J. Austral. Math. Soc. **21** (Series B) (1980), 486–495

A SCHEDULING PROBLEM WITH A SIMPLE GRAPHICAL SOLUTION

A. F. BEECHAM and A. C. HURLEY

(Received 4 June 1979)

(Revised 19 June 1979)

Abstract

It is shown that a problem which arose in the scheduling of two simultaneous competitions between a number of golf clubs may be reduced to that of 4-colouring the edges of a certain bipartite graph which has 4 edges meeting at each vertex. This colouring problem is solved by an analysis in terms of directed cycles, which is simple to carry through in a practical case and is easily extended to the problem with 4 replaced by 2^m. The more general colouring problem with 4 replaced by any positive integer is solved by relating it to the marriage problem enunciated by Philip Hall and to the latin multiplication technique of Kaufmann but, in practical applications, this approach involves severe computational difficulties.

1. Introduction

Every autumn the Pennant and Minor Pennant competitions of the Victorian Golf Association (V.G.A.) are conducted concurrently on six consecutive Saturdays. Most member clubs field both a Pennant and a Minor Pennant team and until 1974 the method of scheduling was such that, despite extensive trial-and-error, clubs were frequently allotted home games in both competitions on the same day. Many clubs found this undesirable since the opportunities for ordinary members to play were severely restricted, with Minor Pennant on the course in the morning and Pennant in the afternoon. Here we show how such a state of affairs may be avoided and, indeed, has been avoided since 1975. We hope that the method of scheduling we describe may have other applications, especially in situations where it is not merely inconvenient but physically impossible to conduct two games at the same club on the same day.

Our discussion is, as far as possible, self-contained; the graph-theoretic concepts we use may be found in the books by Harary *et al.* [2] and Ore [4].

2. The problem

A competition involving teams from several clubs is often run on the following lines. Each year, according to a system of promotion and relegation, the teams are allocated to N sections, each containing n teams. Each team plays a series of $2(n-1)$ home-and-away matches against the other teams in its section. All these $Nn(n-1)$ matches may be scheduled by attaching n labels $A, B, C, \ldots$ to the n teams in each section, by making one random draw, and by applying the same draw pattern to all sections. An example of such a draw pattern for $n=4$ is shown in Table 1.

This procedure is straightforward and causes no difficulties so long as only one competition between the participating clubs is in question. In this case, clearly, half the available facilities, those of the clubs playing away on a given date, are not used in the competition—which suggests the possibility of running a second competition between the same clubs without duplicating the use of facilities. Obviously, two such competitions, for example the V.G.A.'s Pennant and Minor Pennant, can be scheduled, as we have described, for the same dates but, if the schedulings are done independently, it is almost certain that some of the clubs will be allotted two home matches on the same day. Our problem is to schedule the double competition so that all such clashes are avoided.

TABLE 1

Example of a draw pattern for $n=4$ (used by V.G.A. for 1975 draw)

Date	S_1		S_2		S_3		S_4		S_5		S_6	
Home teams	A	C	A	B	A	B	B	D	C	D	D	C
Away teams	B	D	C	D	D	C	A	C	A	B	A	B

3. Reduction to graphical form

The draw pattern of Table 1 has a symmetry property, which we may exploit to reduce our problem to graphical form: namely, the letters A and D are arranged in such a way that they never appear as home teams on the same date; similarly for the pair B and C. Suppose now that we have a complete draw for the double competition, that is, an allocation

$$L(T_i), \quad i=1,\ldots,4N,$$

of letters $L = A$, B, C or D to each of the Pennant teams T_i and an allocation

$$L(t_i), \quad i=1,\ldots,4N,$$

to each of the Minor Pennant teams, t_i. Suppose, further, that this draw allocates

https://doi.org/10.1017/S0334270000002174 Published online by Cambridge University Press

the same letter to the two teams (T_i, t_i) from each club† so that

$$L(T_i) \leftrightarrow L(t_i), \quad i = 1, \ldots, 4N. \tag{1}$$

A draw satisfying conditions (1) is, of course, the most unsatisfactory possible; it clearly leads to $2N$ clashes of home-and-away matches on each date. However, if we now apply the permutation $(AD)(BC)$ to one of the two allocations, say that for the Minor Pennant, we obtain from any complete draw satisfying (1) a complete draw which has the correspondences

$$\left.\begin{array}{lll} A(T_i) \leftrightarrow D(t_i), & D(T_i) \leftrightarrow A(t_i), & \\ B(T_i) \leftrightarrow C(t_i), & C(T_i) \leftrightarrow B(t_i), & i = 1, \ldots, 4N. \end{array}\right\} \tag{2}$$

The above-mentioned property of the draw pattern of Table 1 then ensures that the new complete draw shows no clashes on any date. Conversely, starting with a complete draw satisfying the conditions (2) and applying the permutation $(AD)(BC)$ to the Minor Pennant draw we obtain a complete draw satisfying conditions (1).‡

Our problem of avoiding all clashes is thus, in all cases, reduced to that of carrying out a complete draw satisfying conditions (1). This transformed problem has a convenient graphical representation. The N sections of the Pennant are represented by N vertices arranged for convenience in a vertical column, and the sections of the Minor Pennant by an adjoining vertical column of N vertices. Edges are now inserted; each of these links that section which contains a Pennant team, T_i, in the left column (or part) with that section in the right column (or part) which contains the Minor Pennant team, t_i, with the same subscript i, $i = 1, \ldots, 4N$. The result is a bipartite graph which contains N vertices in each part (left, right) and a total of $4N$ edges, four edges intersecting at each vertex; we refer to it as a V_4 graph. An example, which has $N = 8$, and is based on the composition of the V.G.A. Pennant and Minor Pennant sections for 1975, is shown in Fig. 1. We note that multiple edges joining the same two vertices may arise and that these are shown with the multiple edges separated.

If a complete draw satisfying conditions (1) exists, each of the edges of our V_4 graph may be labelled A, B, C or D in such a way that edges labelled A, B, C and

† Although the same number of teams compete in each competition, some clubs may field a team in one competition only. It is easy to allow for this. One simply allots the subscripts $i = 1, \ldots, 4N$ to the Pennant (T_i) and Minor Pennant (t_i) teams in any way such that teams from the same club have the same subscript (see Table 3 and caption).

‡ There is a different way of transforming from maximum clashes to zero clashes which, although clumsier for Table 1, is applicable to any draw pattern and, indeed, to the more general case of n teams per section. Here we retain the coincidence of labels (1) but use two distinct draw patterns, that for the second competition being derived from that for the first by interchanging the complete home, and away, rows.

D meet at each vertex. In the usual graph-theoretic terminology the labels are interpreted as colours and, since no two edges of the same colour meet at any vertex, we have, in fact, an admissible (four-)colouring of the V_4 graph. Conversely, given an admissible colouring of a V_4 graph such as Fig. 1, we may derive a complete draw satisfying conditions (1).

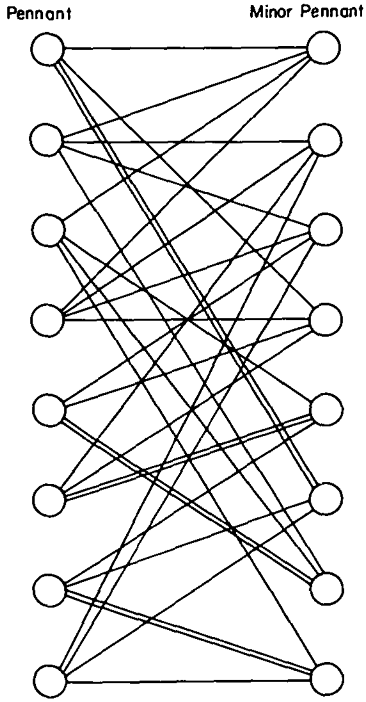


Fig. 1. The V_4 graph based on the composition of the V.G.A. Pennant and Minor Pennant sections for 1975.

These considerations may immediately be generalized to the case of a V_n graph, which we define as an undirected bipartite graph with n edges meeting at each vertex. For convenience, we refer to the two parts as left and right, as in Fig. 1, and we note that any V_n graph necessarily has the same number of vertices, say N, in each of the two parts.

A complete draw satisfying the conditions (1) expresses our original V_4 as the union ($\cup$) of the four edge-disjoint V_1 sub-graphs, each specified by one of the labels A, B, C, D:

$$V_4 = V_1(A) \cup V_1(B) \cup V_1(C) \cup V_1(D). \tag{3}$$

Correspondingly, any admissible n-colouring of a V_n graph is equivalent to an expression for V_n as the union of n edge-disjoint V_1 sub-graphs:

$$V_n = V_1 \cup V_1 \ldots \cup V_1 \ldots \quad (n \text{ terms}). \tag{4}$$

4. Reduction of a V_{2m} graph

The practical solution of our scheduling problem is based on the application of the following theorem and corollary to the V_4 graph derived from the Pennant and Minor Pennant sections (see Section 3 and Fig. 1). We derive the theorem and corollary for the more general case of any V_{2m} graph (see [2], Corollary 12.5a, page 331).

THEOREM 1. *Each of the edges of any V_{2m} graph may be directed with arrows (to the right or to the left) in such a way that the resulting digraph appears as the union of a number of edge-disjoint, directed cycles.*

PROOF. The proof proceeds by the explicit construction of the cycles. We start at any vertex, say vertex i in the left part (column), select at random one of the $2m$ edges, direct it with arrows to the right and follow it to, say, vertex j in the right part (column). We now select at random one of the $2m-1$ remaining undirected edges at vertex j, direct it with arrows to the left and follow it to a vertex in the left part, from which we select at random an undirected edge, and so on. We continue this process as far as possible. Immediately after the start, vertex i has an odd number $(2m-1)$ of undirected edges whereas all other vertices have an even number, $2m$. As two additional edges are directed at each visit to a vertex, the parities, odd or even, of these numbers remain unaltered as we proceed. Consequently, this first stage of directing the edges must terminate at the mth return to the starting vertex i and the directed edges must form m or more directed cycles (see Fig. 2). Since an even number of edges are marked at each vertex visited, the edges, if any, left undirected in the first stage form a sub-graph with an even number of edges at each vertex. In the second stage we choose one of these at random as a starting vertex and proceed to direct the edges of the sub-graph as above. Again, we obtain a number of directed cycles and a residual undirected sub-graph with an even number of edges at each vertex. All edges at the starting vertex of each stage are directed at the end of that stage. Consequently, after at most $2N$ stages, all edges of V_{2m} have been directed and the resulting digraph appears as the union of all the cycles formed in the successive stages. From the method of construction all these cycles are edge-disjoint as required. In Fig. 2, stage 1 of the procedure is illustrated for the V_4 graph of Fig. 1.

COROLLARY 1. *Any V_{2m} graph may be expressed as the union of two edge-disjoint V_m graphs.*

PROOF. We label the right-directed edges of Theorem 1 with the symbol 0 and the left-directed edges with the symbol 1. Since each directed cycle contributes

https://doi.org/10.1017/S0334270000002174 Published online by Cambridge University Press

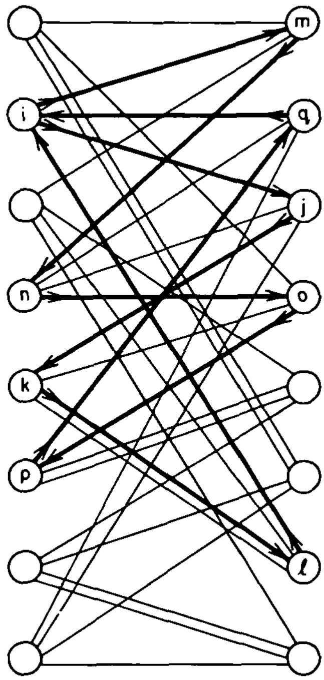


Fig. 2. An illustration of Theorem 1 for the V_4 graph of Fig. 1. The arrows indicate those edges which have been directed at the completion of stage 1 with the mth (here second) return to the starting vertex i. At this stage the directed lines form two directed cycles, namely (i j k l) and (i m n o p q).

one right-directed and one left-directed edge at each of its vertices and since the entire digraph consists of entire cycles, we have just m edges marked (0) and m edges marked (1) meeting at each vertex of our V_{2m}. The complete set of edges marked (0) forms some V_m sub-graph, say $V_m(0)$, the edges marked (1) form another V_m sub-graph, $V_m(1)$, and V_{2m} may be expressed as the union of these two edge-disjoint V_m sub-graphs:

$$V_{2m} = V_m(0) \cup V_m(1). \tag{5}$$

After the separation shown in equation (5) has been effected, the graphs $V_m(0)$, $V_m(1)$ and their edges are all undirected.

5. A practical solution

From equation (5) we may express any V_4 graph, such as that of Fig. 1, as the union of two V_2 graphs:

$$V_4 = V_2(0) \cup V_2(1). \tag{6}$$

https://doi.org/10.1017/S0334270000002174 Published online by Cambridge University Press

Furthermore, the edge labels (0, 1) obtained by the procedure outlined in the proof of Theorem 1 may also be attached to the corresponding Pennant and Minor Pennant teams (T, t). This labelling is the first of two steps necesssary to assign the draw pattern letters, A, B, C and D, to all the teams in the double competition.

Now equation (5) may also be applied to each of the V_2 graphs of equation (6) and the resulting V_1 graphs may be distinguished by additional labels (0, 1), to give

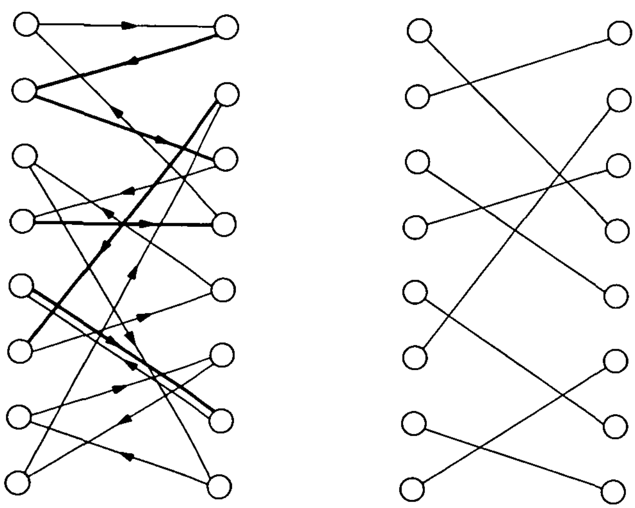
$$V_4 = V_1(00) \cup V_1(01) \cup V_1(10) \cup V_1(11). \tag{7}$$

Again, the additional labels may be derived from the directions of the digraphs obtained by applying the procedure of Theorem 1 separately to $V_2(0)$ and to $V_2(1)$. The complete labels (00, 01, 10, 11) may also be attached to the corresponding Pennant and Minor Pennant teams (T, t). In terms of our original scheduling problem these labels have the properties:

(a) At each vertex, that is, in each Pennant or Minor Pennant section, there is just one team with each label.

(b) In every case, linked teams, that is, Pennant and Minor Pennant teams from the same club, have the same label, 00, 01, 10 or 11.

Two of the sub-graphs $V_2(0)$ and $V_1(01)$ obtained from the V_4 graph of Fig. 1 are shown in Fig. 3.



$V_2(0)$ $V_1(01)$

Fig. 3. The undirected graph $V_2(0)$ is derived from the right-directed edges of the digraph constructed from V_4 as in Theorem 1; the edges contributed by the cycles of stage 1 (Fig. 2) are distinguished as heavier lines. When each edge of $V_2(0)$ is directed in accordance with Theorem 1 (central arrowheads), the left-directed edges yield the undirected graph $V_1(01)$.

https://doi.org/10.1017/S0334270000002174 Published online by Cambridge University Press

If we interpret the labels 00, 01, 10, 11 directly as colours we have a solution of the 4-colouring problem for a V_4 graph like that of Fig. 1 (see equation (3)). If, however, we interpret the labels for Pennant and Minor Pennant teams as shown in Table 2, we obtain an immediate solution to our original scheduling problem expressed in Table 1 and the conditions (2).

TABLE 2

Interpretation of labels to avoid clashes

Label derived graphically	00	01	10	11
Pennant label (T)	A	B	C	D
Minor Pennant label (t)	D	C	B	A

The foregoing solution of our scheduling problem is eminently practical and retains a large random element, both through the selection of a draw pattern and the random selection of edges at each vertex in tracing out the cycles. One may also reduce the whole scheduling operation to a simple routine procedure containing no reference to any graph-theoretic concepts or, indeed, to any graph. In this form it has been used by the Victorian Golf Association since 1975 to schedule their Pennant and Minor Pennant competitions. The complete draw obtained in this way for the 1975 competitions is shown in Table 3.

6. Colouring any V_n graph with n colours

For any V_{2^m} graph the repeated application of equation (5) leads to an expression in terms of 2^m edge-disjoint V_1 sub-graphs:

$$V_{2^m} = V_1 \cup V_1 \cup V_1 \dots \cup V_1 \dots \quad (2^m \text{ terms}), \tag{8}$$

that is, to an admissible 2^m colouring of the V_{2^m} graph.

For $n \neq 2^m$ (any m) this "halving" procedure based on equation (5) gives an incomplete analysis, but a reduction like equation (8), and hence an admissible n-colouring, may be obtained by a different route which employs a solution to the Marriage Problem due to Philip Hall [1].

The Marriage Problem $M(N)$: Each of a set of N boys is to marry a girl he knows.

THEOREM 2. (From [1].) *The Marriage Problem* $M(N)$ *is soluble if and only if every set of* k *boys collectively knows at least* k *girls* ($1 \leqslant k \leqslant N$).

A simple consequence of this result is

https://doi.org/10.1017/S0334270000002174 Published online by Cambridge University Press

COROLLARY 2. *Any V_n graph may be expressed as the union of a V_1 sub-graph and a V_{n-1} sub-graph, which are edge-disjoint:*

$$V_n = V_1 \cup V_{n-1}. \tag{9}$$

The proof is left to the reader (see [1]).

TABLE 3

Allocation of a draw pattern following Tables 1 and 2 to the V.G.A. Pennant (T_i) and Minor Pennant (t_i) teams as in 1975. Competing clubs are numbered *1–32*. Two clubs, *7* and *32*, did not field Minor Pennant teams. The two clubs shown as *7** and *32** in the Minor Pennant did not field Pennant teams

		Pennant T_i			Minor Pennant t_i		
Division 1	Section 1	*1*	10	*C*	*C*	01	*5*
		2	01	*B*	*B*	10	*12*
		3	11	*D*	*D*	00	*4*
		4	00	*A*	*A*	11	*16*
	Section 2	*5*	01	*B*	*D*	00	*30*
		6	11	*D*	*C*	01	*24*
		7	10	*C*	*A*	11	*6*
		8	00	*A*	*B*	10	*15*
Division 2	Section 1	*9*	01	*B*	*B*	10	*17*
		10	11	*D*	*A*	11	*29*
		11	00	*A*	*C*	01	*14*
		12	10	*C*	*D*	00	*8*
	Section 2	*13*	00	*A*	*D*	00	*13*
		14	01	*B*	*B*	10	*21*
		15	10	*C*	*C*	01	*2*
		16	11	*D*	*A*	11	*19*
Division 3	Section 1	*17*	10	*C*	*A*	11	*22*
		18	00	*A*	*D*	00	*23*
		19	11	*D*	*C*	01	*9*
		20	01	*B*	*B*	10	*28*
	Section 2	*21*	10	*C*	*B*	10	*1*
		22	11	*D*	*D*	00	*26*
		23	00	*A*	*C*	01	*31*
		24	01	*B*	*A*	11	*3*
Division 4	Section 1	*25*	11	*D*	*D*	00	*18*
		26	00	*A*	*A*	11	*10*
		27	01	*B*	*C*	01	*20*
		28	10	*C*	*B*	10	*7**
	Section 2	*29*	11	*D*	*A*	11	*25*
		30	00	*A*	*D*	00	*11*
		31	01	*B*	*C*	01	*27*
		32	10	*C*	*B*	10	*32**

https://doi.org/10.1017/S0334270000002174 Published online by Cambridge University Press

The repeated application of equation (9) leads immediately to the extension of equation (8) to general n,

$$V_n = V_1 \cup V_1 \ldots V_1 \ldots \quad (n \text{ terms}), \tag{10}$$

that is, to an admissible n-colouring of any V_n.

A practical algorithm to effect the separation (9) may be found from the following transformation of V_n. First, direct all edges to the right, then merge left and right vertices. This leads to a regular digraph, R_{2n}, the in-degree and out-degree of each vertex being n; and a V_1 sub-graph of our original V_n maps one–one onto a Hamiltonian circuit (or, at least, a vertex-covering set of completely disjoint cycles) in R_{2n}. For this transformed problem practical algorithms and, indeed, general computer programs have been constructed by, for example, the latin multiplication technique of Kaufmann [3].

These techniques are expensive, however, the complication increasing about exponentially with the number of vertices. By contrast, the complexity of the halving procedure of equation (5) increases only linearly with the number of vertices. Consequently, even when $n \neq 2^m$, it is economical to use equation (5) as much as possible in preference to equation (9). For example, if $n = 2^m + 1$, one selects and colours a V_1 sub-graph via equation (9) and colours the remaining V_{2^m} graph by the halving process of equation (5); whereas, if $n = 3.2^m$, one uses the halving procedure of equation (5) to express V_m as the union of 2^m V_3 sub-graphs which are analysed using equation (9) and latin multiplication.

Acknowledgements

The authors are indebted to Professor C. B. Preston and Dr. C. J. Ash for information on the Marriage Problem and also to Professor C. A. Hurst and Dr. J. K. Mackenzie for helpful discussions and constructive criticism of earlier drafts.

References

[1] P. Hall, "On representatives of subsets", *J. Lond. Math. Soc.* 10 (1935), 26–30.

[2] F. Harary, R. Z. Norman and D. Cartwright, *Structural models: An introduction to the theory of directed graphs* (Wiley, New York, London, Sydney, 1965), p. 386.

[3] A. Kaufmann, *Graphs, dynamic programming and finite games* (Academic Press, New York, London, 1967), p. 271.

[4] O. Ore, *Graphs and their uses* (Random House, New York, 1963).

CSIRO Division of Chemical Physics
P.O. Box 160
Clayton
Victoria 3168