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THE INDUCTIVE STEP OF THE SECOND BRAUER-THRALL CONJECTURE

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Introduction. In this paper we are going to use a result of H. Harada and Y. Sai concerning composition of nonisomorphisms between indecomposable modules and the theory of almost split sequences introduced in the representation theory of Artin algebras by M. Auslander and I. Reiten to obtain the inductive step in the second Brauer-Thrall conjecture.

Section 1 is devoted to giving the necessary background in the theory of almost split sequences.

As an application we get the first Brauer-Thrall conjecture for Artin algebras. This conjecture says that there is no bound on the length of the finitely generated indecomposable modules over an Artin algebra of infinite type, i.e., an Artin algebra such that there are infinitely many nonisomorphic indecomposable finitely generated modules. This result was first proved by A. V. Roiter [**8**] and later in general for Artin rings by M. Auslander [**2**] using categorical methods. Recently the same result is obtained for Artin rings by using the same techniques as in this paper by first showing that almost split sequences exist for Artin rings of finite type. This is done by K. Yamagata [**9**].

In Section 3 we give the main result which is as follows:

Suppose Λ is an Artin algebra such that there exists a positive integer n with the property that there exist $\aleph \geqq \aleph_0$ nonisomorphic indecomposable modules of length n where $\aleph_0$ stands for the cardinality of a countable set. Then there exist infinitely many positive integers n_i such that there exist $\aleph$ nonisomorphic indecomposable modules of length n_i.

This result is a part of the second Brauer-Thrall conjecture about Artin algebras of infinite type. The conjecture is as follows:

Let Λ be an Artin algebra of cardinality $\aleph \geqq \aleph_0$ and of infinite type. Then there exist infinitely many positive integers n_i with $\aleph$ nonisomorphic indecomposable modules of length n_i.

This was first proved for Artin algebras over perfect fields by L. A. Nazarova and A. V. Roiter [**7**] and later for algebras over arbitrary fields by C. M. Ringel (unpublished).

The result which we give has no restriction on the centre and we are using different techniques based upon the theory of almost split sequences. As a consequence of this result, we get a result of M. Auslander [**2**] which he calls the One and a Half Brauer-Thrall Conjecture, and is the following:

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If the number of nonisomorphic finitely generated indecomposable modules over an Artin algebra Λ is $\aleph \geqq \aleph_0$, then for each positive integer n, the number of nonisomorphic finitely generated indecomposable modules of length more than n is $\aleph$.

1. A Result of Harada and Sai and a resume from the theory of almost split sequences. We start out with a lemma of Harada and Sai [**6**, Lemma 1.2] which we give without proof.

LEMMA 1.1. *Let Λ be any ring and $\{M_i\}_{i=1,2,\ldots}$ a collection of indecomposable modules of length less than or equal to m and $f_i : M_i \to M_{i+1}$ nonisomorphism between them. Then there exists an m' such that $f_{m'} \circ \ldots \circ f_1 = 0$.*

This result can easily be extended to objects of finite length in an abelian category.

In the rest of this paper Λ will always be an artin algebra; that is, an Artin ring which is finitely generated as a module over its center which is also an Artin ring. We recall from [**3**] that an exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ in mod Λ, the category of finitely generated Λ-modules, is said to be an *almost split* sequence if it is not split, A and C are indecomposable in mod Λ and we have the following equivalent properties:

a) If $f : X \to C$ is not a splittable epimorphism (i.e., there is no $g : C \to X$ such that $fg = 1_C$), there is an $h : X \to B$ such that $p \circ h = f$.
b) If $g : A \to Y$ is not a splittable monomorphism, there is a $j : B \to Y$ such that $j \circ i = g$.

Now we want to list some properties taken from [**3**, **4**]. First we have the existence theorem [**3**, Propositions 4.1, 4.2].

THEOREM 1.2. *For each noninjective (nonprojective) indecomposable module A (C) in* mod Λ *there is an almost split sequence $0 \to A \to B \to C \to 0$ which is unique up to isomorphism.*

To construct the almost split sequence for a noninjective (nonprojective) Λ-module A (C) one has to know what C (A) is in the almost split sequence $0 \to A \to B \to C \to 0$. This is done in [**3**] and is as follows: If Λ is the algebra and R its centre, then $D = \operatorname{Hom}_R(-, I_0(R/\underline{r}))$ will give a duality between mod Λ and mod Λ^{op} where $\underline{r}$ is the radical of R, I_0 stands for the injective envelope and Λ^{op} is the opposite ring of Λ. Another relation between mod Λ and mod Λ^{op} can be obtained by the following process. If M is in mod Λ, let $P_1 \to P_0 \to M \to 0$ be a minimal projective resolution. Then Coker $(\operatorname{Hom}(f, \Lambda))$ is in mod Λ^{op}. This relation is called the *transpose* and denoted Tr. It will induce a functor between mod Λ and mod Λ^{op} where mod Λ is mod Λ modulo maps that factor through projectives. Now from Proposition 4.2 in [**3**] we have the following lemma.

LEMMA 1.3. *Let* Λ *be an Artin algebra and suppose* $0 \to A \to B \to C \to 0$ *is an almost split sequence. Then* $A = \mathrm{Tr}DC$ *and* $C = D\mathrm{Tr}A$.

If we now in an almost split sequence $0 \to A \to B \to C \to 0$ write $B = \amalg B_i$ with the B_i indecomposable, then all the induced maps $A \to B \to B_i$ have the property that if the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[dr, "f"'] & B \arrow[r] & B_i \\ & D \arrow[ur, "g"'] & \end{tikzcd}$$

commutes, then f is either a splittable monomorphism or g a splittable epimorphism. This suggests the following definition taken from [**4**].

A map $f : B \to C$ in mod Λ (Λ an Artin algebra) is called *irreducible* if f is neither a split epimorphism nor a split monomorphism, and whenever $f = g \circ h$ for some $h : B \to D$ and $g : D \to C$ either h is a split monomorphism or g is a split epimorphism.

We now have the following result about irreducible maps [**4**, Proposition 3.3].

PROPOSITION 1.4. *Let* Λ *be an Artin algebra and* A *an indecomposable* Λ*-module. Then*

a) *If* A *is noninjective, there exists an irreducible map* $f : A \to B'$ *if and only if* B' *is a summand in* B *where*

$$0 \to A \to B \to D\mathrm{Tr}A \to 0$$

is the almost split sequence for A. *Further, all maps* $A \to B \xrightarrow{p} B'$ *with* p *a split epimorphism are irreducible.*

b) *If* A *is injective, there exists an irreducible map* $f : A \to B'$ *if and only if* B' *is a summand in* $A/\mathrm{soc}\, A$ *when* soc A *stands for the socle of* A. *Further, all maps*

$$A \to A/\mathrm{soc}\, A \xrightarrow{p} B'$$

with p *a split epimorphism are irreducible. (The dual statements hold for* A *nonprojective, respectively* A *projective.)*

c) *All irreducible maps between indecomposable modules are either proper epimorphisms or proper monomorphisms. (i.e., nonisomorphisms).*

We will point out as a trivial corollary of this the following:

COROLLARY 1.5. *Let* Λ *be an Artin algebra; then for each indecomposable module* A *in* mod Λ *there is only a finite number of indecomposable modules* B *with irreducible maps* $f : A \to B$.

2. Preliminary results. To use the results in Section 1 we have to get some chain

$$\ldots \to M_i \xrightarrow{f_i} M_{i+1} \to \ldots$$

of modules and maps between them which satisfy the properties in Lemma 1.1.

Definition. For indecomposable modules A and B in mod Λ we say that they are *connected by irreducible maps* if there exists a chain of indecomposable modules $\ldots \to M_i \to M_{i+1} \to \ldots$ and irreducible maps $f : M_i \to M_{i+1}$ such that $A \simeq M_n$, $B \simeq M_m$ for some n, m and $f_n \circ \ldots \circ f_m$ is nonzero. We say that such a chain is *connecting* A and B.

LEMMA 2.1. *Let A be an indecomposable module in* mod Λ. *Then there exists only a finite number of indecomposable modules B_i which are connected to A by irreducible maps such that the lengths of the modules in a chain connecting A and B are bounded by an integer n.*

Proof. The length of a chain connecting A to B with the length of all the modules in the chain less than n is bounded by Lemma 1.1. From Corollary 1.5 we have that there is only a finite number of nonisomorphic modules connected to A in each step which fulfill the proof of the lemma.

PROPOSITION 2.2. *Let A and B be indecomposable modules and suppose there exists a nonzero map $f : A \to B$ which is neither a sum of compositions of irreducible maps nor an isomorphism. Then there exists an infinite chain*

$$A \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \ldots$$

of indecomposable modules A_i and irreducible maps $f_i : A_i \to A_{i+1}$ such that any composition $f_i \circ \ldots \circ f_0$ is nonzero.

Proof. We have that $f : A \to B$ is not a splittable monomorphism. So let

$$0 \to A \to A_1 \to \operatorname{Tr} DA \to 0$$

be the almost split sequence if A is noninjective and look at $A_1 = A/\operatorname{soc} A$ if A is injective. In both cases we get that $f : A \to B$ can be factored through A_1, so in the rest of the proof let the successor of an indecomposable module C be the middle term of the almost split sequence of C, if C is noninjective and $C/\operatorname{soc} C$ if C is injective; and denote it by $S(C)$. Now look at the commuting diagram

$$\begin{tikzcd} A \arrow[r, "h_0"] \arrow[d, "t"] & S(A) = \amalg A_{i,0} \arrow[dl, "g_0"] \\ B & \end{tikzcd}$$

where the $A_{i,0}$ are indecomposable. Let $g_{i,0} = g_0 j_i$ and $h_{i,0} = p_i h_0$ where $j_i : A_{i,0} \to \amalg A_{k,0}$ is the inclusion and $p_i : \amalg A_{i,0} \to A_{i,0}$ is the projection. From Proposition 1.4 it follows that the $h_{0,i}$'s are irreducible. Now let $f_1 = f - \amalg g_{i,0}h_{i,0}$ where the sum is taken over those $g_{i,0}$ which are either irreducible maps or isomorphisms. Since f is not a sum of compositions of irreducible maps $f_1 \neq 0$ and further, $f_1 = \amalg g_{i,0}h_{i,0}$ where now the sum is taken over the $g_{i,0}$ which are neither irreducible nor isomorphism. We now have that the following diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f_1"'] & \amalg A_{i,0} \arrow[dl, "\amalg g_{i,0}"] \\ B & \end{tikzcd}$$

where the sum is taken over i such that $g_{i,0}$ is neither an irreducible map nor an isomorphism. These maps $g_{i,0}$ can, of course, be factored further. We then get the following commuting diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f_1"'] & \amalg A_{i,0} \arrow[r] \arrow[dl, "\amalg g_{i,0}"] & \amalg S(A_{i,0}) = \amalg A_{i,1} \arrow[dll, "g_1"] \\ B & & \end{tikzcd}$$

Now the same argument as before can be applied to each of the $g_{i,0}$. So we get an arbitrarily large commuting diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f_n"'] & \amalg A_{i,0} \arrow[r] & \amalg A_{i,1} \arrow[r] & \amalg A_{i,2} \arrow[r] & \cdots \arrow[r] & \amalg A_{i,n} \arrow[dlllll, "g_n"] \\ B & & & & & \end{tikzcd}$$

with $f_n \neq 0$. Therefore the composition

$$A \to \amalg A_{i,0} \to \amalg A_{i,1} \to \ldots \to \amalg A_{i,n}$$

is also nonzero. From this it follows that there has to be a nonzero composition

$$A \to A_{i_0,0} \to A_{i_1,1} \to \ldots \to A_{i_n,n}$$

where all the $A_{i_j,j}$ are indecomposable and all maps $A_{i_j,j} \to A_{i_{j+1},j+1}$ are irreducible. This completes the proof of the proposition.

Now an easy corollary of this proposition is the following:

COROLLARY 2.3. *Let Λ be an Artin algebra of finite type. Then all the maps between indecomposable modules are sums of compositions of irreducible maps or isomorphisms.*

Proof. Suppose to the contrary that there exists a nonzero map $f : A \to B$ which is not a sum of compositions of irreducible maps, and is not an isomorphism with A and B indecomposable. Then Lemma 2.2 can be applied to produce an infinite chain of irreducible maps between indecomposable modules with nonzero composition. This will by Lemma 2.1 give that there is no bound on the length of the finitely generated modules, which gives the desired contradiction.

Now the first Brauer-Thrall conjecture for Artin algebras follows easily.

THEOREM 2.4. *Suppose Λ is an Artin algebra of infinite type. Then there is no bound on the length of the finitely generated indecomposable Λ-modules.*

Proof. Let Λ be of infinite type and $\{M_i\}$ an infinite collection of nonisomorphic indecomposable finitely generated modules. Then, since there is only a finite number of simple Λ-modules, there exists a simple Λ-module S which is a summand in $M_i/\underline{r}M_i$ for infinitely many modules M_i in $\{M_i\}$ ($\underline{r} = \text{Rad } \Lambda$). So we have that there exist infinitely many nonisomorphic indecomposable modules M_i; and nonzero maps $f_i : M_i \to S$ for a simple Λ-module S. If now all of these maps are sums of compositions of irreducible maps, then by Lemma 2.1 there can be no bound on the length of indecomposable modules. The rest of the proof follows from Proposition 2.2.

3. The main result. To prove the main result, we need one more lemma.

LEMMA 3.1. *Let Λ be an Artin algebra, A and B indecomposable finitely generated Λ-modules, and suppose there is an irreducible map $f : A \to B$. Then*

$$|l(A) - l(B)| \leqq l(A) \cdot m^2$$

where $m = \max(l(\Lambda_\Lambda), l({}_\Lambda\Lambda))$ and $l(M)$ is the length of M as Λ-module.

Proof. This is obvious if A is injective since then $l(B) < l(A)$. From Section 1 we have that if A is not injective then B is a summand in the middle term of the almost split sequence

$$0 \to A \to C \to \text{Tr}DA \to 0.$$

So $|l(A) - l(B)| \leqq \max|l(A), l(\text{Tr}DA)|$. Now $lDA = lA$ since D is a duality between mod Λ and mod Λ^{op}. Further

$$l\text{Tr}DA \leqq l \operatorname{Hom}_\Lambda (P, \Lambda)$$

where $P_1 \to P_0 \to DA \to 0$ is a minimal projective resolution of the right Λ-module DA, and

$$l \operatorname{Hom}_\Lambda (P_1, \Lambda) \leqq n \cdot l(\Lambda_\Lambda)$$

where n is the number of indecomposable summands in a direct sum decomposition of P_1 since $\operatorname{Hom}_\Lambda(-, \Lambda)$ is a duality between right and left projectives

Λ-modules. And finally

$$n \leqq l(P_0) \leqq l(DA)\cdot l({}_\Lambda\Lambda).$$

So we get that $|l(A) - l(B)| \leqq l(A)\cdot m^2$ which we were to prove.

We are now able to prove the main result.

THEOREM 3.2. *Suppose Λ is an Artin algebra such that there exist $\aleph \geqq \aleph_0$ nonisomorphic indecomposable modules of length n, where $\aleph_0$ stands for the cardinality of a countable set. Then there exist infinitely many positive integers n_i with $\aleph$ nonisomorphic indecomposable modules of length n_i.*

Proof. Suppose n is a positive integer with $\aleph \geqq \aleph_0$ nonisomorphic indecomposable modules $\{M_i\}$ of length n. If we now for each $k \in \mathbf{Z}_+$ and $M_i \in \{M_i\}$ can associate an indecomposable module M_i' with $l(M_i') \in [k, f(k)]$ for a function $f : \mathbf{Z} \to \mathbf{Z}$ such that each M_i' corresponds only to a finite number of the M_i's we will be done. To do this we have to divide the proof into two cases and use Lemma 2.1 and Proposition 2.2.

Suppose n is an integer with $\aleph \geqq \aleph_0$ nonisomorphic indecomposable modules of length n and suppose ad absurdum that there exists a positive integer $m > n$ such that for all positive integers $k \geqq m$ the number of nonisomorphic indecomposable modules of length k is properly less than $\aleph$.

Let $\{M_i\}$ be a collection of $\aleph$ nonisomorphic indecomposable modules of length n. Then we know that there exists a simple Λ-module S such that S is a summand in $M_i/\underline{r}M_i$ for $\aleph$ of the M_i's and therefore there exist nonzero maps $f_i : M_i \to S$ for $\aleph$ of the M_i's. Now we divide the proof into two cases.

1) Either $\aleph$ of these maps are not sums of compositions of irreducible maps or
2) $\aleph$ of the maps are sums of compositions of irreducible maps.

It is enough to consider these two cases since at least one of them occurs because none of the maps can be isomorphisms.

In the first case we get by Proposition 2.2 for $\aleph$ of the M_i's a chain

$$M_i \to M_{i,0} \to M_{i,1} \to \dots \to M_{i,2m+1} \to$$

of arbitrary length of indecomposable modules $M_{i,j}$ and irreducible maps $M_{i,j} \to M_{i,j+1}$ with the composition of the maps nonzero. Let m' be the number corresponding to m according to Lemma 1.1. Then after m' steps there has to be in each of the chains a module of length at least m. By Lemma 2.1 such a module can only occur in the first m' steps of the chains associated to only a finite number of the M_i's. Further by Lemma 3.1, the length of the modules in the first m' steps of a chain associated to any M_i is bounded by $n^{m'}p^{2m'} = f(m)$ where $p = \max(l_\Lambda(\Lambda), l(\Lambda_\Lambda))$. Therefore, from the M_i we have in Case 1 proved the existence of $\aleph$ nonisomorphic indecomposable modules with length in an interval $[m, f(m)]$. But then there exists an integer k, $m \leqq k \leqq f(m)$ such that there exists $\aleph$ nonisomorphic indecomposable modules of length k which gives the contradiction in Case 1.

Case 2 is even simpler because now we know that $\aleph$ of the M_i's are connected to S by a chain of irreducible maps with nonzero composition. (In this case $\aleph = \aleph_0$ is countable.) Then by Corollary 1.5 only a finite number of these chains can be shorter than m' and the same argument as in Case 1 can be applied on the rest of the chains.

So in both cases we get that there is no bound on the integers k such that there exist $\aleph$ nonisomorphic indecomposable modules of length k which fulfill the proof of the theorem.

Now as an easy consequence of this theorem, we get the "one and a half Brauer-Thrall conjecture."

COROLLARY 3.3. *Let Λ be an Artin algebra. If the number of nonisomorphic finitely generated indecomposable Λ-modules is $\aleph \geqq \aleph_0$ where $\aleph_0$ is the cardinality of a countable set, then for each positive integer n the number of nonisomorphic indecomposable finitely generated Λ-modules of length more than n is $\aleph$.*

Proof. This follows trivially from the theorem.

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