BULL. AUSTRAL. MATH. SOC. 20M20, 54H15
VOL. 44 (1991) [417–427]

THE PRIMES OF $S(R)$

K. D. MAGILL, JR.

Dedicated to my friends and former colleagues,
Josephine Mitchell and Lowell Schoenfeld

$S(R)$ is the semigroup, under composition, of all continuous selfmaps of the space R of real numbers. We show that the primes of $S(R)$ are precisely those continuous selfmaps which are surjective and have exactly two local extrema. Additional results are then derived from this. For example, if f is any surjective continuous selfmap of R with $n \geqslant 2$ local extrema, then there exist homeomorphisms $\{h_i\}_{i=1}^{m}$ from R onto R such that $m \leqslant 1 + n/2$ and

$$f = h_1 \circ P \circ h_2 \circ P \circ \cdots \circ h_{m-1} \circ P \circ h_m$$

where P is the polynomial defined by $P(x) = x^3 - x$. It follows from this that the homeomorphisms together with the polynomial P generate a dense subsemigroup of $S(R)$ where the topology on $S(R)$ is the compact-open topology.

An element a of a semigroup S with identity is said to be *prime* if it is not a unit and if $a = bc$, then either a is a unit or b is a unit. The symbol $S(X)$ denotes the semigroup, under composition, of all continuous selfmaps of the topological space X. The units of $S(X)$ are simply the homeomorphisms from X onto X. There are many spaces X for which $S(X)$ contains no primes whatsoever. The next result provides us with a class of such spaces.

PROPOSITION 1. *If X is homeomorphic to a proper retract of itself, then $S(X)$ has no primes.*

PROOF: Let g be a homeomorphism from X onto a proper retract Y of X. Then there exists a mapping $v \in S(X)$ which maps X onto Y such that $v(x) = x$ for each $x \in Y$. Now $g \in S(X)$ and for any $f \in S(X)$, $f \circ g^{-1} \circ v \in S(X)$ and $f = (f \circ g^{-1} \circ v) \circ g$. But g is not a unit and neither is $f \circ g^{-1} \circ v$. ∎

It follows from the latter result that no $S(I^N)$ contains any primes where I^N is the Euclidean n-cell. In particular, $S(I)$ has no primes where I is the closed unit interval. On the other hand, $S(R)$ does contain primes where R is the space of real

Received 3 December 1990

numbers and it is our purpose here to characterise those primes. Although $S(I)$ has no primes, the semigroup of all continuous surjections of I does have primes and these were characterised by Young in [2].

Recall that a mapping f from a space X to a space Y is said to be *light* if $f^{-1}(f(x))$ is totally disconnected for each $x \in X$. The symbol $\operatorname{Ran}(f)$ will be used to denote the range of a function f.

LEMMA 2. *Every prime in $S(R)$ is surjective and light.*

PROOF: Suppose $f \in S(R)$ is not surjective. We may assume that $\operatorname{Ran}(f) = [a, \infty)$. Define $g(x) = |x - a| + a$. Then $f = g \circ f$ but neither f nor g is a unit. Now suppose f is not light. Then there exists an interval $[a, b]$ such that $f(x) = c$ for each $x \in [a, b]$. Choose any function $g \in S(R)$ such that $g(x) = x$ for $x \notin (a, b)$, $g[a, b] = [a, b]$ and g is not injective on $[a, b]$. Then $f = f \circ g$ but neither f nor g is a unit. ∎

LEMMA 3. *Suppose $f \in S(R)$ is prime. Then both the sets $f(-\infty, 0)$ and $f(0, \infty)$ are unbounded.*

PROOF: Suppose $f(-\infty, 0)$ is a bounded set. Then according to the previous lemma, f must map $(0, \infty)$ onto R. Choose any $a > 0$ such that $f(a) > f(x)$ for all $x \leqslant 0$ and let $b = \min f^{-1}(f(a))$. Then $b > 0$ and we define $t(x) = x + f(b) - b$. We then define

$$g(x) = \begin{cases} t(x) & \text{for } x \leqslant b \\ f(x) & \text{for } x \geqslant b \end{cases}.$$

$$h(x) = \begin{cases} t^{-1} \circ f(x) & \text{for } x \leqslant b \\ x & \text{for } x \geqslant b. \end{cases}$$

Now t maps $(-\infty, b]$ homeomorphically onto $(-\infty, f(b)]$. Suppose $x \leqslant b$. Then $f(x) \leqslant f(b)$ and $g \circ h(x) = g\big(t^{-1}(f(x))\big) = f(x)$. On the other hand, for $x \geqslant b$, we have $g \circ h(x) = g(x) = f(x)$. Consequently $f = g \circ h$. But g is not a unit since it is not injective on $(0, \infty)$ (f must map $(0, \infty)$ onto R) and h is not a unit since it is not surjective. This, of course, is a contradiction and we must conclude that $f(-\infty, 0)$ is unbounded. Now define $t(x) = -x$ and let $g = f \circ t$. Then g is prime since f is and the previous argument shows that $g(-\infty, 0)$ is unbounded. This concludes the proof since $f(0, \infty) = g(-\infty, 0)$. ∎

LEMMA 4. *Suppose $f \in S(R)$ is prime. Then $f(-\infty, 0) \neq R$ and $f(0, \infty) \neq R$.*

PROOF: Suppose $f(0, \infty) = R$. According to Lemma 3, $f(-\infty, 0)$ is an unbounded set. We will assume that it is unbounded from below. We assert that there exists an interval $[a, b]$ such that

$$f \text{ is not injective on } [a, b] \text{ and } f(a) \neq f(b) \tag{4.1}$$

and either

$$f(a) \leqslant f(x) \leqslant f(b) \text{ for all } x \in [a, b] \tag{4.2}$$

or

$$f(b) \leqslant f(x) \leqslant f(a) \text{ for all } x \in [a, b]. \tag{4.3}$$

Since $f(0, \infty) = R$, the set $f^{-1}(y)$ is unbounded from above for each $y \in R$. Choose $0 < c_1 < c_2 < b_1$ such that $f(c_1) = f(c_2) = 0$. Choose $m_1 < \min\{f(x): x \in [0, b_1]\}$ and let b_2 be any point such that $b_2 > b_1$ and $f(b_2) = m_1$. Now choose $M_1 > \max\{f(x): x \in [0, b_2]\}$ and let $b_3 = \min[f^{-1}(M_1) \cap [0, \infty)]$. Note that $b_3 > b_2$ and $f(x) \leqslant M_1 = f(b_3)$ for $x \in [0, b_3]$. Next let $m_2 = \min\{f(x): x \in [0, b_3]\}$ and note that $m_2 \leqslant m_1 < f(0)$. Consequently, $f^{-1}(m_2) \cap (-\infty, 0) = f^{-1}(m_2) \cap (-\infty, 0] \neq \emptyset$ since $f(-\infty, 0)$ is unbounded from below. Let $a_1 = \max[f^{-1}(m_2) \cap (-\infty, 0]]$ and observe that $f(x) \geqslant m_2$ for $x \in [a_1, b_3]$. We consider two cases:

CASE 1. $f(x) \leqslant M_1$ for all $x \in [a_1, b_3]$.

In this case, we take $a = a_1$ and $b = b_3$ and we have $f(a) \leqslant f(x) \leqslant f(b)$ for $a \leqslant x \leqslant b$. That is, (4.2) is satisfied.

CASE 2. $f(x) > M_1$ for some $x \in [a_1, b_3]$.

In this case, we let $M_2 = \max\{f(x): x \in [a_1, b_3]\}$. Note that if $x \in [a_1, b_3]$ and $f(x) = M_2$, then $x < 0$ since $f(x) \leqslant M_1 < M_2$ for $x \in [0, b_3]$. Let $a = \max[f^{-1}(M_2) \cap (-\infty, 0]]$ and let b be any point in $[0, b_3]$ such that $f(b) = m_2$. Note that $b > b_1$ since $f(x) < m_1 \leqslant m_2$ for $x \in [0, b_1]$ and we have $a < c_1 < c_2 < b$. Moreover, $f(b) \leqslant f(x) \leqslant f(a)$ for $x \in [a, b]$ and, in this case, (4.3) is satisfied. In both cases, both c_1 and c_2 belong to $[a, b]$ and thus, f is not injective on $[a, b]$ since $f(c_1) = f(c_2)$. In addition, $f(a) \neq f(b)$ since $m_2 < M_1 < M_2$. This verifies the assertion.

Next, let t be the linear map from R onto R such that $t(a) = f(a)$ and $t(b) = f(b)$. If (4.2) is satisfied, t will map $[a, b]$ homeomorphically onto $[f(a), f(b)]$ while t will map $[a, b]$ homeomorphically onto $[f(b), f(a)]$ in the event (4.3) is satisfied. Define

$$g(x) = \begin{cases} f(x) & \text{for } x \leqslant a \\ t(x) & \text{for } a \leqslant x \leqslant b \, . \\ f(x) & \text{for } x \geqslant b \end{cases}$$

$$h(x) = \begin{cases} x & \text{for } x \leqslant a \\ t^{-1} \circ f(x) & \text{for } a \leqslant x \leqslant b \\ x & \text{for } x \geqslant b. \end{cases}$$

Since by assumption, $f(0, \infty) = R$, it follows that $g \mid [b, \infty) = f \mid [b, \infty)$ is not injective and hence that g is not a unit. On the other hand, h is not a unit either since f is not injective on $[a, b]$. We have reached a contradiction since $f = g \circ h$.

Suppose again that $f(0, \infty) = R$ but now consider the case where $f(-\infty, 0)$ is not bounded from above. Let $k = w \circ f$ where $w(x) = -x$ for all x. Then $k(0, \infty) = R$ and $k(-\infty, 0)$ is unbounded from below. The previous argument allows us to conclude that $k = g \circ h$ where neither g nor h is a prime and we arrive at a contradiction here also. We now conclude that

(4.4) If f is prime, then $f(0, \infty) \neq R$.

Now suppose $f(-\infty, 0) = R$ and define $k = f \circ w$ where again, $w(x) = -x$ for all x. Then $k(0, \infty) = R$ and thus $k = g \circ h$ for two nonunits g and h according to (4.4). Evidently $f = g \circ (h \circ w)$ so that if f is prime, then we cannot have $f(-\infty, 0) = R$ either. This verifies the lemma. $\square$

The latter lemma tells us that functions such as $f(x) = x \sin x$ and $g(x) = x \cos x$ are not prime. Of course f can be factored in a more direct way since it is an even function. Specifically, if h is any even function then $h = h \circ k$ where $k(x) = |x|$. The next lemma is an immediate consequence of Lemmas 2, 3 and 4.

Lemma 5. *Suppose $f \in S(R)$ is prime. Then exactly one of the following two conditions must hold:*

(5.1)
$$f(-\infty, 0) = (-\infty, a) \text{ or } f(-\infty, 0) = (-\infty, a] \text{ for some } a \text{ and}$$
$$f(0, \infty) = (b, \infty) \text{ or } f(0, \infty) = [b, \infty) \text{ for some } b;$$

or

(5.2)
$$f(-\infty, 0) = (a, \infty) \text{ or } f(-\infty, 0) = [a, \infty) \text{ for some } a \text{ and}$$
$$f(0, \infty) = (-\infty, b) \text{ or } f(0, \infty) = (-\infty, b] \text{ for some } b.$$

Our next lemma strengthens the preceding one.

Lemma 6. *Let f be a prime. Then there exist points a and b such that $a < b$, f is injective on $(-\infty, a]$ and on $[b, \infty)$ and either f assumes a local maximum at a and a local minimum at b or it assumes a local minimum at a and a local maximum at b.*

Proof: We first note the (easily demonstrated) fact that

(6.1) Every surjective function in $S(R)$ which is not a unit has at least one local maximum and one local minimum.

In particular, each prime must have at least one local maximum and one local minimum in view of Lemma 2. Since f is prime, either (5.1) or (5.2) holds. Assume first that (5.1) holds and let $c = \operatorname{lub} f(-\infty, 0)$. We consider two cases.

CASE 1. $f(a) = c$ for some $a < 0$.

In this case, $f(x) \leqslant f(a)$ for all $x \leqslant 0$. In particular, f has a local max at a and $f(x) \leqslant f(a)$ for all $x \leqslant a$. Let t be any linear map such that $t(a) = f(a)$ and which maps $(-\infty, a]$ onto $(-\infty, f(a)]$ and define

$$g(x) = \begin{cases} t(x) & \text{for } x \leqslant a \\ f(x) & \text{for } x \geqslant a. \end{cases}$$

$$h(x) = \begin{cases} t^{-1} \circ f(x) & \text{for } x \leqslant a \\ x & \text{for } x \geqslant a. \end{cases}$$

One readily verifies that $f = g \circ h$ and since f has a local maximum at a, and $f(0, \infty)$ is unbounded from above, f cannot be injective on $[a, \infty)$. Thus, g is not a unit and consequently, h must be. It follows from this and (5.1) that, in this case, f is increasing on $(-\infty, a]$.

CASE 2. $f(x) < c$ for each $x < 0$.

Let $a = \min f^{-1}(c) \cap [0, \infty)$. Again, we have $f(x) \leqslant f(a)$ for $x \leqslant a$ and we define the functions g and h just as before. This time, however, we do not know whether or not f is injective on $[a, \infty)$ but at least one of g and h must be a unit and this implies that either

(6.2) $\qquad f$ is injective on $(-\infty, a]$ or

(6.3) $\qquad f$ is injective on $[a, \infty)$.

The former must happen for h to be a unit and the latter for g to be a unit. Suppose (6.2) holds. In this situation, we have $c = f(0)$ and $a = 0$. Then there exist points c_1 and c_2 such that $0 \leqslant c_1 < c_2$ and f assumes a local maximum at c_1 and a local minimum at c_2. Let c_3 be a point in $[0, c_2]$ at which f assumes its absolute maximum value on that interval. Then $0 \leqslant c_3 < c_2$, f has a local maximum at c_3 and $f(x) \leqslant f(c_3)$ for $x \leqslant c_3$ since f is increasing on $(-\infty, 0]$. This time, let t be a linear map such that $t(c_3) = f(c_3)$ and which maps $(-\infty, c_3]$ onto $(-\infty, f(c_3)]$ and define the functions g and h as follows:

$$g(x) = \begin{cases} t(x) & \text{for } x \leqslant c_3 \\ f(x) & \text{for } x \geqslant c_3. \end{cases}$$

$$h(x) = \begin{cases} t^{-1} \circ f(x) & \text{for } x \leqslant c_3 \\ x & \text{for } x \geqslant c_3. \end{cases}$$

Again, we have $f = g \circ h$. Moreover g is not a unit since f assumes a local min on $[c_3, \infty)$. Thus h is a unit and f must, in this case, be injective on $(-\infty, c_3]$.

Now assume (6.3) holds. In this instance there exist points c_1 and c_2 such that $c_1 < c_2 \leqslant a$ and f assumes a local maximum at c_1 and a local minimum at c_2. Since $f(-\infty, 0)$ is unbounded from below, there exists a point $c_3 < c_1$ such that $f(c_3) < f(x)$ for $x \in [c_1, a]$. Let c_4 be a point at which f assumes its absolute minimum value on the interval $[c_3, a]$. Then, $c_3 \leqslant c_4 < c_1$ and $f(x) \geqslant f(c_4)$ for $x \geqslant c_4$ since f is increasing on $[a, \infty)$. Now let t be a linear map such that $t(c_4) = f(c_4)$ and which maps $[c_4, \infty)$ onto $[f(c_4), \infty)$. Then define g and h as follows:

$$g(x) = \begin{cases} f(x) & \text{for } x \leqslant c_4 \\ t(x) & \text{for } x \geqslant c_4. \end{cases}$$

$$h(x) = \begin{cases} x & \text{for } x \leqslant c_4 \\ t^{-1} \circ f(x) & \text{for } x \geqslant c_4. \end{cases}$$

Again, we have $f = g \circ h$, but h is not a unit since f assumes a local minimum at the point $c_2 > c_4$. This means g must be a unit and therefore f is injective on $(-\infty, c_4]$. Since (5.1) holds, this means f is increasing on $(-\infty, c_4]$.

Next, let c_5 be a point at which f assumes its absolute maximum on the interval $[c_4, c_2]$. Then $c_4 \leqslant c_5 < c_2$ and $f(x) \leqslant f(c_5)$ for $x \leqslant c_5$ since f is increasing on $(-\infty, c_4]$. Now let t be a linear map such that $t(c_5) = f(c_5)$ and which maps $(-\infty, c_5]$ onto $(-\infty, f(c_5)]$ and this time define g and h as follows:

$$g(x) = \begin{cases} t(x) & \text{for } x \leqslant c_5 \\ f(x) & \text{for } x \geqslant c_5. \end{cases}$$

$$h(x) = \begin{cases} t^{-1} \circ f(x) & \text{for } x \leqslant c_5 \\ x & \text{for } x \geqslant c_5. \end{cases}$$

Then $f = g \circ h$ and g is not a unit since f has a local minimum at $c_2 > c_5$. Consequently, h must be a unit which means f is injective on $(-\infty, c_5]$. This, in turn, means f is increasing on $(-\infty, c_5]$ and since $f(c_5)$ is the absolute maximum value for f on $[c_4, c_2]$, it follows that f assumes a local maximum at c_5.

At this point, we have shown that if a prime f satisfies (5.1) then there exists a point a such that f assumes a local maximum at a and f is injective on $(-\infty, a]$. The proof that there is a point b such that f assumes a local minimum at b and f is injective on $[b, \infty)$ is quite similar and for that reason we omit the details. Moreover we must have $a < b$, otherwise f would be injective on R and would be a unit. As for the case where f satisfies (5.2), define $t(x) = -x$ and let $g = t \circ f$. Then g satisfies (5.1) and there exist points a and b such that $a < b$, g has a local maximum at a and a local minimum at b and g is injective on both $(-\infty, a]$ and $[b, \infty)$. Consequently,

https://doi.org/10.1017/S0004972700029920 Published online by Cambridge University Press

in this case, f is injective on those same sets but it has a local minimum at a and a local maximum at b. $\square$

DEFINITION 7: Let f be a selfmap of R. If there exists a number K such that $f(x) < f(y)$ for $x < y$ and $|x|$, $|y| > K$, we say that f is *eventually increasing.* If $f(x) > f(y)$ for $x < y$ and $|x|$, $|y| > K$, we say that f is *eventually decreasing.*

DEFINITION 8: A selfmap of R is a *generalised odd degree polynomial* if it is continuous, surjective and has only a finite number of local extrema.

For any $f \in S(X)$, we will denote by $L(f)$ the collection of all those points at which f assumes a local extreme value and the cardinality of a set A will be denoted by $|A|$.

DEFINITION 9: For a generalised odd degree polynomial f, we let $\operatorname{Ord}(f) = |L(f)|$.

LEMMA 10. *Let f be a generalised odd degree polynomial with $\operatorname{Ord}(f) > 0$ and let y be any point in R whatsoever. Then $f^{-1}(y)$ contains a point which does not belong to $L(f)$.*

PROOF: Every point in R is of the form $f(a)$ for some a. If $a \notin L(f)$, it is evident that $f^{-1}(f(a))$ contains a point which does not belong to $L(f)$. The remaining case is where $a \in L(f)$. Now f is either eventually increasing or eventually decreasing. We will assume the former and we will also assume that f has a local maximum at the point a. The remaining cases are similar. Let a_1 be the greatest point in $f^{-1}(f(a)) \cap L(f)$ at which f assumes a local maximum. Since f is surjective, there is a point $a_2 > a_1$ such that $f(a_2) = f(a_1)$ and $a_2 \in f^{-1}(f(a)) \setminus L(f)$. $\square$

LEMMA 11. *Let f and g be any two generalised odd degree polynomials such that $\operatorname{Ord}(f) = \operatorname{Ord}(g) = 2$. Then there exist homeomorphisms h and k from R onto R such that $f = h \circ g \circ k$.*

PROOF: We first consider the case where both f and g are eventually increasing. Let $L(f) = \{a, b\}$ and let $L(g) = \{c, d\}$ where $a < b$ and $c < d$. Let h be any (necessarily increasing) homeomorphism from R onto R such that $h(g(c)) = f(a)$ and $h(g(d)) = f(b)$. Now let $g_1 = g \mid (-\infty, c]$, $g_2 = g \mid [c, d]$ and $g_3 = g \mid [d, \infty)$ and then define

$$k(x) = \begin{cases} g_1^{-1} \circ h^{-1} \circ f & \text{for } x \leqslant a \\ g_2^{-1} \circ h^{-1} \circ f & \text{for } a \leqslant x \leqslant b \\ g_3^{-1} \circ h^{-1} \circ f & \text{for } x \geqslant b. \end{cases}$$

One verifies that k is continuous. Moreover h must be increasing and f is increasing on $(-\infty, a]$ and $[b, \infty)$ and decreasing on $[a, b]$ while g is increasing on $(-\infty, c]$ and $[d, \infty)$ and decreasing on $[c, d]$. It readily follows that k is increasing on each of the intervals $(-\infty, a]$, $[a, b]$ and $[b, \infty)$. Thus, k is a homeomorphism from R onto R and it follows easily that $f = h \circ g \circ k$.

Now suppose f is eventually decreasing and g is eventually increasing. Then $t \circ f$ is eventually increasing where $t(x) = -x$ and by our previous considerations, $t \circ f = h \circ g \circ k$ for two (increasing) homeomorphisms h and k. Consequently $f = (t^{-1} \circ h) \circ g \circ k$. If both f and g are eventually decreasing, then $t \circ f$ and $t \circ g$ are eventually increasing and $t \circ f = h \circ t \circ g \circ k$ for appropriate homeomorphisms h and k and thus, $f = (t \circ h \circ t) \circ g \circ k$. $\square$

Let P be the polynomial function which is defined by $P(x) = x^3 - x$. We are now in a position to state and prove the main result of this paper.

THEOREM 12. *The following statements about a function f in $S(R)$ are equivalent.*

(12.1) *f is prime.*

(12.2) *f is surjective and has exactly one local maximum and one local minimum. In other words, f is a generalised odd degree polynomial with* $\mathrm{Ord}(f) = 2$.

(12.3) *There exist homeomorphisms h and k from R onto R such that $f = h \circ P \circ k$.*

PROOF: We first show that (12.1) implies (12.2). Since f is prime, either (5.1) of (5.2) holds and we assume first that (5.1) holds. It follows from our assumption and Lemma 6 that there are points a and b such that $a < b$, f has a local maximum at a and a local minimum at b and f is increasing on both the intervals $(-\infty, a]$ and $[b, \infty)$. Let a_1 be a point at which f assumes its absolute maximum value on $[a, b]$ and let b_1 be a point at which f assumes its absolute minimum value on $[a, b]$. The points a_1 and b_1 must be distinct since, by Lemma 2, f is light. Suppose $b_1 < a_1$. Let t be a linear map such that $t(a_1) = f(a_1)$ and which carries $(-\infty, a_1]$ onto $(-\infty, f(a_1))]$ and define

$$g(x) = \begin{cases} t(x) & \text{for } x \leqslant a_1 \\ f(x) & \text{for } x \geqslant a_1. \end{cases}$$

$$h(x) = \begin{cases} t^{-1} \circ f(x) & \text{for } x \leqslant a_1 \\ x & \text{for } x \geqslant a_1. \end{cases}$$

Since $f(x) \leqslant f(a_1)$ for $x \leqslant a_1$, it follows that $f = g \circ h$. Now g is not a unit since $a_1 < b$ and f assumes a local minimum at b. On the other hand, h is not a unit either since $b_1 < a_1$ and f also assumes a local minimum at b_1. This, of course, is a contradiction and we conclude that $a_1 < b_1$. Thus, we have $a \leqslant a_1 < b_1 \leqslant b$. Suppose $a < a_1$ and let g and h be the maps we just defined previously. Here again, g is not a unit since f has a local minimum at $b > a_1$ and h is not a unit since f has a local maximum at $a < a_1$. With this contradiction, we conclude that $a_1 = a$. In a similar manner, one can verify that $b_1 = b$ and we have now shown that

(12.4)
$$f \text{ is increasing on both } (-\infty, a] \text{ and } [b, \infty) \text{ and } f(b) \leqslant f(x) \leqslant f(a) \text{ for } a \leqslant x \leqslant b.$$

Now let t be the linear map which maps $[a, b]$ onto $[f(b), f(a)]$ such that $t(a) = f(a)$ and $t(b) = f(b)$ and define

$$g(x) = \begin{cases} f(x) & \text{for } x \leqslant a \\ t(x) & \text{for } a \leqslant x \leqslant b \\ f(x) & \text{for } x \geqslant b \end{cases}$$

$$h(x) = \begin{cases} x & \text{for } x \leqslant a \\ t^{-1} \circ f(x) & \text{for } a \leqslant x \leqslant b \\ x & \text{for } x \geqslant b. \end{cases}$$

Now $f = g \circ h$ and g is not a unit since it has a local maximum at a and a local minimum at b. Consequently h must be a unit and therefore f is injective on $[a, b]$ which implies that f is decreasing on $[a, b]$. We have now shown that (12.2) holds whenever (5.1) holds. Suppose (5.2) holds and consider the function $t \circ f$ where t is defined by $t(x) = -x$. Then $t \circ f$ is a prime and satisfies (5.1). Thus (12.2) holds for $t \circ f$ and therefore, also for f. This concludes the verification that (12.1) implies (12.2).

It follows immediately from Lemma 11 that (12.2) implies (12.3) and it is evident that (12.3) implies (12.2) so the proof will be complete when we show that (12.2) implies (12.1). Suppose, then, (12.2) holds and suppose $f = g \circ h$. If $k \in S(R)$ is surjective and $L(k) \neq \emptyset$, then $|L(k)| > 1$. Evidently $L(f) = L(h) \cup h^{-1}[L(g)]$ and, of course, $|L(f)| = 2$. Now g is surjective since f is and it follows from Lemma 10 that either $L(g) = \emptyset$ and $|L(h)| = 2$ or $|L(g)| = 2$ and $L(h) = \emptyset$. In the first case, g is a unit. If the second case holds, h must be surjective since f is surjective and $|L(g)| = 2$ (one really only needs $L(g)$ to be finite). Therefore, in the second case, h is a unit since not only is h surjective, but $L(h) = \emptyset$ as well. This proves the theorem. □

Corollary 13. *A continuous selfmap of R is a generalised odd degree polynomial if and only if it is the product of a finite number of primes. Moreover, the number of primes in any factorisation of a generalised odd degree polynomial f cannot exceed* $\operatorname{Ord}(f)/2$.

Proof: It is a straightforward matter to verify that any product of primes is a generalised odd degree polynomial. Suppose, on the other hand, f is a generalised odd degree polynomial. If it is prime, we are finished. If it is not prime, then $f = f_1 \circ f_2$ where neither f_1 nor f_2 is a unit. Then $L(f_1) \neq \emptyset \neq L(f_2)$ and since $L(f) = L(f_2) \cup f_2^{-1}[L(f_1)]$, we have $\mathrm{Ord}(f_1) < \mathrm{Ord}(f)$ and, $\mathrm{Ord}(f_2) < \mathrm{Ord}(f)$. In fact, since $\mathrm{Ord}(f)$, $\mathrm{Ord}(f_1)$ and $\mathrm{Ord}(f_2)$ are all even integers, we conclude that $\mathrm{Ord}(f_1) \leqslant \mathrm{Ord}(f) - 2$ and $\mathrm{Ord}(f_2) \leqslant \mathrm{Ord}(f) - 2$. This same argument is then used for each of the factors f_1 and f_2 which are not prime and then repeated for subsequent nonprime factors. The process terminates when all the factors are prime. Now we want to show that if $f = f_1 \circ f_2 \circ \cdots \circ f_n$, and the functions f_i are all prime, then $\mathrm{Ord}(f) \geqslant 2n$. This is immediate if $n = 1$. Suppose it holds for $n = m$ and let $f = g \circ f_{m+1}$ where $g = f_1 \circ f_2 \circ \cdots \circ f_m$. Then $L(f) = L(f_{m+1}) \cup f_{m+1}^{-1}[L(g)]$. According to Lemma 10, for each point $a \in L(g)$, there exists a point $b \in f_{m+1}^{-1}(a) \setminus L(f_{m+1})$ and since $\mathrm{Ord}(g) \geqslant 2n$ by the induction hypothesis, we have

$$\mathrm{Ord}(f) = |L(f)| = \left|L(f_{m+1}) \cup f_{m+1}^{-1}[L(g)]\right| \geqslant |L(f_{m+1})| + |L(g)| \geqslant 2 + 2n$$

and the assertion has been verified by induction. We have therefore shown that the number of factors in any prime factorisation of a generalised odd degree polynomial f cannot exceed $\mathrm{Ord}(f)/2$ and the proof is complete. ∎

EXAMPLE 14: The previous corollary tells us that no factorisation of f can contain more than $\mathrm{Ord}(f)/2$ primes and there are certainly instances where this number is attained. However, there are also instances where the number of primes in a prime factorisation of f is actually less than $\mathrm{Ord}(f)/2$. All this is illustrated in what follows. Define

$$h(x) = \begin{cases} x+2 & \text{for } x \leqslant -1 \\ -x & \text{for } -1 \leqslant x \leqslant 1 \\ x-2 & \text{for } 1 \leqslant x \end{cases}$$

$$k(x) = \begin{cases} x+4 & \text{for } x \leqslant -2 \\ -x & \text{for } -2 \leqslant x \leqslant 2 \\ x-4 & \text{for } 2 \leqslant x. \end{cases}$$

Now let $f = h \circ k$ and let $g = k \circ h$. Both h and k are primes and $\mathrm{Ord}(h) = \mathrm{Ord}(k) = 2$. One can verify that $\mathrm{Ord}(f) = 4$ while $\mathrm{Ord}(g) = 8$. In fact, $L(f) = \{-4, -1, 1, 4\}$ and $L(g) = \{-5, -3, -2, -1, 1, 2, 3, 5\}$. Moreover, one can show that $f(x) = x + 6$ for $x \leqslant -4$, $f(x) = x - 6$ for $x \geqslant 4$ and the graph of f on the interval $[-4, 4]$ consists of successive straight line segments joining the points $(-4, 2)$, $(-1, -1)$, $(1, 1)$ and $(4, -2)$ in order. As for the function g, $g(x) = x + 6$ for $x \leqslant -5$, $g(x) = x - 6$ for $x \geqslant 5$ and graph of g on the interval $[-5, 5]$ consists of successive straight line segments joining the points $(-5, 1)$, $(-3, -1)$, $(-2, 0)$, $(-1, -1)$, $(1, 1)$, $(2, 0)$, $(3, 1)$ and $(5, -1)$ in order.

Corollary 15. *Let f be any generalised odd degree polynomial which is not a unit and let g be any prime. Then there exist n units $\{h_i\}_{i=1}^{n}$ of $S(R)$, that is, homeomorphisms from R onto R, such that $f = h_1 \circ g \circ h_2 \circ g \circ \cdots \circ h_{n-1} \circ g \circ h_n$ where $n \leqslant 1 + \mathrm{Ord}(f)/2$.*

Proof: By Corollary 13, $f = f_1 \circ f_2 \circ \cdots \circ f_m$ where each f_i is prime and $m \leqslant Ord(f)/2$. By Lemma 11, there exists for each f_i two homeomorphisms h_i and k_i such that $f_i = h_i \circ g \circ k_i$. Thus $f = t_1 \circ g \circ t_2 \circ g \circ t_3 \cdots \circ t_m \circ g \circ t_{m+1}$ where $t_1 = h_1$, $t_{m+1} = k_m$ and $t_i = k_{i-1} \circ h_i$ for $2 \leqslant i \leqslant m$. Then $n \leqslant 1 + \mathrm{Ord}(f)/2$ where $n = m + 1$. ∎

Corollary 16. *Let $G(R)$ denote the group of units of $S(R)$ and let f be any prime of $S(R)$. Then the semigroup generated by $G(R)$ together with the element f is precisely the semigroup of all generalised odd degree polynomials. Consequently, $G(R)$, together with any prime f generates a dense subsemigroup of $S(R)$ where $S(R)$ is provided with the compact-open topology.*

Proof: Denote by $\langle G(R), f\rangle$ the semigroup generated by $G(R)$ together with the prime f. It is immediate that $\langle G(R), f\rangle$ consists of generalised odd degree polynomials. On the other hand, every generalised odd degree polynomial belongs to $\langle G(R), f\rangle$ in view of Corollary 13. Finally, it is well known that given any function $g \in S(R)$, any bounded closed interval $[a, b]$ and any $\varepsilon > 0$, there exists a generalised odd degree polynomial f such that $|f(x) - g(x)| < \varepsilon$ for $x \in [a, b]$. It follows that $\langle G(R), f\rangle$ is dense in $S(R)$ with respect to the compact–open topology. ∎

The generators $G(R)$ together with f in the previous corollary are far from being a minimal generating set so far as obtaining a dense subsemigroup of $S(R)$ is concerned. Subbiah showed in [1] that $S(R)$ has a dense subsemigroup with only two generators.

References

[1] S. Subbiah, 'A dense subsemigroup of $S(R)$ generated by two elements', *Fund. Math.* **117** (1983), 85–90.

[2] S.W. Young, 'Finitely generated semigroups of continuous functions on $[0, 1]$', *Fund. Math.* **68** (1970), 297–305.

Department of Mathematics
SUNY at Buffalo
106 Diefendorf Hall
Buffalo NY 14214-3093
United States of America