal to $\mathrm{vol}(L \cap \Lambda)$ if and only if

$$Y - Z \in L^* + \Lambda^* + \mathfrak{q}.$$

This implies the lemma.

We now make the following precise assumptions on the p-adic norm on $\mathfrak{g}$ and on the small compact-open subgroups. We assume that $G \subset GL(n, k)$, an embedding given by a faithful representation of k-groups. On $M_n(k)$, we have the p-adic norm

$$|x| = \max|x_{ij}|$$

where x is the matrix (x_{ij}). It restricts to a norm on $\mathfrak{g}$.

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Set $K_0 = \exp L_0$, L_0 being a small lattice in $\mathfrak{g}$. We may assume, by taking L_0 small enough:

(1) $K_0^{1/2} \subset GL(n, R_k)$

(2) $|(\mathrm{Ad}(\exp \lambda) - 1)X| \leqq |\lambda|\,|X|$

for $\lambda \in \dfrac{1}{2}L_0$, $X \in \mathfrak{g}$

(3) Θ is (G, K_0)-admissible at γ and $K_0\gamma K_0 \subset U$.

Likewise, we define $K_M = \exp \Lambda$, where Λ is a small lattice in $\mathfrak{m}$, well adapted with respect to M ([**8**, Section 12]) such that

(1) $\Lambda \subset L_0$, $K_M \subset U_M$

(2) $|(\mathrm{Ad}(\gamma m) - 1)Y| \geqq c|Y|$ for some $c > 0$

if $m \in K_M$, $Y \in \mathfrak{q}$.

(3) $\left|\dfrac{1}{2}\Lambda\right| < 1$ and $|(\mathrm{Ad}(m) - 1)Z| \leqq \left|\dfrac{1}{2}\Lambda\right|\,|Z|$

for $m \in K_M^{1/2}$, $Z \in \mathfrak{m}$.

It is easy to check that these conditions are satisfied for Λ small enough. We now define, for ν a positive integer,

$$L_\nu = \tilde{\omega}^\nu L_0, \quad K_\nu = \exp L_\nu.$$

Recall that $\mathscr{N}$ is the nilpotent variety in $\mathfrak{g}$. Let Φ_ν be the set of $\mu \in \mathscr{E}^{1/2}(K_\nu)$ such that

(1) $\mathscr{O}_\mu \cap \mathscr{N} \neq \emptyset$

(2) $\mathrm{Ad}(\gamma m)\mathscr{O}_\mu \cap \mathscr{O}_\mu \neq \emptyset$ for some $m \in K_M$.

LEMMA 10. *Let V be a neighborhood of $\mathscr{N} \cap S \cap \mathfrak{m}$ in $S \cap \mathfrak{m}$. Then there is ν_0 with the following property. Assume $\nu \geqq \nu_0$, $\mu \in \Phi_\nu$ and $|X| \geqq q^{2\nu}$ for some $X \in \mathscr{O}_\mu$. Then*

$$p_{\mathfrak{m}}\mathscr{O}_\mu \subset kV.$$

We will need an auxiliary lemma:

LEMMA 11. *There is $c_1 \geqq \max\left(\left|\dfrac{1}{2}L_0\right|, |L_0^*|\right)$ with the following property. If $\nu \geqq 0$, $\mu \in \Phi_\nu$, then there is $X_0 \in \mathscr{O}_\mu$ such that*

$$|p_{\mathfrak{q}}X_0| \leqq c_1 \max(q^\nu, q^{-\nu}|X_0|)$$

and

$$|X_0 - X| \leqq c_1 \max(q^\nu, q^{-\nu}|X_0|)$$

for all $X \in \mathscr{O}_\mu$.

Proof. By definition of Φ_ν, there is $m \in K_M$ such that

$$\mathrm{Ad}(\gamma m)\mathcal{O}_\mu \cap \mathcal{O}_\mu \neq \emptyset.$$

So let $X_0 \in \mathcal{O}_\mu$ be such that

$$\mathrm{Ad}(\gamma m)X_0 \in \mathcal{O}_\mu.$$

By definition of the orbit, there is then $k \in K_\nu^{1/2}$ such that

$$\mathrm{Ad}(\gamma m)X_0 - \mathrm{Ad}(k)X_0 \in L_\nu^*$$

or

$$(\mathrm{Ad}(\gamma m) - 1)X_0 - (\mathrm{Ad}(k) - 1)X_0 \in L_\nu^*.$$

By condition (2) on Λ, we have

$$c|p_{\mathfrak{q}}X_0| \leqq |(\mathrm{Ad}(\gamma m) - 1)p_{\mathfrak{q}}X_0|$$
$$\leqq \max\{|p_{\mathfrak{q}}L_\nu^*|, |p_{\mathfrak{q}}(\mathrm{Ad}\, k - 1)X_0|\}.$$

Thus

$$|p_{\mathfrak{q}}X_0| \leqq c_2 \max\left\{q^\nu|L_0^*|, q^{-\nu}\left|\frac{1}{2}L_0\right| |X_0|\right\}$$

using the definition of L_ν. Here

$$c_2 = c^{-1}|p_{\mathfrak{q}}|.$$

Setting

$$c_1 = \max(1, c_2) \max\left(\left|\frac{1}{2}L_0\right|, |L_0^*|\right)$$

gives the first inequality.

For the second, choose $X \in \mathcal{O}_\mu$. Then there exist $k \in K_\nu^{1/2}$ and $\lambda \in L_\nu^*$ such that

$$X = \mathrm{Ad}(k)X_0 + \lambda$$

whence

$$X - X_0 = (\mathrm{Ad}(k) - 1)X_0 + \lambda;$$

therefore

$$|X - X_0| \leqq \max\left(\left|\frac{1}{2}L_\nu\right| |X_0|, |L_\nu^*|\right)$$
$$\leqq c_1 \max(q^{-\nu}|X_0|, q^\nu).$$

COROLLARY. *Assume that* $c_1 < q^{\nu - r}$ *and* $|X| \geqq q^{2\nu}$ *for some* $X \in \mathcal{O}_\mu$, *where*

$$q^r = \max(|p_{\mathfrak{m}}|, |p_{\mathfrak{q}}|).$$

Then

(a) $|p_{\mathfrak{m}}X| = |X|$

(b) $|X| = |X'|$ *for any* $X' \in \mathcal{O}_\mu$

(c) $|p_{\mathfrak{m}}X - X'| \leqq c_1 q^{-\nu+r}|X|$ *for any* $X' \in \mathcal{O}_\mu$.

(*Here* $|p_{\mathfrak{m}}|$, $|p_{\mathfrak{q}}|$ *are defined by the* sup *norm for operators*).

Proof. If $X' \in \mathcal{O}_\mu$, we can write

$$X' = \mathrm{Ad}(k)X + \lambda,\ k \in K_\nu^{1/2},\ \lambda \in L_\nu^*.$$

Since $K_0^{1/2} \subset GL(n, R_k)$, we have

$$|\mathrm{Ad}(k)X| = |X|.$$

Moreover

$$|\lambda| \leqq |L_\nu^*| \leqq c_1 q^\nu < q^{2\nu - r} \leqq |X|.$$

Hence $|X'| = |X|$. In particular, with X_0 as in the Lemma, $|X_0| = |X|$. The first inequality in the Lemma yields

$$|p_{\mathfrak{q}}X_0| \leqq c_1 \max(q^\nu, q^{-\nu}|X_0|) < |X_0|.$$

This implies that $|p_{\mathfrak{m}}X_0| = |X_0|$. Moreover

$$|X_0 - X| \leqq c_1 \max(q^\nu, q^{-\nu}|X_0|) \leqq c_1 q^{-\nu}|X_0| < q^{-r}|X_0|.$$

Hence $|p_{\mathfrak{q}}(X_0 - X)| < |X_0|$. This, and the majoration of $|p_{\mathfrak{q}}X_0|$, imply

$$|p_{\mathfrak{q}}X| < |X_0| = |X|.$$

So we conclude that $|p_{\mathfrak{m}}X| = |X|$.
We still have to prove (c). Write

$$(*)\quad p_{\mathfrak{m}}X - X' = p_{\mathfrak{m}}(X - X_0) - (X' - X_0) - p_{\mathfrak{q}}X_0.$$

We have proved that

$$|X_0 - X| \leqq c_1 q^{-\nu}|X_0|.$$

Hence

$$|p_{\mathfrak{m}}(X_0 - X)| \leqq c_1 q^{-\nu+r}|X_0|.$$

Also, since $|X'| = |X|$, the assumptions of the Corollary apply to X', so

$$|X_0 - X'| \leqq c_1 q^{-\nu}|X_0|$$

as proved above for X. Furthermore,

$$|p_{\mathfrak{q}}X_0| \leqq c_1 q^{-\nu}|X_0|$$

as proved above.

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Adding terms in (*), we get

$$|p_{\mathfrak{m}}X - X'| \leqq c_1 q^{-\nu+r}|X_0| = c_1 q^{-\nu+r}|X|.$$

This is (c).

We can now prove Lemma 10. Let $\mathfrak{g}(R)$ be the set $\{\,|X| \leqq R\}$. (The same notation will apply to $\mathfrak{m}$.) We may choose $\epsilon > 0$ such that

$$(\mathcal{N} \cap S + \mathfrak{g}(\epsilon)) \cap \mathfrak{m} \subset V.$$

Assume $\nu_0 \geqq 0$ is such that

$$c_1 q^{-\nu_0+r} \leqq \epsilon.$$

Then if ν, μ, X are as in Lemma 10, choose $a \in k$ such that $|a| = |X|$. Since $\mu \in \Phi_\nu$, we may choose $Y \in \mathcal{O}_\mu \cap \mathcal{N}$. For ϵ small,

$$c_1 < q^{\nu_0 - r} \leqq q^{\nu - r}.$$

The corollary to Lemma 11 applies. So

$$|Y| = |X| = |a|, \quad |p_{\mathfrak{m}}X - Y| \leqq c_1 q^{-\nu+r}|X| \leqq \epsilon|a|.$$

If $Y_1 = a^{-1}Y$, $X_1 = a^{-1}X$, we have

$$Y_1 \in \mathcal{N} \cap S, \quad |p_{\mathfrak{m}}X_1 - Y_1| \leqq \epsilon.$$

Thus

$$p_{\mathfrak{m}}X_1 \in (\mathcal{N} \cap S + \mathfrak{g}(\epsilon)) \cap \mathfrak{m} \subset V.$$

Thus $p_{\mathfrak{m}}X \in kV$. By part (b) of the corollary, $|X'| = |X|$ for any $X' \in \mathcal{O}_\mu$. Therefore this applies to X', and

$$p_{\mathfrak{m}}\mathcal{O}_\mu \subset kV.$$

3.7 *Fourier expansion of θ.* Recall that the distribution θ is defined on U_M. We have $K_M \subset U_M$, and a character of K_M extends by 0 to a function on U_M. If $\delta \in \mathcal{E}^{1/2}(K_M)$, we set

$$\theta(\delta) = \theta(\overline{\xi}_{\delta_0})$$

where δ_0 is any element in the orbit δ; this does not depend on δ_0 since θ is M-invariant and $K_M^{1/2} \subset M$.

Let $F_\nu(\nu \geqq 0)$ be the set of all $\delta \in \mathcal{E}^{1/2}(K_M)$ such that

$$\mathcal{O}_\delta \subset \mathfrak{m}(q^{2\nu+r}).$$

The purpose of this section is to prove:

LEMMA 12. *Assume $\nu > 0$ is large enough. Then if $\delta \in \mathcal{E}^{1/2}(K_M)$ does not belong to F_ν, and $\theta(\delta) \neq 0$, we have*

(a) $|Z| > q^{2\nu}$ *for any* $Z \in \mathcal{O}_\delta$

(b) $\mathcal{O}_\delta \subset kV$.

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We start with a lemma that is often useful.

LEMMA 13 ([**12**, Lemma 2.4]). *Assume* $\omega \subset \mathfrak{g}$ *is compact. Then there is a lattice* $L \subset \mathfrak{g}$ *such that*

$$\mathrm{Ad}(G)\omega \subset \mathcal{N} + L.$$

Proof. We may assume G connected. Let A be a maximal split torus in G, M its centralizer, K a Bruhat-Tits subgroup of G adapted to A. Fix a lattice $L_0 \subset \mathfrak{g}$ adapted to (K, A) ([**8**, Section 12]). It is enough to consider ω equal to a lattice $L = \tilde{\omega}^\nu L_0$. Let

$$\mathfrak{g} = \mathfrak{m} + \mathfrak{n}^+ + \mathfrak{n}^-, \quad \mathfrak{m} = \mathrm{Lie}(M),$$

be a triangular decomposition of $\mathfrak{g}$. Let M^+ be the set of elements of M which contract $\mathfrak{n}^+$. By the Cartan decomposition [**3**, Section 3.5], $G = KM^+K$. Then $\mathrm{Ad}(K)L = L$

$$\begin{aligned}\mathrm{Ad}(M^+K)L &= \mathrm{Ad}(M^+)L \\ &= \mathrm{Ad}(M^+)(\,(L \cap \mathfrak{m}) + (L \cap \mathfrak{n}^+) + (L \cap \mathfrak{n}^-)\,) \\ &\subset a(L \cap \mathfrak{m}) + (L \cap \mathfrak{n}^+) + \mathfrak{n}^- \\ &\subset aL + \mathfrak{n}^-\end{aligned}$$

for some $a \in k$ independent of L, since the eigenvalues of $\mathrm{Ad}(M)$ on $\mathfrak{m}$ are bounded. Then

$$\begin{aligned}\mathrm{Ad}(G)L &\subset a\,\mathrm{Ad}(K)L + \mathrm{Ad}(K)\mathfrak{n}^- \\ &\subset aL + \mathcal{N}.\end{aligned}$$

This implies the lemma.

Now fix $\nu_0 \geqq 0$ large enough that

(1) $\mathrm{Ad}(G)\mathcal{O}_0 \subset \mathcal{N} + L^*_{\nu_0}$ (cf. Lemma 13)

(2) $q^{2\nu_0} \geqq |\Lambda^*| \left|\frac{1}{2}\Lambda\right|^{-1}$

(3) $c_1 q^{r-\nu_0} < 1$ where c_1 is as in Lemma 11.

(4) Lemma 10 is satisfied.

LEMMA 14. *Assume* $\nu \geqq \nu_0$ *and* $\delta \in \mathcal{E}^{1/2}(K_M)$ *does not belong to* F_ν. *Then*

$$\mathcal{O}_\delta \cap \mathfrak{m}(q^{2\nu+r}) = \emptyset.$$

Proof. There is $Z \in \mathcal{O}_\delta$ such that $|Z| > q^{2\nu+r}$. Assume $Z' \in \mathcal{O}_\delta$, so

$$Z' = \mathrm{Ad}(m)Z + \lambda, \quad m \in K_M^{1/2}, \lambda \in \Lambda^*.$$

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Then

$$Z' - Z = (\mathrm{Ad}(m) - 1)Z + \lambda$$

$$|Z' - Z| \leqq \max\left(\left|\frac{1}{2}\Lambda\right| |Z|, |\Lambda^*|\right).$$

Now

$$|Z| > q^{2\nu+r} \geqq q^{2\nu_0} \geqq |\Lambda^*| \left|\frac{1}{2}\Lambda\right|^{-1} \quad \text{and} \quad \left|\frac{1}{2}\Lambda\right| < 1.$$

So

$$|Z' - Z| \leqq \left|\frac{1}{2}\Lambda\right| |Z| < |Z|.$$

This shows that $|Z'| = |Z|$.

We can now prove Lemma 12. Assume $\nu \geqq \nu_0$. Applying Lemma 5 to the representations in the orbit δ, we see that if $\theta(\delta) \neq 0$ there is $\mu \in \mathscr{E}^{1/2}(K_\nu)$ such that δ interacts with μ and $\Theta_\mu(\gamma m) \neq 0$ for some $m \in K_M$. Using Lemma 9, we rewrite this as

(1) $p_{\mathrm{m}}\mathscr{O}_\mu \cap \mathscr{O}_\delta \neq \emptyset$

(2) $\Theta_\mu(\gamma m) \not\equiv 0$ on K_M.

Now let $\mathscr{O}_0 = L_0^*$ be the orbit associated to the trivial representation of K_0. Assume $\mu \in \mathscr{E}^{1/2}(K_\nu)$. Since Θ is (G, K_0)-admissible at γ, we have $\Theta * \xi_\mu = 0$ on γK_0 unless G intertwines μ and 1_{K_0}, i.e., by Proposition 1, unless

$$\mathscr{O}_\mu \cap \mathrm{Ad}(G)\mathscr{O}_0 \neq \emptyset.$$

A fortiori, condition (2) above implies

$$\mathscr{O}_\mu \cap \mathrm{Ad}(G)\mathscr{O}_0 \neq \emptyset.$$

By (1) before Lemma 14 we have

(3) $\mathscr{O}_\mu \cap \mathscr{N} \neq \emptyset$.

By (2), we may choose $m \in K_M$ such that $\Theta_\mu(\gamma m) \neq 0$. We have

$$K_\nu \gamma m K_\nu \subset K_0 \gamma K_0 \subset U;$$

the function

$$(k_1, k_2) \mapsto \Theta_\mu(k_1 \gamma m k_2)$$

is in $\mathscr{A}(\mu) \otimes \mathscr{A}(\mu)$. By Lemma 3, then, γm intertwines $\mathscr{O}_\mu$ with itself, so by Proposition 1:

(4) $\mathrm{Ad}(\gamma m)\mathscr{O}_\mu \cap \mathscr{O}_\mu \neq \emptyset$.

Assertions (3) and (4) mean that μ belongs to Φ_ν (cf. before Lemma 10).

Now using property (1), choose $X \in \mathcal{O}_\mu$ such that

$$Z = p_{\mathfrak{m}} X \in \mathcal{O}_\delta.$$

We claim that $|X| > q^{2\nu}$. For otherwise,

$$|Z| = |p_{\mathfrak{m}} X| \leqq q^{2\nu + r} \quad \text{since } |p_{\mathfrak{m}}| \leqq q^r.$$

Since we assumed that $\delta \in F_\nu$,

$$\mathcal{O}_\delta \cap \mathfrak{m}(q^{2\nu + r}) = \emptyset$$

by Lemma 14: this is a contradiction.

So $|X| > q^{2\nu}$. By the Corollary to Lemma 11 (note that the assumption on c_1 is our assumption (3) before Lemma 14), $|Z| = |X| > q^{2\nu}$. This is part (a) of Lemma 12.

We now prove (b). For $a \in k$ with $|a| = |Z| = |X|$. Since $\mu \in \Phi_\nu$, we then have $a^{-1}Z \in V$ by Lemma 10.

Obviously we may assume that

$$V = \mathcal{N} \cap S \cap \mathfrak{m} + \mathfrak{m}(\epsilon)$$

for some small ϵ. If $Z' \in \mathcal{O}_\delta$,

$$Z' = \mathrm{Ad}(m)Z + \lambda \text{ for some } m \in K_M^{1/2}, \lambda \in \Lambda^*.$$

Then

$$a^{-1}\mathrm{Ad}(m)Z \in V, \quad \text{and} \quad |\mathrm{Ad}(m)Z| = |Z| = |a| > q^{2\nu}$$

since $K_M^{1/2} \subset K_0$ preserves length. Moreover

$$|a^{-1}\lambda| \leqq q^{-2\nu}|\Lambda^*| \leqq \epsilon \quad \text{for large } \nu.$$

Thus

$$a^{-1}Z \in V + \mathfrak{m}(\epsilon) = V,$$

finishing the proof of Lemma 12.

3.8 *A formula for the K_M-expansion.* We keep to the notation of the previous sections. In particular, if $Z \in \mathfrak{m}$, let f_Z denote the characteristic function of $Z + \Lambda^*$. We assume moreover that $\frac{1}{2}\Lambda \subset \mathfrak{m}_0$.

LEMMA 15. *Assume* $\delta \in \mathcal{E}^{1/2}(K_M)$, $Z \in \mathcal{O}_\delta$. *Then*

$$\theta(\delta) = v(\Lambda^*)^{-1} d(\delta)\hat{\theta}_0(f_{-Z}).$$

Here θ_0 is defined as in Section 3.4 by restricting θ to $\exp(\mathfrak{m}_0)$, pulling it back to $\mathfrak{m}_0$ and extending it by 0 to $\mathfrak{m}$; $\hat{\theta}_0$ is its Fourier transform.

Proof. Consider ξ_δ as a function on $\mathfrak{m}$ as follows:

$$\begin{aligned} \xi_\delta(Z) &= \xi_\delta(\exp Z) \quad Z \in \Lambda \\ &= 0 \qquad\qquad\quad Z \notin \Lambda. \end{aligned}$$

Define

$$\hat{\mathcal{O}}_\delta(Z) = \int_{\mathcal{O}_\delta} \chi(B(Z, Z'))dZ'$$

where dZ' is Haar measure on $\mathfrak{m}$. Since $\mathcal{O}_\delta$ is Λ^*-invariant,

$$\hat{\mathcal{O}}_\delta(Z) = 0 \quad \text{for } Z \notin \Lambda.$$

On the other hand, if $Z \in \Lambda$,

$$\hat{\mathcal{O}}_\delta(Z) = \sum_{Z' \in \mathcal{O}_\delta/\Lambda^*} v(\Lambda^*)\chi(B(Z, Z')).$$

By Theorem 4(1), this is the expression of $\xi_\delta(Z)d(\delta)v(\Lambda^*)$. In other terms, we have the Kirillov formula

$$\hat{\mathcal{O}}_\delta = v(\Lambda^*)d(\delta)\xi_\delta.$$

Now, writing $\check{f}(Z)$ for $f(-Z)$, we have

$$\theta(\overline{\xi}_\delta) = \theta_0(\xi_\delta^\vee) = \frac{\hat{\theta}_0(\check{\mathcal{O}}_\delta)}{v(\Lambda^*)d(\delta)}.$$

Now

$$\mathcal{O}_\delta^\vee = \sum_{Z' \in \mathcal{O}_\delta/\Lambda^*} f_{-Z'}.$$

Since θ_0 is M-invariant, we deduce that

$$\theta(\overline{\xi}_\delta) = |\mathcal{O}_\delta/\Lambda^*|\hat{\theta}_0(f_{-Z}).$$

We now remember that $\theta(\delta)$, by definition, is equal to $\theta(\overline{\xi}_{\delta_0})$ for some $\delta_0 \in \delta$. Thus

$$\theta(\delta) = (v(\Lambda^*)d(\delta))^{-1}N(\delta)^{-1}\theta(\overline{\xi}_\delta),$$

$N(\delta)$ being the number of elements of $\delta \in \mathcal{E}^{1/2}(K_M)$. On the other hand ([**11**, Theorem 1]; this is easily deduced from Theorem 4):

$$|\mathcal{O}_\delta/\Lambda^*| = N(\delta)d(\delta)^2.$$

Whence the result.

3.9 *Proof of Lemma* 6. We can now prove Lemma 6, thus completing the proof of the main results. We rely on Lemmas 12 and 15. Take ν such that Lemma 12 holds. Assume $Z \in \mathfrak{m}$, with $|Z| > q^{2\nu+r}$ and $Z \notin kV$. By Lemma 15, $\hat{\theta}_0(f_Z) = 0$ if and only if $\theta(\delta) = 0$, δ being associated to the orbit of $-Z$. Since

$$|-Z| > q^{2\nu+r},$$

δ does not belong to F_ν. Since we have assumed $Z \notin kV$, we see that $\mathcal{O}_\delta \not\subset kV$. Lemma 12 then implies that $\theta(\delta) = 0$. This is Lemma 6.

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