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MONOLITHIC BRAUER CHARACTERS

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Abstract

Let G be a group, p be a prime and $P \in \mathrm{Syl}_p(G)$. We say that a p-Brauer character φ is monolithic if $G/\ker\varphi$ is a monolith. We prove that P is normal in G if and only if $p \nmid \varphi(1)$ for each monolithic Brauer character $\varphi \in \mathrm{IBr}(G)$. When G is p-solvable, we also prove that P is normal in G and G/P is nilpotent if and only if $\varphi(1)^2$ divides $|G : \ker\varphi|$ for all monolithic irreducible p-Brauer characters φ of G.

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1. Introduction

All groups throughout this note are finite. For notation and terminology, we refer the reader to [7] for character theory and [13] for Brauer character theory. Let G be a group and p be a prime number. We write $\mathrm{Irr}(G)$ and $\mathrm{IBr}(G)$ for the sets of irreducible (complex) characters and irreducible (p-)Brauer characters of G, respectively.

Recall that a group G is a *monolith* if it has only one minimal normal subgroup. A (complex) character χ is said to be *monolithic* if $G/\ker\chi$ is a monolith. In a similar fashion, we will now define a p-Brauer character φ of G to be *monolithic* if $G/\ker\varphi$ is a monolith.

Suppose that G is a group and P is a Sylow p-subgroup of G for some prime p. Itô showed that P is normal and abelian if and only if p does not divide the degree of every irreducible character of G, provided this is true for every simple group (see [7, Theorem 12.33]). In 1986, Michler used the classification of finite simple groups to show that the condition is true for simple groups, so the Itô–Michler theorem asserts that P is a normal abelian Sylow p-subgroup of a group G if and only if $p \nmid \chi(1)$ for every $\chi \in \mathrm{Irr}(G)$ (see [9, 12]). Navarro [14] gave an excellent discussion of the variations and extensions of the Itô–Michler theorem. Berkovich and Zhmud proved

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in [1] that $P \lhd G$ and P is abelian if and only if $p \nmid \chi(1)$ for every monolithic $\chi \in \mathrm{Irr}(G)$. Interestingly, there seems to be a parallel between results that can be proved using only monolithic characters and results about solvable groups that can be proved using only monomial characters. For example, Pang and Lu proved in [15] that if G is solvable and p does not divide $\chi(1)$ for all monomial $\chi \in \mathrm{Irr}(G)$, then P is normal in G.

The Brauer character version of the Itô–Michler theorem says that P is a normal Sylow p-subgroup of a group G if and only if $p \nmid \varphi(1)$ for every Brauer character $\varphi \in \mathrm{IBr}(G)$ (see [14, Theorem 3.1]). With the hypothesis that G is solvable, we proved in [3] that $P \in \mathrm{Syl}_p(G)$ is normal if and only if $p \nmid \varphi(1)$ for any monomial Brauer character $\varphi \in \mathrm{IBr}(G)$. Motivated by the parallel between the results of Pang and Lu and Berkovich and Zhmud, we replace monomial Brauer characters by monolithic Brauer characters to obtain the following result.

Theorem 1.1. *Let G be a group and let p be a prime. Then p does not divide $\varphi(1)$ for every monolithic Brauer character $\varphi \in \mathrm{IBr}(G)$ if and only if G contains a normal Sylow p-subgroup.*

Combining Theorem 1.1 with the Brauer version of the Itô–Michler theorem, the following corollary is immediate.

Corollary 1.2. *Let G be a group, p a prime and P a Sylow p-subgroup of G. Then the following statements are equivalent:*

(a) *P is normal in G;*
(b) *$p \nmid \varphi(1)$ for every Brauer character $\varphi \in \mathrm{IBr}(G)$;*
(c) *$p \nmid \varphi(1)$ for every monolithic Brauer character $\varphi \in \mathrm{IBr}(G)$.*

In [5], Gagola and the second author characterised nilpotent groups through a character-theoretic condition. In fact, they proved that a group G is nilpotent if and only if $\chi(1)^2$ divides $|G : \ker \chi|$ for all $\chi \in \mathrm{Irr}(G)$. Recently, Lu *et al.* in [11] showed that a group G is nilpotent if and only if $\chi(1)^2$ divides $|G : \ker \chi|$ for all monolithic irreducible characters χ of G. Lu proved in [10] that when G is solvable, G is nilpotent if and only if $\chi(1)^2$ divides $|G : \ker \chi|$ for all monomial characters $\chi \in \mathrm{Irr}(G)$.

The authors with Cossey and Tong-Viet proved in [2] that $\varphi(1)^2$ divides $|G : \ker \varphi|$ for all $\varphi \in \mathrm{IBr}(G)$ if and only if G has a normal Sylow p-subgroup P and G/P is nilpotent. In [4], we proved that when G is solvable, P is normal in G and G/P is nilpotent if and only if $\varphi(1)^2$ divides $|G : \ker \varphi|$ for all monomial Brauer characters $\varphi \in \mathrm{IBr}(G)$. Again, motivated by the parallels between monomial and monolithic characters that we mentioned above, we prove the following result.

Theorem 1.3. *Let G be a p-solvable group, p a prime number and $P \in \mathrm{Syl}_p(G)$. Then P is normal in G and G/P is nilpotent if and only if $\varphi(1)^2$ divides $|G : \ker \varphi|$ for all monolithic Brauer characters $\varphi \in \mathrm{IBr}(G)$.*

2. Proofs

We first prove Theorem 1.1. Recall that a group is p-closed if and only if it has a normal Sylow p-subgroup.

Proof of Theorem 1.1. Let P be a Sylow p-subgroup of G. Suppose first that P is normal in G. Since $P \subseteq \ker\varphi$ for every $\varphi \in \mathrm{IBr}(G)$,

$$\mathrm{IBr}(G) = \mathrm{IBr}(G/P) = \mathrm{Irr}(G/P),$$

where the second equality holds since G/P is a p'-group. It follows that $p \nmid \varphi(1)$ for every Brauer character $\varphi \in \mathrm{IBr}(G)$.

Conversely, suppose that p does not divide $\varphi(1)$ for every monolithic Brauer character $\varphi \in \mathrm{IBr}(G)$. We work by induction on $|G|$. Suppose that G has two different minimal normal subgroups, say N_1 and N_2. By induction, the quotients G/N_i for $i = 1$ and 2 are p-closed. Let P be a Sylow p-subgroup of G. This implies that PN_i is normal in G for $i = 1, 2$. If N_i is a p-group for either $i = 1$ or 2, then $N_i \leq P$ and so P is normal in G. We may assume that N_1 and N_2 are p'-groups. This implies that N_i is a Hall p-complement of PN_i for each i and, since it is normal, it is the unique Hall p-complement of PN_i. It follows that $PN_1 = PN_2$ would imply that $N_1 = N_2$. Thus, we must have $PN_1 \cap PN_2 < PN_i$ for $i = 1$ or 2. Observe that $P \leq PN_1 \cap PN_2$. Also, $PN_1 \cap PN_2$ will be normal in G. Since N_i is minimal normal, we see that $N_i \cap PN_1 \cap PN_2$ is either 1 or N_i. If it is N_i, then $PN_i \leq PN_1 \cap PN_2$ and this would imply that N_i is contained in both N_1 and N_2, which would imply that $N_1 = N_2$, which is a contradiction. Thus, $N_i \cap PN_1 \cap PN_2 = 1$ and so $P = PN_1 \cap PN_2$. This implies that G is p-closed.

We now assume that G is a monolith and let N be the unique minimal normal subgroup of G. Since p does not divide $\varphi(1)$ for every monolithic Brauer character $\varphi \in \mathrm{IBr}(G)$, we can use induction to see that G/N has a normal Sylow p-subgroup PN/N and so PN is normal in G. If N is a p-group, then $PN = P$ and P is normal in G, as desired.

We suppose that N is a p'-group, so PN is not a p-group. Let H/N be a minimal normal p-subgroup of G/N. We see that $H \cap P$ is a Sylow p-subgroup of H and $H \cap P$ is not normal in H because G is a monolith. By the Brauer character version of the Itô–Michler theorem, there exists a nonlinear Brauer character $\mu \in \mathrm{IBr}(H)$ with $p \mid \mu(1)$. Take an irreducible Brauer constituent φ of μ^G. Then $p \mid \varphi(1)$ and so φ is not monolithic by the hypothesis. Hence, φ is not faithful. It follows that $N \leq \ker\varphi$ and, thus, $N \leq \ker\mu$. Therefore, $\mu \in \mathrm{IBr}(H/N)$ is the principal Brauer character, which is a contradiction.

Consequently, N is divisible by p, but is not a p-group. This implies that N is not solvable and so N is a direct product of isomorphic nonabelian simple groups of order divisible by p. Thus, there exists a Brauer character $\theta \in \mathrm{IBr}(N)$ so that $p \mid \theta(1)$ by the Brauer character version of the Itô–Michler theorem. Let φ be an irreducible Brauer constituent of θ^G. We claim that φ is faithful, that is, $\ker\varphi = 1$. Otherwise, $N \subseteq \ker\varphi$. This implies that the set of p-regular elements of N is contained in $\ker\theta$.

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However, we know that $\ker\theta = 1$, so this would imply that N is a p-group, which is a contradiction. Hence, φ is monolithic and $p \nmid \varphi(1)$ by the hypothesis. However, since θ is an irreducible Brauer constituent of φ_N, it follows that $p \mid \varphi(1)$ as p divides $\theta(1)$. This contradiction completes the proof. □

We now prove Theorem 1.3.

Proof of Theorem 1.3. Suppose that G has a normal Sylow p-subgroup P and G/P is nilpotent. From [2, Lemma 3.4], $\varphi(1)^2$ divides $|G : \ker\varphi|$ for all Brauer characters $\varphi \in \mathrm{IBr}(G)$ and so it certainly holds for the monolithic Brauer characters.

Conversely, suppose that $\varphi(1)^2$ divides $|G : \ker\varphi|$ for all monolithic Brauer characters $\varphi \in \mathrm{IBr}(G)$. We work by induction on $|G|$ to complete the proof. Assume first that $\mathbf{O}_p(G) > 1$. The hypothesis holds for $G/\mathbf{O}_p(G)$ and so, by induction, $G/\mathbf{O}_p(G)$ has a normal Sylow p-subgroup, meaning that G has a normal Sylow p-subgroup. Moreover, induction (applied to $G/\mathbf{O}_p(G)$) shows that $G/\mathbf{O}_p(G)$ is nilpotent. Thus, we may assume that $\mathbf{O}_p(G) = 1$. It suffices to prove that G is a p'-group since if this is true we will have $\mathrm{Irr}(G) = \mathrm{IBr}(G)$ and we will be done by [11, Theorem 1.2]. Thus, we work to prove that G is a p'-group.

Let M be a minimal normal subgroup of G. Since $\mathbf{O}_p(G) = 1$ and G is p-solvable, it follows that M is a p'-subgroup of G. Observe that G/M satisfies the hypothesis. By induction, G/M will have a normal Sylow p-subgroup N/M and G/N will be nilpotent. If P is a Sylow p-subgroup of G, then $N = MP$. The Frattini argument implies that

$$G = N\mathbf{N}_G(P) = MP\mathbf{N}_G(P) = M\mathbf{N}_G(P).$$

If $M_1 \neq M$ is another minimal normal subgroup of G, then M_1 would also be a p'-subgroup of G. Note that M is the normal Hall p-complement of N, so $N \cap M_1 \leq M$. If $M_1 \cap N > 1$, then $M_1 \cap N = M$ since M is minimal normal in G. This implies that $M \leq M_1$ and so $M = M_1$ since M_1 is minimal normal. This is a contradiction, so $M_1 \cap N = 1$. Since M_1 and N are normal subgroups, M_1 centralises N and M_1 centralises P as $P \subseteq N$. Applying the previous argument with M_1 in place of M, we see that $G = M_1\mathbf{N}_G(P)$ and, since M_1 centralises P, it follows that $G = \mathbf{N}_G(P)$ and P is normal in G. Since $\mathbf{O}_p(G) = 1$, we have $P = 1$ and G is a p'-group, as desired.

Therefore, we may assume that G is a monolith and M is the unique minimal normal subgroup of G. If M is not abelian, then

$$M = S_1 \times S_2 \times \cdots \times S_k,$$

where $S_i \cong S$ and S is a nonabelian simple group. Since M is unique and nonabelian, $\mathbf{C}_G(M) = 1$ and G is isomorphic to a subgroup of $\mathrm{Aut}(M) = \mathrm{Aut}(S) \wr \mathbf{S}_k$, where $\mathbf{S}_k$ is the symmetric group on k letters. Using [5, Proposition 2.1], there exist a prime q and a character $\sigma \in \mathrm{Irr}(S)$ such that $\sigma(1)_q = |S|_q$ and $|\mathrm{Out}(S)|_q < |S|_q$. Write

$$\theta = \underbrace{\sigma \times \cdots \times \sigma}_{k}.$$

By Brauer character theory, there exists a Brauer character $\varphi \in \mathrm{IBr}(G)$ that lies over θ. It follows by the Fong–Swan theorem that there is a character $\chi \in \mathrm{Irr}(G)$ such that

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$\chi^o = \varphi$, where χ^o is the restriction of χ to the set of p-regular elements of G. Notice that χ also lies over θ and $\ker \chi = 1$. By [2, Lemma 3.3], it follows that φ is monolithic. Observe that $\sigma(1)^k$ divides $\varphi(1)$. Since σ has q-defect 0, we see that $|M|_q$ divides $\varphi(1)_q$. Using the same computation as in Case 1 of the proof of Theorem A of [5],

$$|G|_q < {|M|_q}^2 \leq \varphi(1)^2$$

and so $\varphi(1)^2$ does not divide $|G|$, which is a contradiction.

If M is abelian, it follows that M is an elementary abelian r-group for some prime $r \neq p$. Since $G = M\mathbf{N}_G(P)$, it follows that $M \cap N_G(P)$ is a normal subgroup of G. As $\mathbf{O}_p(G) = 1$, we have that $M \cap N_G(P) = 1$. Observe that $\mathbf{N}_G(P)$ will normalise $\mathbf{C}_P(M)$ and obviously M normalises $\mathbf{C}_P(M)$. Hence, $\mathbf{C}_P(M)$ is normal in $G = M\mathbf{N}_G(P)$. Since $M \cap P = 1$ and M is the unique minimal normal subgroup of G, we conclude that $\mathbf{C}_P(M) = 1$. Hence, P acts faithfully on M and thus it also acts faithfully on $\mathrm{Irr}(M) = \mathrm{IBr}(M)$.

Notice that P acts coprimely and faithfully on $\mathrm{IBr}(M)$. Applying Isaacs' large orbit result [8, Theorem B], we see that there exists a Brauer character $\lambda \in \mathrm{IBr}(M)$ such that $|\mathbf{C}_P(\lambda)| < \sqrt{|P|}$, which implies that $|P : \mathbf{C}_P(\lambda)| > \sqrt{|P|}$.

Denote by T the inertia group of λ in G and write $S = M\mathbf{C}_P(\lambda)$. Notice that S is the stabiliser of λ in N and $S \leq T$. In particular, $S = T \cap N$. It follows that S/M is the Sylow p-subgroup of T/M because N/M is the Sylow p-subgroup of G/M. From the fact that $|N : S| = |P : \mathbf{C}_P(\lambda)|$, we conclude that

$$|G : T|_p = |N/M : S/M| = |P : \mathbf{C}_P(\lambda)| > \sqrt{|P|} = \sqrt{|G|_p}.$$

In particular, $|G : T|^2$ does not divide $|G|$.

Note that M is complemented in T. It follows from [6, Lemma 1] that there exists a Brauer character $\mu \in \mathrm{IBr}(T)$ such that $\mu_M = \lambda$. By the Clifford correspondence (see [13, Theorem 8.9]), it follows that $\varphi = \mu^G \in \mathrm{IBr}(G)$. Since M is the unique minimal normal subgroup of G, it follows that $\ker \varphi = 1$. Otherwise, $M \subseteq \ker \varphi$ and then $M \subseteq \ker \mu$, so μ_M is the principal Brauer character of M, which is impossible. Therefore, φ is monolithic. Notice that $\varphi(1) = |G : T|$. By the previous paragraph, it follows that $|G : T|^2$ does not divide $|G|$, which is a contradiction with the hypothesis that $\varphi(1)^2$ divides $|G : \ker \varphi|$ for all monolithic irreducible p-Brauer characters of G. Therefore, G is a p'-group, which is the desired conclusion. □

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