MODULAR ANNIHILATOR ALGEBRAS

BRUCE A. BARNES

1. Introduction. In a recent paper **(7)** Yood developed the beginnings of a theory of modular annihilator algebras. In this paper we extend his work on these algebras.

The definition of modular annihilator algebra is algebraic in nature (see §4); in fact the algebra need not be assumed even topological. However, a significant number of important normed algebras are modular annihilator algebras. A list of examples is given in §8.

The theory of modular annihilator algebras is related to the theory of certain important topological algebras. In §5 we consider the relationships between dual and annihilator algebras and modular annihilator algebras, and in §7, the relationship between completely continuous normed algebras and modular annihilator algebras.

Except for §2, which is introductory in nature, the remaining sections, 3, 4, and 6, are concerned with the elementary properties of modular annihilator algebras, especially the structure of ideals.

2. Notation and preliminaries. Notation and definitions not explicitly given are taken from Rickart's book **(5)**.

Throughout this paper, A will denote a real or complex algebra. R_A is the radical of A, and S_A is the socle of A.

If F is any subset of elements of A, $L(F)$ is the left annihilator of F (i.e., $L(F) = \{x \in A \mid xy = 0 \text{ for all } y \in F\}$), and $R(F)$ is the right annihilator of F.

Simple facts about primitive and right primitive ideals are used repeatedly; see **(5**, pp. 53–54**)**. In particular we use:

(2.1) If P is a primitive (right primitive) ideal of A, and M and N are left (right) ideals of A such that $M \cdot N \subset P$, then either $M \subset P$ or $N \subset P$ **(5**, Theorem (2.2.9) (iv), p. 54**)**.

Π_A is the structure space of A, the space of all primitive ideals of A with the hull-kernel topology; see **(5**, pp. 77–78**)**.

We denote the set of all minimal idempotents of A as E_A. For the basic properties of minimal ideals and minimal idempotents used in this paper, see **(5**, pp. 45–47**)**.

3. The ideals P^e. Assume that Ae is a minimal left ideal of A and $e \in E_A$. Then $A(1 - e)$ is a maximal modular left ideal of A. With these assumptions we define:

Received January 25, 1965.

DEFINITION. *P^e is the primitive ideal*

$$P^e = \{x \in A \mid xA \subset A(1-e)\}.$$

The ideals P^e will be important in subsequent sections. The next propositions describe the special properties of these ideals.

PROPOSITION 3.1. *Assume that A has no nilpotent left or right ideals and that $e \in E_A$. Then*

(1) $L(Ae) = R(eA) = P^e = \{x \in A \mid Ax \subset (1-e)A\}$,

(2) *P^e is minimal primitive and minimal right primitive.*

Proof. (1) If $x \in P^e$, then $xA \subset A(1-e)$ by the definition of P^e, and therefore $(xA)e = x(Ae) = 0$. Thus $x \in L(Ae)$. If $x \in L(Ae)$, then in particular $xe = 0$, and therefore $x = x(1-e)$. We have shown that

$$P^e \subset L(Ae) \subset A(1-e).$$

Since $L(Ae)$ is a two-sided ideal, $P^e = L(Ae)$.

A similar proof shows that $R(eA) = \{x \in A \mid Ax \subset (1-e)A\}$. Let Q be this right primitive ideal. $P^e(eA) = 0$, and therefore $P^e \subset Q$ or $eA \subset Q$ by (2.1). But then $P^e \subset Q$ since $e \notin Q$. Similarly, $Q \subset P^e$. This completes the proof of (1).

(2) By (1), P^e is right primitive and $L(P^e)$ is a non-zero ideal. Assume that Q is primitive or right primitive and $Q \subset P^e$. Since $L(P^e) \cdot P^e = 0$, by (2.1) either $P^e \subset Q \subset P^e$ or $L(P^e) \subset Q \subset P^e$. But if $L(P^e) \subset P^e$, then $L(P^e)^2 = 0$, contradicting the assumption that A has no nilpotent ideals. Therefore $P^e = Q$.

PROPOSITION 3.2. *Assume that A has no nilpotent left or right ideals. Let I be a two-sided ideal of A. Then either $I \subset P^e$ for some e or $S_A \subset I$.*

Proof. If $e \in E_A$, then either $eI = eA$ or $eI = 0$, that is either $eA \subset I$ or $I \subset P^e$ (Proposition 3.1). This proves the proposition.

PROPOSITION 3.3. *Let A have no nilpotent left or right ideals. Then $L(S_A)$ is the intersection of all the p^e in A. If M is a non-zero left (right) ideal of A, then either M contains a minimal left (right) ideal or $M \subset L(S_A)$.*

Proof. $x(S_A) = 0$ if and only if $x(Ae) = 0$ for every $e \in E_A$. Then by Proposition 3.1, $x(S_A) = 0$ if and only if $x \in P^e$ for every e. Thus $L(S_A)$ is the intersection of the P^e.

Assume now that M is a non-zero left ideal of A such that

$$M \not\subset L(S_A) = R(S_A).$$

Then $(Ae)M \neq 0$ for some $e \in E_A$. Therefore there exists $u \in M$ such that $eu \neq 0$, and it follows that Aeu is a minimal left ideal and $Aeu \subset M$. The proof for right ideals is similar.

Various forms of Proposition 3.3 are well known.

The next theorem generalizes a result of Yood (**7**, 3.3 Lemma, p. 38).

THEOREM 3.4. *Assume $R(A) = 0$. Let M be a maximal left ideal of A such that $R(M) \not\subset R_A$. Then there exists $e \in E_A$ such that $R(M) = eA$ and $M = A(1 - e)$.*

Proof. Assume that there exists $x \in R(M)$ such that x is left quasi-singular. Then $M = M(1 - x) \subset A(1 - x)$ and $A(1 - x)$ is proper in A. Therefore $M = A(1 - x)$. Now $y \in R(M)$ if and only if $A(1 - x)y = 0$ if and only if $y = xy$. Since $x \in R(M)$, $x^2 = x$ and $R(M) = xA$. Since M is maximal, $x \in E_A$.

We complete the proof of the theorem with a lemma.

LEMMA. *If N is a right ideal of A such that every element of N is left quasi-regular, then $N \subset R_A$.*

Proof. Assume that N is as above and that $r \in N$, $r \notin R_A$. Let $a \to T_a$ be an irreducible representation of A on the linear space X such that $T_r \neq 0$. Then there exist $u, v \in X$ such that $T_r(v) = u \neq 0$. Since $a \to T_a$ is irreducible on X, there exists $b \in A$ such that $T_b(u) = v$. Then

$$(T_{-arb} + T_a + T_{rb})(u) = u \neq 0 \qquad \text{for any } a \in A.$$

Therefore rb is not left quasi-regular. From this contradiction it follows that $N \subset R_A$.

COROLLARY (Yood). *Assume that A has no nilpotent left or right ideals, and let M be a maximal modular left ideal of A. Then if $R(M) \neq 0$, there exists $e \in E_A$ such that $M = A(1 - e)$ and $R(M) = eA$.*

Proof. If $R(M) \subset R_A$, then $R(M) \subset M$. But then $R(M)^2 = 0$ and $R(M) = 0$, a contradiction. Therefore $R(M) \not\subset R_A$. Note that $R(A) = 0$ since A has no nilpotent right ideals.

4. Modular annihilator algebras.

DEFINITION. *A is a modular annihilator algebra if for every maximal modular left ideal M and every maximal modular right ideal N,*

(1) *$R(M) \neq 0$ and $R(A) = 0$,*

(2) *$L(N) \neq 0$ and $L(A) = 0$.*

If A satisfies (1), A is a right modular annihilator algebra.

The definition given by Yood does not include the hypotheses that $L(A) = R(A) = 0$. However, he considers only modular annihilator algebras that have no nilpotent left or right ideals; see (**7**, p. 37).

THEOREM 4.1. *Let A be a modular annihilator algebra, and assume that M is a maximal modular left ideal of A such that $R(M) \not\subset R_A$. Then there exists $e \in E_A$ such that $M = A(1 - e)$ and $R(M) = eA$, a minimal right ideal.*

Proof. By Theorem 3.4, $M = A(1 - e)$ and $R(M) = eA$ for some $e \in E_A$. It remains to be shown that eA is a minimal right ideal.

$(1 - e)A$ is contained in some maximal modular right ideal N. But $L(N) \neq 0$ and $L(N) \subset Ae$. Also Ae is a minimal left ideal of A since $M = A(1 - e)$ is maximal. Therefore $L(N) = Ae$, and since $R(A) = 0$, $N = (1 - e)A$. Hence $(1 - e)A$ is maximal, implying that eA is minimal.

For the remainder of this section we assume that A *has no nilpotent left or right ideals.*

THEOREM 4.2. *Let A be a right modular annihilator algebra. Then:*

(1) *Every maximal modular left ideal of A is of the form $A(1 - e)$ for some $e \in E_A$, and every primitive ideal is of the form $L(Ae)$.*

(2) $L(S_A) = R(S_A) = R_A$.

(3) *If M is a left (right) ideal such that $M \not\subset R_A$, M contains a minimal left (right) ideal.*

(4) *Every irreducible representation of A is equivalent to the left regular representation of A on some minimal left ideal.*

(5) *Every primitive ideal is both maximal and minimal primitive.*

(6) Π_A *is discrete in the hull-kernel topology.*

Proof. (1) follows from the corollary to Theorem 3.4 and Proposition 3.1.

(2) and (3): By (1), every primitive ideal is a P^e. (2) and (3) then follow directly from Proposition 3.3.

(4) Any irreducible representation of A is equivalent to the regular representation of A on $A - M$ for M some maximal modular left ideal of A. But by (1), $M = A(1 - e)$ for some $e \in E_A$, and it follows that the regular representation of A on $A - M$ is equivalent to the left regular representation of A on Ae.

(5) Every primitive ideal is a P^e. Therefore by Proposition 3.1 (2), every primitive ideal is both maximal and minimal primitive.

(6) By (5), it is clear that $\{P\} = h(k(\{P\}))$ for any primitive ideal P. Let E denote the complement of $\{P\}$ in Π_A. If $Q \in E$, either $P \subset Q$ or $L(P) \subset Q$. Therefore $L(P) \subset Q$ since $P \not\subset Q$ by (5). Hence $L(P) \subset k(E)$ and since $\{P\} \notin h(L(P))$, $P \notin h(k(E))$. It follows that $E = h(k(E))$.

We state a result of Yood which we use in the next theorem:

(4.1) *If M is a maximal modular left ideal of A, either $S_A \subset M$ or $R(M) = 0$* (**7**, 3.3 Lemma, p. 38).

THEOREM 4.3. *The following are equivalent:*

(1) *A is a right modular annihilator algebra.*

(2) Π_A *is discrete, and* $L(S_A) = R_A$.

(3) Π_A *is discrete, and every left ideal M such that $M \not\subset R_A$ contains a minimal left ideal.*

https://doi.org/10.4153/CJM-1966-055-6 Published online by Cambridge University Press

(4) *$h(S_A)$ is empty.*

(5) *Every primitive ideal is of the form $L(Ae)$ for some $e \in E_A$.*

(6) *Every maximal modular left ideal is of the form $A(1 - e)$ for some $e \in E_A$.*

(7) *Every irreducible representation of A is equivalent to the left regular representation of A on some minimal left ideal.*

Proof. (1) implies (2): Theorem 4.2 (2) and (6).

(2) implies (3): Proposition 3.3.

(3) implies (4): Choose $P \in \Pi_A$ and let E be the complement of $\{P\}$ in Π_A. Since Π_A is discrete, $h(k(E)) = E$. Therefore $k(E) \not\subset P$, and hence $k(E) \not\subset R_A$. Therefore $k(E)$ contains a minimal left ideal N. But $k(E) \cap P = R_A$, and since $N \not\subset R_A$, $N \not\subset P$. Therefore $P \notin h(S_A)$.

(4) implies (5): Let P be any primitive ideal. $S_A \not\subset P$, and thus by Proposition 3.2, $P \subset P^e$ for some e. But P^e is minimal primitive (Proposition 3.1 (2)), and therefore $P = P^e = L(Ae)$ (Proposition 3.1 (1)).

(5) implies (6): Let M be a maximal modular left ideal of A. Either $R(M) \neq 0$ or $S_A \subset M$ by (4.1). But if $S_A \subset M$, S_A is contained in some primitive ideal P. Since $P = L(Ae)$ for some $e \in E_A$, $P \cap Ae = 0$. Therefore $S_A \not\subset M$, and by the corollary to Theorem 3.4, $M = A(1 - e)$ for some $e \in E_A$.

(6) implies (7): This follows directly from the proof of Theorem 4.2 (4).

(7) implies (5) implies (6) implies (1): Immediate.

We note here that Yood has proved the following result:

(4.2) *If A has no left or right nilpotent ideals and A is a right modular annihilator algebra, then A is a modular annihilator algebra* (**7**, Theorem 3.4, p. 38).

5. Ideals in modular annihilator algebras. In the first theorem of this section we characterize annihilator ideals in a semi-simple modular annihilator algebra.

THEOREM 5.1. *Let A be a semi-simple modular annihilator algebra. Then*

(1) *A proper left (right) ideal M is the intersection of maximal modular left (right) ideals of A if and only if M is a left (right) annihilator ideal.*

(2) *A proper two-sided ideal I has the property that $I = k(h(I))$ if and only if I is an annihilator ideal.*

Proof. By Theorem 4.2 (1), every maximal modular left ideal of A and every primitive ideal of A is a left annihilator ideal. Since an intersection of annihilator ideals is an annihilator ideal, the "only if" parts of (1) and (2) are immediate.

Assume now that M is a proper left annihilator ideal of A. Then $R(M) \neq 0$, and therefore there exists $e \in E_A \cap R(M)$ (Theorem 4.2 (3)). Thus

$$M = L(R(M)) \subset A(1 - e),$$

which is a maximal modular left ideal of A. Let K be the intersection of all maximal modular left ideals containing M. Since every maximal modular left ideal is a left annihilator ideal (Theorem 4.1 (1)), K is a left annihilator ideal. Therefore to complete the proof of (1), it is sufficient to prove that $K \cdot R(M) = 0$, since then $R(M) \subset R(K)$, and consequently

$$K = L(R(K)) \subset L(R(M)) = M.$$

Assume that $K \cdot R(M) \neq 0$. Then there exists $k \in K$ such that $k \cdot R(M)$ is a non-zero right ideal of A, and therefore $k \cdot R(M)$ contains a minimal idempotent e (Theorem 4.2 (3)). Then $e = kx$ for some $x \in R(M)$, $e = kxe$, and $xe \in R(M)$.

There exists $f \in E_A$ such that $fA = xeA$, and in particular $xe = fxe$. $fA \subset R(M)$, thus $M \subset A(1 - f)$, and finally $K \subset A(1 - f)$. Now $kf = 0$, $xe = fxe$, and $e = kxe$; it follows that $e = kfxe = 0$, a contradiction. This completes the proof of (1).

Assume that I is a proper two-sided annihilator ideal of A. Then by (1), $I = \bigcap A(1 - e_\alpha)$ for some set of $e_\alpha \in E_A$. Therefore

$$I \subset \bigcap P^{e_\alpha} \subset \bigcap A(1 - e_\alpha) = I$$

for this same set of e_α. Hence $I = k(h(I))$.

The next theorem represents certain important topological algebras in terms of modular annihilator algebras. Part (1) includes the folowing result of Yood for a topological algebra A with no nilpotent left or right ideals: If every maximal modular left ideal of A is closed, and if A has dense socle, then A is a modular annihilator algebra (**7**, Lemma 3.11, p. 41).

THEOREM 5.2. *Let A be a topological algebra in which every maximal modular left ideal is closed. If A has no nilpotent left or right ideals, then*

(1) *A has dense socle if and only if A is a modular annihilator algebra, and for every proper closed two-sided ideal I of A, $h(I)$ is non-empty.*

If also A is semi-simple, then

(2) *A is an annihilator algebra if and only if A is a modular annihilator algebra, every proper closed left ideal of A is contained in a maximal modular left ideal of A, and every proper closed right ideal of A is contained in a maximal modular right ideal of A.*

(3) *A is dual if and only if A is a modular annihilator algebra, every proper closed left ideal of A is the intersection of maximal modular left ideals of A, and every proper closed right ideal of A is the intersection of maximal modular right ideals of A.*

Proof. Assume that A has no left or right nilpotent ideals. Suppose that A has dense socle and let M be a maximal modular left ideal of A. Either $S_A \subset M$ or $R(M) \neq 0$ by (4.1). Since M is closed, $S_A \not\subset M$. Therefore $R(M) \neq 0$, and A is a right modular annihilator algebra. Then (4.2) implies that A is a modular annihilator algebra. Now let I be a proper closed two-sided ideal of A. Either $I \subset P^e$ for some e or $S_A \subset I$ (Proposition 3.3). Since I is proper, $S_A \not\subset I$, and hence $h(I)$ is non-empty.

Conversely, if A is a modular annihilator algebra, and for every proper closed ideal I of A, $h(I)$ is non-empty, then in particular, if $\bar{S}_A$ is proper, $h(\bar{S}_A)$ is non-empty. Then by Theorem 4.3 (4), $A = \bar{S}_A$.

For the remainder of the proof we assume that A is semi-simple.

If A is an annihilator algebra, by the assumption that every maximal modular left ideal is closed, A is a right modular annihilator algebra, and hence a modular annihilator algebra. Let M be a proper closed left (right) ideal of A. Then $R(M) \neq 0$ $(L(M) \neq 0)$ and therefore there exists a minimal idempotent $e \in R(M)$ $(e \in L(M))$. It follows that $M \subset A(1 - e)$ $(M \subset (1 - e)A)$.

Conversely, if A is a modular annihilator algebra and every proper closed left (right) ideal M is contained in a maximal modular left (right) ideal, then clearly $R(M) \neq 0$ $(L(M) \neq 0)$.

If A is dual, then A is an annihilator algebra, and hence a modular annihilator algebra by (2). Then by Theorem 5.1, every proper closed left (right) ideal is the intersection of maximal modular left (right) ideals.

The converse of this follows directly from Theorem 5.1.

6. Maximal modular ideals. "Ideal" will be understood to mean "two-sided ideal" for the purposes of this section. The object of this section is to give necessary and sufficient conditions for a primitive ideal in a modular annihilator algebra to be a maximal modular ideal. It is always true that a maximal modular ideal is primitive.

Various forms of the following lemma are well known.

LEMMA 6.1. *Let M be a finite-dimensional left (right) ideal in a semi-simple algebra A. Then there exists an idempotent $u \in S_A$ such that $M = Au$ $(M = uA)$.*

Proof. We do the proof for left ideals. Let M be a finite-dimensional left ideal of A. Then there exists a minimal idempotent $e \in M$ (or $M = 0$, but this case is trivial). Let u be an idempotent such that $u \in S_A \cap M$ and $M \cap A(1 - u)$ is minimal among ideals of the form $M \cap A(1 - v)$, where $v^2 = v \in M \cap S_A$. Such ideals exist because $M \cap A(1 - e)$ is one, and a minimal ideal of this form exists by the finite dimensionality of M.

Either (1) $M \cap A(1 - u) = 0$, or (2) there exists $f \in E_A$ such that $Af \subset M \cap A(1 - u)$.

Assume that (2) holds. We define $v = u + f - uf$. Note that $fu = 0$, $vf = f$, and $vu = u$. It follows that $v^2 = v$, and clearly $v \in M \cap S_A$. If $m \in M \cap A(1 - v)$, $mv = 0$, and therefore $0 = mvu = mu$. Hence

$$M \cap A(1 - v) \subset M \cap A(1 - u).$$

But $f \in M \cap A(1-u)$ and $fv = v$. This contradicts the choice of u and therefore (1) must hold. Then $M \cap A(1-u) = 0$, and if $m \in M$,

$$m(1-u) \in M \cap A(1-u),$$

and therefore $M = Mu = Au$. Hence the lemma.

We use this lemma to give us a sufficient condition that a primitive ideal in a modular annihilator algebra be maximal modular.

PROPOSITION 6.2. *Let A be a semi-simple modular annihilator algebra. If P is a primitive ideal of A such that A/P is finite dimensional, P is a maximal modular ideal of A.*

Proof. Let P be as stated. A/P is finite dimensional, and therefore has an identity by Lemma 6.1. Therefore P is a modular ideal. But $P = P^e$ for some e (Theorem 4.2 (1)), and $P \subset M$ a maximal modular ideal of A. Since $P = P^e$ is maximal primitive (Theorem 4.2 (5)), $P = M$.

Most of the remainder of this section will be devoted to proving the converse of Proposition 6.2.

For the next proposition we use in part arguments from (**1**, p. 37).

PROPOSITION 6.3. *If A is a semi-simple, normed modular annihilator algebra with an identity, then A is finite dimensional.*

We start the proof of this proposition with a lemma:

LEMMA. *Let A be a semi-simple algebra such that A is the sum of a finite number of minimal left ideals. Then there exists an integer N such that if*

$$Af_1 + Af_2 + \ldots + Af_n$$

is a direct sum, where $f_i \in E_A$, then $n \leqslant N$.

Proof. Since A is the sum of a finite number of minimal left ideals, we may choose N to be the minimal number of minimal left ideals which have sum A. Let $\{e_i\}$, $1 \leqslant i \leqslant N$, be a subset of E_A such that $A = Ae_1 + \ldots + Ae_N$. Assume now that $\{f_i\}$, $1 \leqslant i \leqslant n$, is a subset of E_A such that

$$Af_1 + Af_2 + \ldots + Af_n$$

is a direct sum. There exist $y_i \in A$ such that $f_1 = y_1 e_1 + \ldots + y_N e_N$. Let j be such that $y_j e_j \neq 0$. Then $Ae_j = Ay_j e_j$ and $Ae_j \subset Af_1 + \sum_{i \neq j} Ae_i$. Therefore this sum is equal to A.

Again there exist z and z_i, $i \neq j$, $1 \leqslant i \leqslant N$, such that

$$f_2 = zf_1 + \textstyle\sum_{i \neq j} z_i e_i.$$

Let k be such that $z_k e_k \neq 0$; such a k exists because we have assumed that $Af_1 + \ldots + Af_n$ is a direct sum. Thus as above

$$A = Af_1 + Af_2 + \sum_{i \neq j,\, i \neq k} Ae_i.$$

Proceeding in this manner, we can substitute each Af_i for some Ae_m. It follows from this and the assumption that $\sum Af_i$ is direct that $n \leqslant N$.

Proof of Proposition 6.3. The assumption that A has an identity implies that every proper ideal of A has a non-zero left annihilator. But $L(S_A) = 0$ since A is semi-simple and by Theorem 4.2 (2). Therefore $A = S_A$. In particular, $1 = x_1 e_1 + \ldots + x_k e_k$ for some $e_i \in E_A$ and $x_i \in A$. It follows that A is a finite sum of minimal left ideals (and also a finite sum of minimal right ideals). We choose N as in the Lemma.

Next we show that for $e \in E_A$, eA is finite dimensional over the division ring eAe. Suppose that $a_1, \ldots, a_n$ are in eA and linearly independent over eAe. Let x_i, $1 \leqslant i \leqslant n$, be elements of A such that $x_1 a_1 + \ldots + x_n a_n = 0$. Note that if $x_i a_i \neq 0$, then $Ax_i a_i = Ax_i ea_i = Aea_i = Aa_i$, and hence Aa_i is a minimal left ideal of A. Now $Ax_1a_1 \subset Ax_2a_2 + \ldots + Ax_na_n$. Assume that $x_1 a_1 \neq 0$. Then $a_1 \in Ax_1 a_1 = Aa_1$ and so there exist $y_2, \ldots, y_n \in A$ such that $a_1 = y_2 a_2 + \ldots + y_n a_n$. It follows that $a_1 = ey_2 ea_2 + \ldots + ey_n ea_n$ and this contradicts the linear independence of the a_i. Therefore $x_1 a_1 = 0$, and by repeating the proof as above, $x_i a_i = 0$ for $1 \leqslant i \leqslant n$. Thus the sum of the minimal ideals $Aa_1 + \ldots + Aa_n$ is direct and $n \leqslant N$.

This shows that eA is finite dimensional over eAe. But also eAe is the field of the real numbers, the complex numbers, or the quaternions, and in any case finite dimensional over the scalar field of A. Since A is a finite sum of minimal right ideals, A is finite dimensional.

THEOREM 6.4. *Let A be a semi-simple, normed modular annihilator algebra. Then a primitive ideal P of A is maximal modular if and only if A/P is finite dimensional.*

Proof. If A/P is finite dimensional, P is maximal modular by Proposition 6.2. Conversely, if P is maximal modular, A/P is a semi-simple, normed modular annihilator algebra with identity. Then, by Proposition 6.3, A/P is finite dimensional.

THEOREM 6.5. *Let A be a semi-simple, normed modular annihilator algebra. Then the following are equivalent:*

(1) *P^e is a maximal modular ideal of A.*

(2) *Ae and eA are finite dimensional.*

(3) *There exists an idempotent $u \in S_A$ such that*

$$P^e = A(1 - u) = (1 - u)A.$$

Proof. Assume (1). Then by Theorem 6.4 A/P^e is finite dimensional. But Ae and eA may be embedded isomorphically in A/P^e, and hence (2).

Assume (2), and set $I = AeA$. Since Ae and eA are finite dimensional, I is finite dimensional. By Lemma 6.1, there exists an idempotent $u \in I \cap S_A$ such that $I = Au = uA$. By Proposition 3.1, $I = AeA \subset L(P^e)$, and hence $P^e u = uP^e = 0$. Then

$$P^e \subset A(1 - u) = (1 - u)A \subset A(1 - e)$$

since $ue = e$ (u is an identity for I). Therefore

$$P^e = A(1 - u) = (1 - u)A.$$

Assume (3). Then P^e is modular and hence contained in a maximal modular ideal. It follows that P^e is maximal modular since P^e is maximal primitive.

7. Completely continuous algebras. Let A be any algebra. Let $a \to T_a$ be the left regular representation of A on A (for $b \in A$, $T_a(b) = ab$), and let $a \to Q_a$ be the representation of A on A given by $Q_a(b) = ba$ for $b \in A$.

DEFINITION. *If A is a normed algebra, let $J(A)$ denote the set of those $a \in A$ such that T_a and Q_a are completely continuous operators on A. If $A = J(A)$, A is a completely continuous algebra.*

$J(A)$ is a two-sided ideal, and if A is a Banach algebra it is easy to verify that $J(A)$ is closed; see **(3)**.

We note the following information that we shall use in this section: *Let X be a normed linear space and assume that T is a completely continuous operator on X. Then:*

(7.1) *The null space of $(I - T)^n$ is finite dimensional, I the identity operator on X* **(6**, Theorem 5.5-C, p. 278**)**.

(7.2) *Either $(I - T)$ is invertible or there exists $x \neq 0$ in X such that $(I - T)(x) = 0$* **(6**, Theorem 5.5-F, p. 281**)**.

(7.3) *There exists an integer N such that if $n, m \geqslant N$, then the null space of $(I - T)^n$ is identical with the null space of $(I - T)^m$* **(6**, Theorem 5.5-E, p. 279**)**.

THEOREM 7.1. *Let A be a semi-simple normed algebra.*

(1) *If A is a modular annihilator algebra, then every primitive ideal of A is maximal modular if and only if $S_A \subset J(A)$.*

(2) *If A is a Banach algebra with dense socle, then every primitive ideal of A is maximal modular if and only if A is a completely continuous algebra.*

Proof. (1) Assume that A is a modular annihilator algebra. By Theorem 6.5 (2), every primitive ideal of A is maximal modular if and only if for every $e \in E_A$, Ae and eA are finite dimensional. Since operators with finite-dimensional range are completely continuous, T_e and Q_e are completely continuous if eA and Ae are finite dimensional. Conversely, if $e \in J(A)$, eA and Ae are finite dimensional since eA is the null space of $(I - T_e)$ and Ae is the null space of $(I - Q_e)$; see (7.1). It follows that Ae and eA lie in $J(A)$ if and only if P^e is maximal modular. Hence (1).

(2) Assume that A is a Banach algebra with dense socle. Then A is a modular annihilator algebra by Theorem 5.2 (1). Thus every primitive ideal of A is maximal modular if and only if $S_A \subset J(A)$ by (1). But $\bar{S}_A = A$ and $J(A)$ is closed. Therefore every primitive ideal of A is maximal modular if and only if $J(A) = A$. Hence (2).

We devote the remainder of this section to proving the following theorem:

THEOREM 7.2. *Let A be a semi-simple normed algebra such that T_a is completely continuous for all $a \in A$. Then A is a modular annihilator algebra.*

We first note that if A has an identity 1, T_1 is the identity operator on A, and T_1 is completely continuous so that A must be finite dimensional. An easy application of Lemma 6.1 proves that all semi-simple finite-dimensional algebras are modular annihilator algebras.

To prove the theorem in the case where A has no identity, we need several lemmas. Much of the information we have concerning A is in terms of the spectrum of T_a as an operator on A. We cannot assume that the spectrum of T_a in the algebra of bounded operators on A is the same as the spectrum of a in A.

If B is an algebra and $b \in B$, we denote the spectrum of b in B by $\mathrm{Sp}_B(b)$. If X is a normed linear space, we denote the algebra of completely continuous operators on X and the algebra of bounded operators on X by $C(X)$ and $B(X)$ respectively.

With this notation our problem is that $\mathrm{Sp}_A(a)$ need not be the same as $\mathrm{Sp}_{B(A)}(T_a)$. Denote by A_1 the usual extension of A to a normed algebra with identity. Elements of A_1 will be written as $\lambda + a$, λ a scalar and $a \in A$. Let $||\cdot||$ be the norm on A. The norm on A_1 is then, as usual,

$$||\lambda + a|| = |\lambda| + ||a||.$$

We define $a \rightarrow K_a$ to be the representation of A on A_1 given by

$$K_a(\lambda + b) = \lambda a + ba.$$

For the remainder of this section we assume that A has no identity and we adopt the notation above.

LEMMA 7.3. $\mathrm{Sp}_A(a) = \mathrm{Sp}_{B(A_1)}(K_a)$.

Proof. See (**5**, Theorem 1.6.9 (ii), p. 32).

LEMMA 7.4. *If $T_a \in C(A)$, then $K_a \in C(A_1)$.*

Proof. It is not difficult to verify that the image of the unit ball in A_1 under K_a is pre-compact. We leave the details to the reader.

LEMMA 7.5. *Let M be a maximal modular left ideal of A. Let $w \in A$ be such that $A(1 - w) \subset M$. Assume furthermore that $K_a \in C(A_1)$ for all $a \in A$. Then:*

(1) *The null space of $(I - K_w)$ is equal to $R(A(1 - w))$, which is non-zero.*

(2) *There exists an integer $n \geqslant 0$ such that for all $k \geqslant 0$,*

$$R(A(1 - w)^{n+k}) = R(A(1 - w)^n).$$

(3) *$R(A(1 - w)^m)$ is finite dimensional for any $m \geqslant 0$.*

Proof. Since $A(1-w) \subset M$, $1 \in \mathrm{Sp}_A(w)$. By Lemma 7.3, $1 \in \mathrm{Sp}_{B(A_1)}(K_w)$. It follows that the null space of $(I - K_w)$ is non-zero by (7.2). If x_1 is in the null space of $(I - K_w)$, $x_1 = \lambda + x$, λ a scalar, $x \in A$, then

$$0 = (\lambda + x) - w(\lambda + x) = \lambda + (x - \lambda w - wx).$$

Thus $\lambda = 0$. Therefore $x_1 = x$, and $x \in R(A(1-w))$. The other inclusion is obvious: thus the null space of $(I - K_w)$ is equal to $R(A(1-w))$. This proves (1).

Since $A(1-w) \subset M$, $A(1-w)^m \subset M$ for any $m \geqslant 0$. Let

$$(1 - w_m) = (1-w)^m.$$

Then $w_m \in A$. By (1), the null space of $(I - K_{w_m})$ is equal to

$$R(A(1-w_m)) = R(A(1-w)^m).$$

Then by (7.3), there exists $n \geqslant 0$ such that

$$R(A(1-w)^n) = R(A(1-w)^{n+k}) \text{ for } k \geqslant 0.$$

Hence (2).

(3) follows directly from (1) and (7.1).

We can now complete the proof of Theorem 7.2 for the case where A has no identity.

Let M be a maximal modular left ideal of A. Either $S_A \subset M$ or $R(M) \neq 0$ by (4.1). We assume that $S_A \subset M$. Let u be such that $A(1-u) \subset M$. Since $T_a \in C(A)$ for all $a \in A$, $K_a \in C(A_1)$ for all $a \in A$ by Lemma 7.4. Now Lemma 7.5 holds, and we choose n such that

$$R(A(1-u)^n) = R(A(1-u)^{n+k}) \text{ for } k \geqslant 0.$$

Let N be the right ideal $R(A(1-u)^n)$.

Since N is non-zero and finite dimensional (Lemma 7.5), there exists an idempotent $w \in S_A$ such that $N = wA$ by Lemma 6.1. By assumption $S_A \subset M$, and therefore setting $(1-v) = (1-u)^n$, we obtain

$$A(1 - (v + w)) \subset M.$$

By Lemma 7.5, $R(A(1-(v+w)))$ is a non-zero right ideal of A. Suppose x is in this ideal. Then $(1-v)x = wx$, and since $wx \in N$,

$$A(1-v)^2 x = A(1-v)wx = 0.$$

It follows that

$$x \in R(A(1-v)^2) = R(A(1-v))$$

by our choice of n and the definition of v. Thus $(1-v)x = 0$ and $wx = 0$. Then $x \in N = wA$, and therefore $x = wx = 0$. This means that

$$R(A(1-(v+w))) = 0,$$

contradicting Lemma 7.5.

It follows that A is a modular annihilator algebra, since M was an arbitrary maximal modular left ideal of A, and we have shown that $R(M) \neq 0$.

COROLLARY. *Let A be a semi-simple, normed completely continuous algebra. Then:*

(1) *Every primitive ideal P of A is maximal modular and*

$$P = A(1 - u) = (1 - u)A$$

for $u = u^2 \in S_A$.

(2) *Π_A is discrete.*

Part (2) of this corollary generalizes a theorem of Kaplansky (**4**, Theorem 5.1, p. 406).

8. Examples.

8.1. Assume that A is a semi-simple topological algebra in which every maximal modular left ideal is closed. Then if A is in addition an annihilator algebra, a dual algebra, or an algebra with dense socle, A is a modular annihilator algebra; cf. Theorem 5.2, or (**7**, Lemma 3.11, p. 41).

8.2. Assume that X is a Banach space. Then the algebra of inessential operators on X, the algebra of compact operators on X, and any two-sided ideal of these algebras are modular annihilator algebras (**2**, Example 7.3, p. 76).

8.3. Let X be the Banach space of all continuous functions on some compact Hausdorff space. Let Y be $L^1(G)$, G the group of n-dimensional Euclidean space or the circle group, all with Haar measure. Then the algebra of all weakly compact operators on X and the algebra of all weakly compact operators on Y are modular annihilator algebras (**2**, Example 7.4, p. 77).

8.4. Any commutative, semi-simple algebra with a discrete space of maximal modular ideals is a modular annihilator algebra (**2**, Proposition 5.4, p. 63).

8.5. Semi-simple, normed completely continuous algebras are modular annihilator algebras (Theorem 7.2).

REFERENCES

1. E. Artin, C. Nesbitt, and R. Thrall, *Rings with minimum condition* (Ann Arbor, 1944).
2. B. A. Barnes, *Modular annihilator algebras*, Doctoral Thesis, Cornell University (1964).
3. M. Freundlich, *Completely continuous elements of a normed ring*, Duke Math. J., *16* (1949), 273–283.
4. I. Kaplansky, *Normed algebras*, Duke Math. J., *16* (1949), 399–418.
5. C. Rickart, *Banach algebras* (Princeton, 1960).
6. A. E. Taylor, *Introduction to functional analysis* (New York, 1958).
7. B. Yood, *Ideals in topological rings*, Can. J. Math., *16* (1964), 28–45.

Cornell University and
University of California, Berkeley