GREEN MANURE AND NITROGENOUS FERTILIZER—
A TWO-SECTOR OPTIMAL-GROWTH MODEL

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Abstract

A continuous-time model of a two-sector trading economy with a finitely-saturating production function and constant population is constructed. To apply it to the farm-economy problem of optimal trading of cereal for chemical nitrogenous fertilizer and of optimal allocation of land for green manure the special assumptions are made of similar technologies in both sectors and of a nitrogen-capital loss proportional to the cereal production. Optimal-control theory is applied to get the pair of controls that maximizes an infinite-horizon integral of utility of consumption. The analysis of the model is shown to reduce to that of the Ramsey-Koopmans one-sector neoclassical model of optimal growth. Calculations of the feedback control and optimal time-paths for the standardized dimensionless model are tabulated for a range of 7 utility functions of constant elasticity of marginal utility of magnitude 1 to 4 and of production functions of degree 2 to 4. Two particular analytic solutions are given. The application of the results both to the farm-economy model and to the Ramsey-Koopmans model is illustrated.

1. Introduction

The problem of planning the best use of agricultural fertilizers over time is an optimal-control problem ([2], [3], [8–13]). Response curves to the principal nutrients have been measured for the most important crops and as they turn out to be approximately quadratic ([1], [6]) the static optimization problem leads to linear equations and is easily solved as, for example, by Abraham and Nair [1]. Using 1973–74 prices, they showed that by 1978–79 it would be optimal for India to use 37% more fertilizer nitrogen than the 5.2 megatonnes the Government planned to have available then ([4] p. 12) and of which 23% would need to be imported ([4] p. 146).

Although the static optimization model is easy to apply to a farm economy, it may be inappropriate where the soil is poor or where there is little capital equipment and no credit. On such a farm some investment from its own resources

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into improving its condition is essential for the long-term best use of the recurrent inputs, but the maximum output initially may be so near the farmer's subsistence level that he must judge carefully any sacrifice of present consumption.

At the 1973–74 price-ratios application of chemical nitrogenous fertilizers was economic on most Indian crops but the 1974 oil price rise doubled the chemical fertilizer price. The likelihood of continued scarcities of fuel, fertilizer and foreign exchange makes it worth considering the long-term economics of both chemical nitrogenous fertilizer use and of direct farm-supplementation of nitrogen by green manuring.

In this study I develop a continuous-time model of cereal production and optimal consumption that is one of the optimal growth of a two-sector economy which can trade with an outside world. One sector produces a consumption good, "cereal", the other a capital good, "nitrogen", which is used by both the capital and consumption-good sectors. The state of the farm is described by an aggregated capital stock variable (the soil-nitrogen stock) which can be increased or decreased. It can be increased by trading some of the cereal for imported nitrogenous fertilizer or by allocating some of the land for home-production of nitrogen by growing on it leguminous green-manure crops which, on being ploughed in, add nitrogen to the soil. The nitrogen is depleted by the production of the cereal crop. The other nutrients are ignored.

Assuming similar technologies for the production of both the green-manure nitrogen and the cereal I show that the analysis of this model can be transformed to that of the Ramsey-Koopmans model of optimal growth of an economy with an expanding population.

Although I describe the model in agricultural terms it is easy to see applications to other two-sector economies having similar technologies in both sectors and in which the capital is depleted by the production of the consumption good.

In §2 I describe and develop the model in control-theory form, and in §3 apply Pontryagin's theorem to get the optimal controls. In §4 I show how the model can be transformed into the Ramsey-Koopmans model. In §5 I synthesize numerically, graph and tabulate the feedback control for the model with an infinite planning horizon for a set of seven utility functions having constant elasticities of marginal utility in the range –1 to –4. In §6 I calculate, graph and tabulate the corresponding time-paths for a range of production functions. In §7 I illustrate and discuss the application of the results.

2. Description of the model

L is the total available area of land, a constant, which is to be allocated by the control variable v_1 into a section $v_1 L$ for the production of green manure and $(1 - v_1)L$ for the production of cereal.

T is the time, the unit of which will be called a "year".

N is the capital stock of nitrogen and $K = N/L$ is the nitrogen concentration (per unit area), assumed the same on both sections of land.

$F(K)$ is the production of cereal in one year on one unit of cereal-producing land so the cereal produced in one year is $(1 - v_1)LF(K)$.

E is the amount of cereal produced that is allocated by the control variable v_2 to be exported in one year to trade for fertilizer:

$$E = v_2(1 - v_1)LF(K). \tag{2.1}$$

Y is the remaining amount of cereal, for consumption in one year:

$$Y = (1 - v_2)(1 - v_1)LF(K). \tag{2.2}$$

I is the amount of nitrogen imported as chemical fertilizer in one year. The export price P_C for cereal and the import price P_N for fertilizer nitrogen are constant and trade is balanced each year:

$$P_C E - P_N I = 0. \tag{2.3}$$

$G(K)$ is the net production of nitrogen in one year on one unit of the green-manure land so the total home-production of nitrogen is $v_1 LG(K)$.

$Q(K)$ is the loss of nitrogen in one year on one unit of the cereal land (due to the growing and removal of the crop) so the total loss due to the harvest is $(1 - v_1)LQ(K)$.

The increase each year in nitrogen is the sum of the imported chemical fertilizer nitrogen and the home-produced green-manure nitrogen *less* the loss due to harvest; that is

$$dN/dT = I + v_1 LG(K) - (1 - v_1)LQ(K); \tag{2.4}$$

that is, using (2.1), (2.3) and the definition of K,

$$\frac{dK}{dT} = (1 - v_1)\left[v_2 \frac{P_C}{P_N} F(K) - Q(K)\right] + v_1 G(K). \tag{2.5}$$

I assume an instantaneous utility function of consumption $U(Y)$ and that the model will have some stationary maximum level of consumption, $\overline{Y}$ (the "bliss" level, the existence and value of which is determined in §4), and that the planner's object is to choose an optimal admissible control v^*,

$$v^* = (v_1^*, v_2^*) \in V \text{ where } V = [0,1] \times [0,1], \tag{2.6}$$

that will maximize the undiscounted total utility integral

$$\int_0^\infty (U(Y) - U(\overline{Y}))\, dT \tag{2.7}$$

where Y is given by (2.2) and is subject to the constraint (2.5).

I now assume the loss of nitrogen due to the removal of the crop is proportional to the size of the crop, that is,

$$Q(K) = \alpha F(K) \quad (\alpha > 0); \tag{2.8}$$

that there is a finite "saturation point" $\overline{K}$ and corresponding "saturation value" $\overline{F}$ where

$$F'(\overline{K}) = 0 \quad \text{and} \quad \overline{F} = F(\overline{K})$$

and that the green manure and cereal crops have similar production functions and saturate at the same point, that is

$$G(K) = \beta F(K) \quad (\beta > 0). \tag{2.9}$$

Let dimensionless constants μ_1 and μ_2 (the prices of green-manure nitrogen and of chemical fertilizer nitrogen relative to that of cereal) be defined by

$$\mu_1 = \alpha/(\alpha + \beta) \quad \text{and} \quad \mu_2 = \alpha P_N/P_C \tag{2.10}$$

and dimensionless variables and functions by

$$\left. \begin{array}{l} k = (1/\overline{K})K; \quad y = (1/\overline{Y})Y; \quad t = (\alpha\overline{Y}/L\overline{K})T; \\ f(k) = (L/\overline{Y})F(k\overline{K}) \quad \text{and} \quad u(y) = U(y\overline{Y}). \end{array} \right\} \tag{2.11}$$

For the utility function $u(y)$ I assume

$$\left. \begin{array}{l} u'(y) > 0; \; u''(y) < 0; \\ u(1) = 0; \; u'(1) = 1; \\ \lim_{y \to 0^+} u(y) = -\infty. \end{array} \right\} \tag{2.12}$$

The first two are the usual assumptions of positive and diminishing marginal utility; the third and fourth set a convenient arbitrary zero point and scale of utility; the fifth assumption excludes a zero-level of consumption being allowed for any time interval.

For the production function $f(k)$, I assume:

$$\left. \begin{array}{ll} f'(k) > 0 \text{ and } f''(k) < 0 & \text{for } 0 \leqslant k < 1; \\ f(0) = 0; \quad \cdot \\ f'(k) = 0 & \text{for } k \geqslant 1. \end{array} \right\} \tag{2.13}$$

Through (2.8) to (2.11), the problem is transformed to a dimensionless one containing only two parameters, μ_1 and μ_2, having some choice of utility function, u, and leaving open the precise specification of $f(k)$. The problem now is the optimal-control problem to

$$\operatorname*{maximize}_{v \in V} \int_0^\infty u(y) \, dt \tag{2.14}$$

where

$$y = (1 - v_1)(1 - v_2)f(k), \tag{2.15}$$

subject to the constraint

$$\frac{dk}{dt} = (v_1/\mu_1 + v_2/\mu_2 - v_1v_2/\mu_2 - 1)f(k) \tag{2.16}$$

and the initial condition

$$k(0) = k_0 > 0 \tag{2.17}$$

where u and f have the properties (2.12) and (2.13) and V is the bounded set defined in (2.6).

3. Application of Pontryagin's theorem

Pontryagin's theorem ([10] p. 19–21) applies precisely to problems like (2.14)–(2.17).

One constructs the Hamiltonian

$$H(k, \mathbf{v}, \Lambda) = u\big((1 - v_1)(1 - v_2)f(k)\big)$$
$$+ \Lambda(v_1/\mu_1 + v_2/\mu_2 - v_1v_2/\mu_2 - 1)f(k) \tag{3.1}$$

and then determines the function $M(k, \Lambda)$ where

$$M(k, \Lambda) = \sup_{\mathbf{v} \in V} H(k, \mathbf{v}, \Lambda). \tag{3.2}$$

Pontryagin's theorem then says that if there is an optimal control $\mathbf{v}^*$ satisfying (2.14)–(2.17) then it satisfies

$$H(k, \mathbf{v}^*, \Lambda) = M(k, \Lambda) \tag{3.3}$$

where k and Λ satisfy

$$\left.\begin{aligned}
\frac{dk}{dt} &= \left.\frac{\partial}{\partial \Lambda} H(k, \mathbf{v}, \Lambda)\right|_{\mathbf{v} = \mathbf{v}^*}; \\
\frac{d\Lambda}{dt} &= -\left.\frac{\partial}{\partial k} H(k, \mathbf{v}, \Lambda)\right|_{\mathbf{v} = \mathbf{v}^*}.
\end{aligned}\right\} \tag{3.4}$$

Because in this case $\partial H/\partial t = 0$

$$H(k, \mathbf{v}^*, \Lambda) \text{ is constant along the optimal path.} \tag{3.5}$$

The function $M(k, \Lambda)$. One finds through an elementary examination (not presented here) the description of $M(k, \Lambda)$ and the corresponding optimal controls. *The case* $\mu_1 < \mu_2$. Let the function $C = C(k, \Lambda)$ be defined by

$$u'\big((1 - C)f(k)\big) = \Lambda/\mu_1 \tag{3.6}$$

then for the region $\{(k, \Lambda) \mid \Lambda/\mu_1 > u'(f(k))\}$ the function M and the optimal control $\mathbf{v}^*$ are given by

$$
\left.
\begin{aligned}
M(k, \Lambda) &= u((1 - C)f(k)) + \frac{\Lambda}{\mu_1}(C - \mu_1)f(k); \\
v_1^* &= C(k, \Lambda); \quad v_2^* = 0;
\end{aligned}
\right\}
\tag{3.7}
$$

and for the remaining region $\{(k, \Lambda) \mid \Lambda/\mu_1 \leqslant u'(f(k))\}$

$$
\left.
\begin{aligned}
M(k, \Lambda) &= u(f(k)) - \Lambda f(k); \\
v_1^* &= 0; \quad v_2^* = 0.
\end{aligned}
\right\}
\tag{3.8}
$$

The case $\mu_1 > \mu_2$. $M(k, \Lambda)$ and the corresponding optimal control are described just as in (3.6)–(3.8) for the case $\mu_1 < \mu_2$ but with the subscripts "1" and "2" interchanged.

The case $\mu_1 = \mu_2 = \mu$. $M(k, \Lambda)$ is described exactly as for the case $\mu_1 < \mu_2$ but the optimal control in the region

$$
\{(k, \Lambda) \mid \Lambda/\mu > u'(f(k))\}
$$

is no longer unique but can be arbitrarily chosen from the set (which includes the control in (3.7))

$$
\{(v_1, v_2) \mid (1 - v_1)(1 - v_2) = 1 - C(k, \Lambda)\}.
\tag{3.9}
$$

This is the only case in which there is a candidate optimal control in the interior of V and even then it is no better than either of the two end-points of the arc of the hyperbola (3.9) that lie on the boundary of V.

4. Analysis of the case $\mu_1 < \mu_2$

It is now apparent that the solutions to the problem will be symmetric under the interchange of subscripts "1" and "2" so it is sufficient to analyse just the case $\mu_1 < \mu_2$. I assume $\mu_1 < \mu_2$ from now on.

I define a new control w in place of v_1 and a corresponding control set W by

$$
w = (1 - v_1)/(1 - \mu_1) \text{ and } W = [0, 1/(1 - \mu_1)]
\tag{4.1}
$$

and a re-scaled time variable τ and a function $\phi(k)$ by

$$
\tau = t/\mu_1 \text{ and } \phi(k) = (1 - \mu_1)f(k).
\tag{4.2}
$$

Using these and the fact (from (3.6) and (3.7)) that $v_2^* = 0$ everywhere the remaining problem reduces to the single control-variable problem:

$$
\left.
\begin{aligned}
&\underset{w \in W}{\text{maximize}} \int_0^\infty u(y) \, d\tau \\
&\text{subject to } dk/d\tau = \phi(k) - y \text{ and } k(0) = k_0 \\
&\text{where } y = w\phi(k).
\end{aligned}
\right\}
\tag{4.3}
$$

This problem now is virtually identical with the optimal growth problem in the well-known modified Ramsey model of Koopmans and others (see, for example: Wan [13] pp. 294–324; Hadley and Kemp [5] pp. 50–71, 76–78; Chakravarty [3] pp. 80–110) in which the population grows exponentially, y and k are the *per capita* consumption and capital and $\phi(k)$ represents the balanced growth consumption *per capita*. In the modified Ramsey model $\phi(k)$ has a unique positive maximum at some finite $\bar{k}$ and eventually becomes negative but here $\phi(k)$ is assumed merely to saturate at some finite $\bar{k}$. This difference is irrelevant if $k_0 < \bar{k}$.

Presuming familiarity with the results for the Ramsey-Koopmans model I shall omit most of the details of the argument.

Pontryagin's method applied to (4.3) leads to a Hamiltonian $h(k, w, \lambda)$ and an optimal control w^* where (referring to (3.1) and (3.2))

$$\left.\begin{aligned}
h(k, w, \lambda) &= u(w\phi(k)) + \lambda(1 - w)\phi(k) \\
&= H(k, 1 - (1 - \mu_1)w, 0, \mu_1\lambda); \\
h(k, w^*, \lambda) &= \sup_{w \in W} h(k, w, \lambda) = M(k, \mu_1\lambda).
\end{aligned}\right\} \tag{4.4}$$

The region of the (k, λ)-plane where $\lambda \geqslant u'(f(k))$ is the region where w^* varies and it is this region that emerges as the region of interest in the infinite-horizon problem. For this region one finds by (3.6), (3.7) and (4.4)

$$\left.\begin{aligned}
h(k, w^*, \lambda) &= u(w^*\phi(k)) + \lambda(1 - w^*)\phi(k) \\
\text{where } w^* &= y^*/\phi(k) \\
\text{where } u'(y^*) &= \lambda
\end{aligned}\right\} \tag{4.5}$$

and the corresponding Hamilton equations

$$dk/d\tau = \phi(k) - y^* \text{ and } d\lambda/d\tau = -\lambda\phi'(k). \tag{4.6}$$

Stationary solutions, where $dk/d\tau = 0$ and $d\lambda/d\tau = 0$, occur only in the region where $\lambda \geqslant u'(f(k))$. From (4.5) and (4.6) $dk/d\tau = 0$ only if $\lambda = u'(\phi(k))$; $d\lambda/d\tau = 0$ only if $\phi'(k) = 0$, that is, only if $k \geqslant 1$. For $k \geqslant 1$ $\phi(k) = \phi(1)$ so at a stationary point $\lambda = u'(\phi(1))$; but $\phi(1)$ is just the consumption y under these conditions and by (2.10) and (2.11) the maximum stationary dimensionless consumption y has the value 1 so

$$\phi(1) = (1 - \mu_1)f(1) = 1. \tag{4.7}$$

From (2.10) and (2.11) the actual maximum stationary consumption is $\bar{Y}$ where

$$\bar{Y} = \left\{\begin{aligned}
(1 - \mu_1)L\bar{F} & \quad \text{if } \mu_1 < \mu_2 \\
(1 - \mu_2)L\bar{F} & \quad \text{if } \mu_2 < \mu_1
\end{aligned}\right\}. \tag{4.8}$$

By (2.12) $u'(1) = 1$ so the bliss point, $(k, \lambda)_b$, the stationary point with the lowest utility-maximizing k, is given by

$$(k, \lambda)_b = (1, 1). \tag{4.9}$$

There is a line of further stationary points at $\lambda = 1$ and $k > 1$, corresponding to over-saturation of the production function. For the assumed, practical, case of $0 < k_0 < 1$ the region $k > 1$ is irrevelant and, in any case, it is easy to show that without a positive time-preference there is no *optimal* solution for $k_0 > 1$ (see, for example, Koopmans [9]).
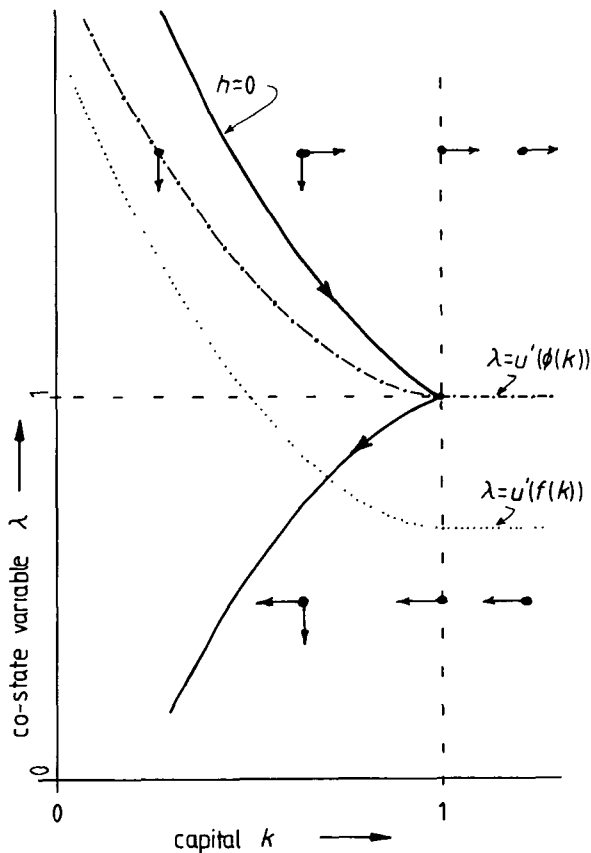

Figure 1. Extremal paths containing the bliss-point, $(1, 1)$, in (k, λ) phase-space.

At $(1, 1)$ h is zero and because in this problem h is constant along an optimal path one looks for paths containing $(1, 1)$ and along which $h = 0$. One finds from (4.5) that $h = 0$ gives for $\lambda \geq u'(f(k))$

$$\left.\begin{array}{r} u(y^*) + u'(y^*)(\phi(k) - y^*) = 0 \\ \text{where } u'(y^*) = \lambda. \end{array}\right\} \tag{4.10}$$

(The first of these is equivalent to the familiar Ramsey-Keynes rule of optimal investment,

$$(dk/d\tau)^* = -u(y^*)/u'(y^*),$$

easily obtainable without the apparatus of Pontryagin's theorem as the first integral of the Euler equation appropriate to (4.3).)

The equations (4.10) define two path-segments in (k, λ) phase-space: one with $\lambda > 1$, negatively sloped and convex, leading towards the bliss point; another with $\lambda < 1$, positively sloped and concave, leading away from the bliss point (see Figure 1). From the assumptions here on the infinite planning horizon, the concavity of $u(y)$ and $\phi(k)$, the saturation of $\phi(k)$, the vanishing of $u(y)$ at the bliss point and that $0 < k_0 < 1$ it can be shown that the path towards the bliss point is the unique optimal path; that on it the bliss values are approached asymptotically as $\tau \to \infty$ and that on it the total utility integral converges (see, for example, the immediately preceding references).

5. Synthesis of the optimal feedback control for an infinite planning horizon

With an infinite planning horizon and with $k_0 < 1$ one solves equations (4.10) for the branch of the path with $\lambda \geq 1$ giving λ and the optimal consumption y^* as functions of k and then the equation $w^* = y^*/\phi(k)$ gives the optimal control w^* directly in terms of the state variable k. As k enters these two equations only through $\phi(k)$ one can regard the current value of $\phi(k)$ as a state variable instead of k. If only the synthesis of the feedback control is needed, rather than a description of $k(\tau)$ or $y(\tau)$, then there is no need to determine or specify in detail the function $\phi(k)$. Apart from the two prices μ_1 and μ_2 the synthesis of the optimal control in terms of the current value of $\phi(k)$ depends only on the utility function.

A convenient family of utility functions u_η having the properties so far assumed is defined by

$$u_\eta(y) = \left\{\begin{array}{ll} (y^{1-\eta} - 1)/(1 - \eta) & \text{for } \eta > 1; \\ \log y & \text{for } \eta = 1. \end{array}\right\} \tag{5.1}$$

These functions u_η have certain theoretical virtues (Chakravarty [3] pp. 22–28) including that of a constant elasticity of marginal utility equal to $-\eta$.

The synthesis of the optimal control using u_η must generally be done numerically, but for $\eta = 2$ an analytic solution is easily found. For $\eta = 2$

$$u_\eta(y) = u_2(y) = 1 - y^{-1} \text{ and } u_2'(y) = y^{-2}. \tag{5.2}$$

Using these in (4.5) and (4.10) one finds the equation for the optimal path towards the bliss point

$$\lambda = 1 / \left(1 - \sqrt{1 - \phi(k)} \right)^2 \tag{5.3}$$

and the optimal consumption y^* and optimal feedback control w^*

$$\left. \begin{aligned} y^* &= 1 - \sqrt{1 - \phi(k)} \,; \\ w^* &= \left(1 - \sqrt{1 - \phi(k)} \right) / \phi(k). \end{aligned} \right\} \tag{5.4}$$

One sees from (5.4) that $w^* \to 1/2$ as $\phi(k) \to 0$ so with this utility function, u_2, one always uses for cereal production at least half the land that is eventually going to be used for cereal production, no matter how poor the land is to begin with. [If $\mu_2 < \mu_1$ then one always keeps for consumption at least half of the

TABLE 1

Optimal consumption ratio w^* as a function of productivity $\phi(k)$ for different utility indices η.

Productivity	Optimal consumption ratio w^*						
$\phi(k)$	$\eta = 1.0$	$\eta = 1.5$	$\eta = 2.0$	$\eta = 2.5$	$\eta = 3.0$	$\eta = 3.5$	$\eta = 4.0$
0.00	0.000	0.333	0.500	0.600	0.667	0.714	0.750
0.05	0.174	0.366	0.506	0.601	0.667	0.714	0.750
0.10	0.205	0.383	0.513	0.604	0.668	0.715	0.750
0.15	0.229	0.398	0.520	0.607	0 669	0.715	0.750
0.20	0.250	0.412	0.528	0.610	0.671	0.716	0.751
0.25	0.271	0.426	0.536	0.615	0.673	0.717	0.751
0.30	0.291	0.440	0.544	0.620	0.676	0.719	0.752
0.35	0.311	0.454	0.554	0.626	0.679	0.721	0.753
0.40	0.331	0.469	0.564	0.632	0.684	0.724	0.755
0.45	0.352	0.484	0.574	0.639	0.689	0.727	0.758
0.50	0.373	0.500	0.586	0.648	0.695	0.731	0.760
0.55	0.396	0.517	0.599	0.657	0.701	0.736	0.764
0.60	0.421	0.536	0.613	0.668	0.710	0.742	0.769
0.65	0.447	0.556	0.628	0.680	0.719	0.750	0.775
0.70	0.477	0.579	0.646	0.694	0.730	0.759	0.782
0.75	0.510	0.605	0.667	0.711	0.744	0.770	0.791
0.80	0.548	0.635	0.691	0.731	0.761	0.784	0.803
0.85	0.594	0.672	0.721	0.756	0.782	0.802	0.819
0.90	0.653	0.719	0.760	0.789	0.810	0.827	0.841
0.95	0.738	0.787	0.817	0.838	0.854	0.866	0.877
1.00	1.000	1.000	1.000	1.000	1.000	1.000	1.000
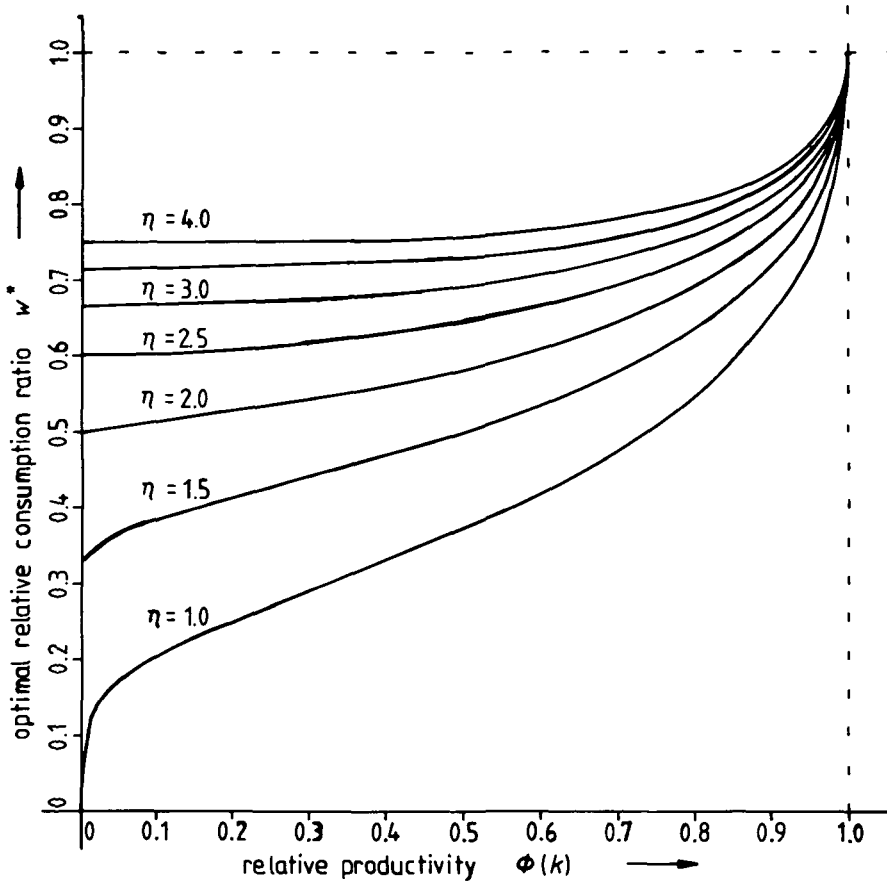
Figure 2. Optimal relative consumption ratio w^* as a function of relative productivity $\phi(k)$ for different utility function indices η.
$\phi(k)$ = productivity relative to the bliss-level of productivity. w^* = optimal consumption ratio relative to the bliss-level ratio. $w^*\phi(k)$ = optimal level of consumption relative to the bliss level.

fraction of the crop that is eventually to be kept for consumption.] One can show for the more general utility function, u_η, that

$$\lim_{\phi(k) \to 0} w^* = 1 - \eta^{-1}$$

which just reaches zero for $\eta = 1$, corresponding to the logarithmic utility function.

The optimal feedback control w^* is shown as a function of the current value of $\phi(k)$ in Table 1 and in Figure 2 for utility functions u_η with $\eta = 1(0.5)4$.

6. Behaviour in time of the optimally controlled system

The path in the (k, λ)-plane is prescribed by (4.10). From this one finds the appropriate value of λ for each value of k (numerically, if necessary) and then uses these in the Hamilton equation for $dk/d\tau$. Quadrature then yields the relation $k = k(\tau)$, from which all the other time-behaviour of the system can be deduced. For this result one has to specify the production function as well as the utility function and usually one must do the calculation numerically however two cases yield easy analytic solutions.

Suppose as in the example in §5 that the utility function u_2 is chosen from the family $\{u_\eta\}$ of (5.1), leading to the optimal-path equation (5.3) and through (4.6) and (5.4) to

$$dk/d\tau = \sqrt{1 - \phi(k)} - (1 - \phi(k)) \tag{6.1}$$

and so to

$$\int_{k_0}^{k} \frac{dk}{\sqrt{1 - \phi(k)} - (1 - \phi(k))} = \tau. \tag{6.2}$$

A family of functions $\{\phi_\nu\}$ $(\nu \geqslant 1)$ having all the properties assumed by (2.13) and (4.2) for ϕ is

$$\phi_\nu(k) = \left\{ \begin{array}{ll} 1 - (1 - k)^\nu & \text{for } 0 < k \leqslant 1; \\ 1 & \text{for } k > 1. \end{array} \right\} \tag{6.3}$$

Suppose $\nu = 2$ fits the data adequately (as in [1] and [6]). For this ϕ equation (6.1) becomes

$$dk/d\tau = (1 - k)k \tag{6.4}$$

and has the solution

$$k(\tau) = 1 \Big/ \left[1 + \frac{1 - k_0}{k_0} e^{-\tau} \right], \tag{6.5}$$

the well-known S-shaped logistic curve (Tinbergen and Bos [12] p. 27). Now using (5.3) and (5.4) one gets:

$$\lambda = 1/k^2; \quad w^* = 1/(2 - k) \quad \text{and} \quad y^* = k; \tag{6.6}$$

from which (using (6.5)) the explicit time-functions can be written down. In this example the optimal consumption increases towards its stationary value exactly in proportion to the increase in the capital stock of nitrogen.

One also finds for $\eta = 2$ and $\nu = 4$:

$$\lambda = \left[k(2-k)\right]^{-2}; \quad y^* = k(2-k);$$
$$w^* = \left[1+(1-k)^2\right]^{-1} \text{ and } dk/d\tau = (1-k)^2 k(2-k)$$

from which one gets

$$\tau = 1/(1-k) + \frac{1}{2}\log\left|k/(2-k)\right| - A$$

where

$$A = 1/(1-k_0) + \frac{1}{2}\log\left|k_0/(2-k_0)\right|.$$

It is convenient to define the dimensioned "optimal time-constant", $\overline{T}$, of the problem by

$$\overline{T} = \frac{\mu}{1-\mu}\frac{\overline{K}}{\alpha\overline{F}} \text{ (years) where } \mu = \min\{\mu_1, \mu_2\}; \tag{6.7}$$

that is, using (2.9-11) and (4.8),

$$\overline{T} = \begin{cases} \overline{K}/G(\overline{K}) & \text{for } \mu_1 < \mu_2; \\ \overline{K}/\left(\dfrac{P_C}{P_N} - \alpha\right)\overline{F} & \text{for } \mu_2 < \mu_1. \end{cases} \tag{6.8}$$

The first, $\overline{K}/G(\overline{K})$, is an average capital-to-output ratio in the production of nitrogen by green manure and similarly the second, $\overline{K}/(P_C/P_N - \alpha)\overline{F}$, is an average net capital-to-output ratio in the production of cereal by fertilizer when the nitrogen capital loss due to harvesting is taken into account and the appropriate prices are used for valuation.

Recalling (4.2) and (2.9-11) one sees from (6.7) that

$$\tau = T/\overline{T}. \tag{6.9}$$

The dimensionless time-variable τ is used to show the time-behaviour results for the infinite-horizon problem for utility indices $\eta = 1(0.5)4$ and production function indices $\nu = 2(1)4$ in Table 2* and Figure 3.

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*The calculations were done using for each (η,ν) $k_0 = 0.05$, 144 steps of size 0.00625 in k, a tolerance of 10^{-7} in the equation-solver for y^* and a Simpson's rule quadrature for τ. For the cases where we have the exact solution $(\eta = 2, \nu = 2,4)$ the final errors in the computation for τ, at $k = 0.95$, were less than 0.02%.

TABLE 2(a)

Time-paths of optimal relative capital k and consumption y for utility indices $\eta = 1\ 0\ (0.5)\ 4.0$ and production function indices $\nu = 2, 3, 4$.

$\nu = 2$

k	$\phi(k)$	$\eta = 1.0$		$\eta = 1.5$		$\eta = 2.0$		$\eta = 2.5$		$\eta = 3.0$		$\eta = 3.5$		$\eta = 4.0$	
		y	τ	y	τ	y	τ	y	τ	y	τ	y	τ	y	τ
.05	.098	.020	.000	.037	.000	.050	.000	.059	.000	.065	.000	.070	.000	.073	.000
.10	.190	.047	.463	.078	.594	.100	.747	.116	.913	.127	1.085	.136	1.261	.143	1.440
.15	.278	.078	.756	.120	.968	.150	1.210	.171	1.471	.187	1.744	.199	2.024	.209	2.307
.20	.360	.113	.981	.164	1.251	.200	1.558	.226	1.888	.245	2.232	.260	2.586	.271	2.944
.25	.438	.152	1.168	.210	1.488	.250	1.846	.279	2.229	.301	2.629	.318	3.040	.331	3.458
.30	.510	.193	1.334	.257	1.695	.300	2.097	.331	2.526	.355	2.972	.373	3.430	.388	3.896
.35	.578	.237	1.486	.305	1.885	.350	2.325	.383	2.793	.408	3.279	.427	3.778	.443	4.285
.40	.640	.283	1.629	.353	2.063	.400	2.539	.433	3.042	.459	3.564	.479	4.098	.495	4.642
.45	.698	.331	1.767	.403	2.235	.450	2.744	.484	3.279	.509	3.834	.529	4.401	.545	4.978
.50	.750	.382	1.903	.454	2.404	.500	2.944	.533	3.511	.558	4.096	.578	4.694	.594	5.301
.55	.798	.435	2.040	.505	2.574	.550	3.145	.582	3.742	.606	4.356	.625	4.983	.640	5.619
.60	.840	.491	2.180	.558	2.748	.600	3.350	.630	3.976	.653	4.619	.671	5.275	.685	5.939
.65	.878	.548	2.327	.611	2.929	.650	3.563	.678	4.220	.699	4.892	.715	5.576	.729	6.267
.70	.910	.607	2.485	.664	3.124	.700	3.792	.725	4.479	.744	5.181	.759	5.894	.771	6.614
.75	.938	.668	2.659	.719	3.339	.750	4.043	.772	4.764	.788	5.498	.801	6.241	.812	6.990
.80	.960	.731	2.860	.774	3.586	.800	4.331	.818	5.089	.832	5.859	.843	6.635	.851	7.416
.85	.978	.796	3.103	.829	3.886	.850	4.679	.864	5.482	.875	6.292	.883	7.108	.890	7.926
.90	.990	.862	3.428	.886	4.284	.900	5.142	.910	6.002	.917	6.866	.923	7.730	.928	8.596
.95	.998	.930	3.954	.943	4.930	.950	5.889	.955	6.841	.959	7.787	.962	8.729	.964	9.667

TABLE 2(b)

$\nu = 3$

k	$\phi(k)$	$\eta = 1.0$		$\eta = 1.5$		$\eta = 2.0$		$\eta = 2.5$		$\eta = 3.0$		$\eta = 3.5$		$\eta = 4.0$	
		y	τ	y	τ	y	τ	y	τ	y	τ	y	τ	y	τ
.05	.143	.032	.000	.056	.000	.074	.000	.086	.000	.095	.000	.102	.000	.107	.000
.10	.271	.076	.332	.117	.423	.146	.527	.167	.639	.183	.755	.194	.875	.204	.997
.15	.386	.125	.551	.179	.701	.216	.867	.243	1.045	.263	1.231	.279	1.423	.291	1.618
.20	.488	.180	.727	.242	.921	.284	1.135	.315	1.362	.338	1.598	.356	1.841	.371	2.089
.25	.578	.237	.880	.305	1.113	.350	1.366	.383	1.633	.408	1.910	.428	2.195	.443	2.485
.30	.657	.297	1.022	.368	1.291	.414	1.578	.448	1.880	.473	2.192	.493	2.512	.510	2.839
.35	.725	.358	1.159	.429	1.461	.476	1.781	.509	2.114	.534	2.458	.554	2.810	.571	3.169
.40	.784	.420	1.296	.490	1.630	.535	1.981	.567	2.345	.592	2.719	.611	3.100	.627	3.488
.45	.834	.482	1.435	.549	1.803	.592	2.185	.622	2.578	.645	2.981	.663	3.391	.678	3.807
.50	.875	.544	1.581	.607	1.983	.646	2.397	.674	2.821	.695	3.252	.712	3.691	.726	4.134
.55	.909	.605	1.738	.662	2.177	.698	2.624	.723	3.079	.742	3.541	.757	4.008	.769	4.479
.60	.936	.664	1.912	.715	2.391	.747	2.874	.769	3.363	.786	3.856	.799	4.353	.809	4.854
.65	.957	.722	2.109	.766	2.634	.793	3.158	.812	3.683	.826	4.211	.837	4.741	.846	5.274
.70	.973	.777	2.341	.813	2.920	.836	3.490	.851	4.058	.863	4.625	.872	5.193	.879	5.761
.75	.984	.829	2.627	.857	3.270	.875	3.897	.887	4.516	.896	5.131	.903	5.743	.909	6.353
.80	.992	.876	2.998	.898	3.725	.911	4.425	.919	5.109	.926	5.784	.931	6.452	.935	7.115
.85	.997	.919	3.523	.933	4.370	.942	5.171	.948	5.946	.952	6.704	.956	7.449	.958	8.184
.90	.999	.956	4.378	.964	5.418	.968	6.384	.972	7.305	.974	8.195	.976	9.063	.977	9.913
.95	1.000	.984	6.265	.987	7.729	.989	9.055	.990	10.293	.991	11.472	.992	12.605	.992	13.703

TABLE 2(c)

ν = 4

k	φ(k)	η = 1.0		η = 1.5		η = 2.0		η = 2.5		η = 3.0		η = 3.5		η = 4.0	
		y	τ	y	τ	y	τ	y	τ	y	τ	y	τ	y	τ
.05	.185	.045	.000	.076	.000	.098	.000	.113	.000	.124	.000	.133	.000	.139	.000
.10	.344	.106	.266	.156	.338	.190	.418	.215	.503	.234	.592	.248	.684	.259	.778
.15	.478	.174	.450	.236	.570	.278	.699	.308	.837	.331	.980	.349	1.128	.363	1.278
.20	.590	.246	.604	.314	.762	.360	.931	.393	1.108	.418	1.292	.438	1.481	.453	1.673
.25	.684	.319	.744	.391	.937	.438	1.140	.471	1.351	.497	1.569	.517	1.792	.533	2.020
.30	.760	.393	.880	.464	1.106	.510	1.340	.543	1.583	.568	1.831	.587	2.085	.603	2.343
.35	.821	.466	1.018	.534	1.277	.578	1.542	.608	1.814	.632	2.092	.650	2.375	.665	2.661
.40	.870	.536	1.163	.600	1.456	.640	1.753	.668	2.055	.690	2.361	.706	2.672	.720	2.986
.45	.908	.604	1.319	.662	1.649	.698	1.979	.723	2.312	.742	2.648	.757	2.988	.769	3.330
.50	.938	.668	1.493	.719	1.863	.750	2.230	.772	2.597	.788	2.965	.801	3.335	.812	3.706
.55	.959	.728	1.693	.771	2.109	.798	2.517	.816	2.921	.830	3.324	.841	3.727	.849	4.131
.60	.974	.782	1.930	.818	2.401	.840	2.856	.855	3.303	.866	3.746	.875	4.188	.882	4.627
.65	.985	.832	2.221	.860	2.758	.878	3.271	.889	3.770	.898	4.262	.905	4.748	.911	5.231
.70	.992	.875	2.595	.897	3.217	.910	3.803	.919	4.368	.926	4.920	.931	5.463	.935	5.999
.75	.996	.913	3.103	.928	3.839	.938	4.524	.944	5.177	.949	5.809	.952	6.426	.955	7.032
.80	.998	.944	3.845	.954	4.749	.960	5.576	.964	6.356	.967	7.104	.970	7.829	.971	8.535
.85	.999	.968	5.058	.974	6.236	.978	7.295	.980	8.280	.982	9.214	.983	10.111	.984	10.978
.90	1.000	.986	7.448	.988	9.164	.990	10.679	.991	12.066	.992	13.364	.992	14.597	.993	15.776
.95	1.000	.996	14.556	.997	17.866	.998	20.729	.998	23.301	.998	25.683	.998	27.906	.998	30.010

7. Illustration

I shall use nominal values of P_N, P_C, α, β, K_0, $\overline{K}$ and $\overline{F}$, using some Indian data for "local-variety" wheat, to illustrate the application and limitations of the results of §§5–6.

The 1973–74 prices of wheat and chemical nitrogenous fertilizer were $P_C = 0.76$ rupees/kg and $P_N = 5.00$ rupees/kg (of N) ([1], Table 1). Typical wheat yield and nitrogen removal rates per crop period are 1570 kg/ha and 56 kg/ha ([7], Table 19, p. 82) and the nitrogen response function is $\rho(x) = 18.9x - 0.166x^2$ where ρ kg/ha is the additional yield due to a dose of x kg/ha of chemical fertilizer nitrogen (quoted in [1]). Assuming this is additional to the "typical
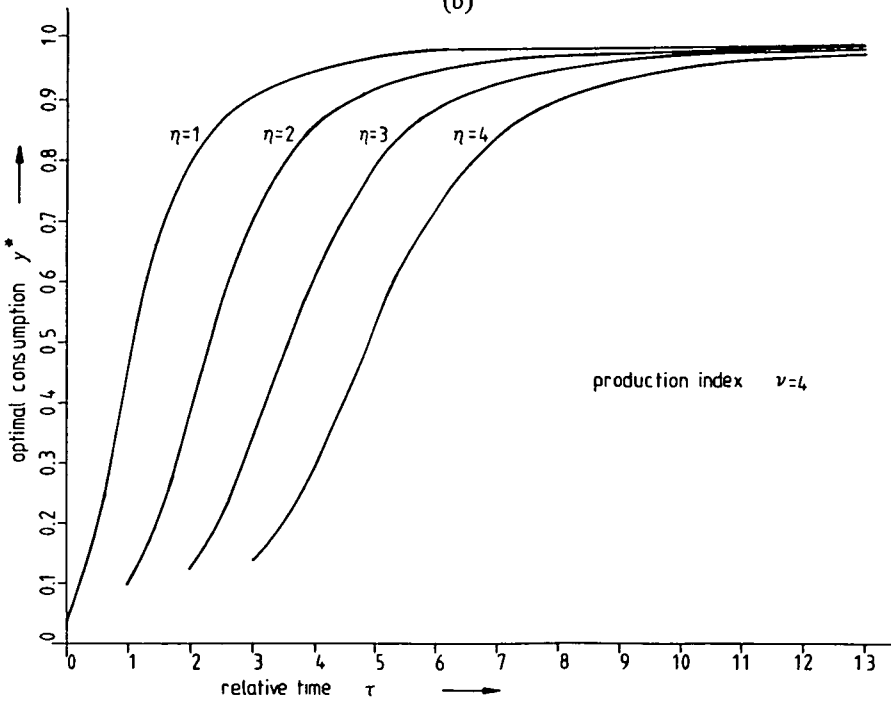

(a)
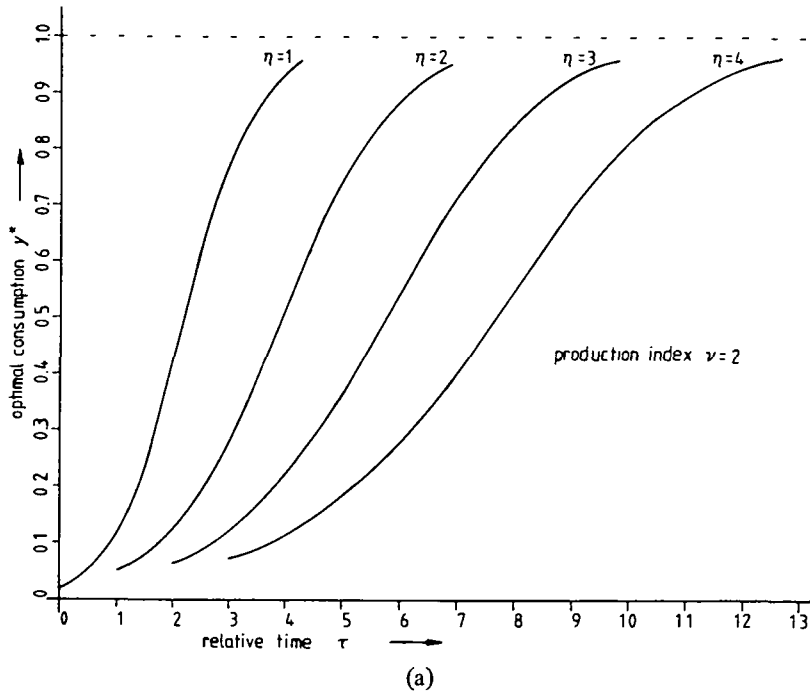
Figure 3. Time-paths of optimal consumption for different utility function indices η and production function indices ν.
   (a) Production index $\nu = 2$.
   (b) $\nu = 3$.
   (c) $\nu = 4$.
(The path for each η-value has been advanced by $\eta - 1$ relative time-units from the tabulated values of Table 2.)
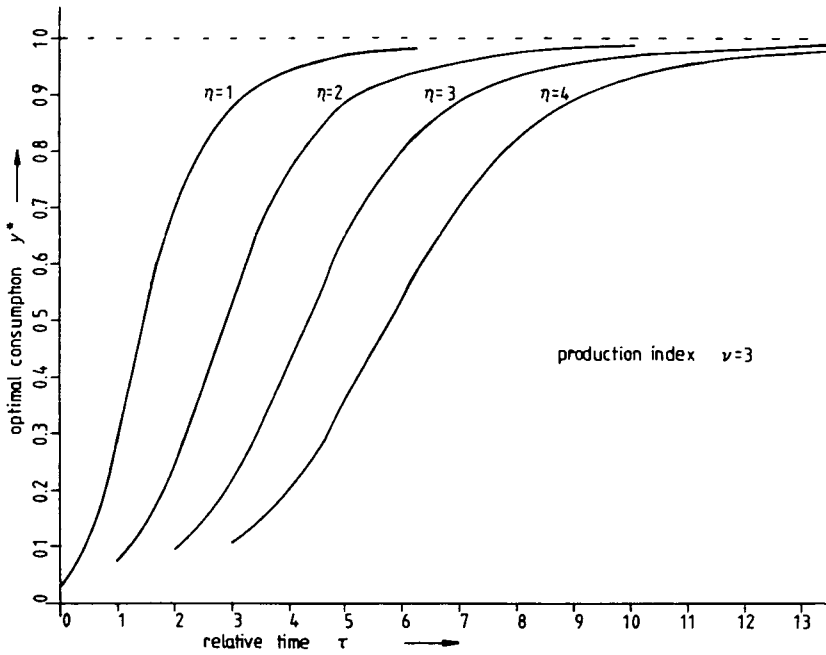
(b)

(c)

Figure 3 (continued)

yield" of [7] and extrapolating one gets the production function $F(K) = (\overline{F}/\overline{K}^2)K(2\overline{K} - K)$ (for $0 \leqslant K \leqslant \overline{K}$) where $\overline{K} = 110$ kg/ha and $\overline{F} = 2100$ kg/ha yr. A crop of leguminous green manure adds 56-90 kg/ha of nitrogen when ploughed under ([7] p. 94) and two to three crops of it can be grown during a single cereal crop period. From this data one estimates that $0.035 \leqslant \alpha \leqslant 0.040$ and $0.070 \leqslant \beta \leqslant 0.172$.

In Table 3 are values for three cases: (a) 1973–74 prices and the values of α and β within the estimated ranges leading to an upper estimate of μ_1; (b) 1973–74 prices and the values of α and β leading to a lower estimate of μ_1; (c) as in (a) but with the chemical fertilizer price 40% higher.

TABLE 3

Parameter values for three illustrative cases

Case	P_C (r/kg)	P_N (r/kg)	α	β	μ_1	μ_2	optimal fertilizer	$\overline{T}$ (yr)
(a)	0.76	5.00	0.040	0.070	0.364	0.263	chemical	0.468
(b)	0.76	5.00	0.035	0.172	0.169	0.230	green manure	0.305
(c)	0.76	7.00	0.040	0.070	0.364	0.368	green manure	0.748

In case (a) $\mu_2 < \mu_1$ so the optimal plan is to trade some of the crop for chemical fertilizer; however suppose the farmer has a utility index $\eta = 1$ and an initial farm yield 0.190 of its saturation value then one sees from Table 2 ($\eta = 1$, $\nu = 2$) that the optimal control w^* starting with 0.246 and increasing to 0.932 will take him to 93% of his bliss-level consumption in a τ-time of 3.491, that is (using $\overline{T} = 0.468$) in 1.63 years or fewer than two harvests! This shows that for these parameter values and a low utility index the continuous-time model is inappropriate if one is interested in that part of the trajectory covering the 20–90% of the bliss-level consumption and one should use a discrete-time model.

In case (b) one sees for the same prices but with an optimistic estimate for the production of green-manure nitrogen (3 crops yielding 90 kg/ha for each crop of cereal) that the optimal plan would now allocate some land for green manure but that again the continuous-time model is inappropriate for examining the lower-level consumption paths for low utility indices.

In case (c), with only a 40% rise in fertilizer price and the same production of green-manure nitrogen as in (a), the optimal plan now allocates land for green manure. If the farmer has a utility index $\eta = 4$ (a much stronger aversion to low consumption) then, referring to Table 2, with the same initial yield of 0.190 $\overline{F}$ as in (a) the optimal control w^* starting at 0.751 and increasing to 0.967 (corresponding to v_1^* going from 0.523 to 0.385) will take him to 96.4% of his bliss-level

consumption in a τ-time of 8.227, that is (using $\overline{T} = 0.7483$) in 6.16 years. For these values of the parameters the continuous-time model is a reasonably good guide to the optimal plan.

To illustrate the application of §§5–6 to the Ramsey-Koopmans model one must convert it to the appropriate form. Suppose the rate of relative population growth is constant at N per year, the *per capita* consumption at time T years is $Y(T)$ \$/man year, the capital-to-labour ratio is $K(T)$ \$/man, the *per capita* production is $G(K)$ \$/man year and is such that $G'(\overline{K}) = N$ for some finite $\overline{K}$. The capital growth equation is $dK/dT = G(K) - NK - Y$ and there is a stationary maximum consumption $\overline{Y} = G(\overline{K}) - N\overline{K}$. Define $\hat{T} = \overline{K}/(G(\overline{K}) - N\overline{K})$ and dimensionless quantities $y = Y/\overline{Y}$, $k = K/\overline{K}$, $t = T/\hat{T}$, $n = N\hat{T}$, $g(k) = (\hat{T}/\overline{K})G(k\overline{K})$ and $\phi(k) = g(k) - nk$ and the capital growth equation becomes $dk/dt = \phi(k) - y$ and one now seeks to maximize $\int_0^\infty u(y)\,dt$ where u satisfies (2.12). The problem is now in the standardized form (4.3). The saving ratio s corresponds to the control variable v_1 in §4 and has the golden-rule value $n/(1 + n)$ so in (4.2) one must set $\mu_1 = n/(1 + n)$. The optimal saving ratio s^* at any time is given by $s^* = 1 - (1 - \mu_1)w^*$; that is, $s^* = 1 - w^*/(1 + n)$.

Suppose, for example, $N = 4\%$ per year and $G(K) = 0.15K - 2.5 \times 10^{-6}K^2$. One finds $\overline{K} = 22 \times 10^3$, $\overline{Y} = 1210$, $\hat{T} = 18.2$ years, $n = 0.727$, $\mu_1 = 0.421$ and the optimal time-constant $\overline{T} = \mu_1\hat{T} = 7.66$ years. From Table 2 with $\nu = 2$ and choosing $\eta = 4$ one sees that if current output corresponds to $K/\overline{K} = k = 0.50$ (that is 1347.5 \$/man year) then starting with $w^* = 0.791$ (or $s^* = 0.542$) and $y = 0.594$ one can progress along the optimal path to $y = 0.964$ (where $w^* = 0.967$ or $s^* = 0.440$) after a τ-interval of 4.366; that is, after $4.366 \times \overline{T} = 33.4$ years the consumption will be 3.6% below the golden-rule value.

8. Conclusion

I have illustrated the application of my model and the calculations from it to a farm economy using parameter values approximating those for Indian local-variety wheat and 1973-74 prices and for this cereal one saw that the model could be useful on poor farms only for a high utility index and a high chemical fertilizer price. The model could be useful all the way down to the lowest indices of utility and production functions only if the empirical parameters P_C, P_N, α, β, $\overline{K}$ and $\overline{F}$ (defined in §2) make the "optimal time-constant" (defined in (6.7)) of the order of 10 years or more. Such a time-constant corresponds to a combination of low average productivity of capital and high relative prices of both the home-produced and foreign capital goods.

In the application illustrated the estimated values of the relative prices μ_1 and μ_2 had a wide spread and straddled the dividing line, $\mu_1 = \mu_2$, between the two

regimes of the optimal plan. This points to the need for good experimental determination of the various functions involved.

The simplifying assumptions made here may be poor approximations; nevertheless for an initially poor and unproductive economy facing high prices for foreign capital goods this study has pointed to the possibility of developing useful practical guide-lines for the long-term planning at the local level of the optimal and self-sufficient path towards the bliss-point of maximum stationary consumption.

9. Acknowledgements

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