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LINEAR SPACES WITH LINE RANGE $\{n-1, n, n+1\}$ AND AT MOST n^2 POINTS

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Abstract

We characterize all finite linear spaces with $p \leqslant n^2$ points where $n \geqslant 8$ for $p \neq n^2 - 1$ and $n \geqslant 23$ for $p = n^2 - 1$, and the line range is $\{n-1, n, n+1\}$. All such linear spaces are shown to be embeddable in finite projective planes of order a function of n. We also describe the exceptional linear spaces arising from $p < n^2 - 1$ and $n \geqslant 4$.

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1. Introduction

A *finite linear space* (FLS) is a finite set of p elements called *points* together with a collection of q sets of points called *lines* such that any two distinct points u and v belong to precisely one common line, denoted uv, and every line contains at least two points.

An FLS is *trivial* if it has fewer than two lines.

In case a point u is an element of a line x, we shall use phrases such as 'u is on x' or 'x passes through u'. Points on the same line are said to be *collinear*, lines through the same point *concurrent*.

The *degree* of a line x, denoted $a(x)$, will be the number of points on x. The *degree* of a point u, denoted $b(u)$, will be the number of lines on u. If $a(x) = k$ we call x a *k-line*. If $b(u) = k$ we call u a *k-point*.

A *finite affine plane* (FAP) of *order* $m \geqslant 2$ is an FLS with m^2 points in which $a(x) = m$, $b(u) = m+1$ for every line x and point u. A *finite projective plane* (FPP) of order $m \geqslant 2$ is an FLS with m^2+m+1 points in which $a(x) = b(u) = m+1$ for every line x and point u.

The *Nwankpa–Shrikhande plane* (Shrikhande FLS) was examined by Totten (1976) where it was shown to be the unique FLS—other than the FAP of order four less one line and all its points—with 12 points and 19 lines where $b(u) = 5$ for all points u, $a(x) = 3$ or 4 for all lines x, and each point is on precisely one 4-line and four 3-lines. It was also shown that this FLS is embeddable in an FPP of order 5. By the *extended Nwankpa–Shrikhande plane*, we shall mean the Nwankpa–Shrikhande plane with one additional point on all 4-lines. The *Nwankpa plane* is the FLS on 11 points with one 5-point line and fifteen 3-point lines obtained from a 6-arc by adding five points corresponding to the partitions of the arc into three 2-point lines.

Recently, de Witte (1979b) and Totten and the author (1979) classified FLS's with $p \leqslant (n+1)^2$ points and line range $\{n, n+1\}$. It was shown that, except possibly for some small values of n, each such FLS is embeddable in an FPP. The purpose of this paper is to classify and establish embeddability in FPP's of all linear spaces with $p \leqslant n^2$ points in which each line has $n-1$, n or $n+1$ points, and for which n is greater than some fixed positive integer. We prove the following.

THEOREM. *Let L be a nontrivial linear space with $p \leqslant n^2$ points and line range $\{n-1, n, n+1\}$. Then if $n \geqslant 23$, L is*

(i) *an FAP of order $n-1$ or n,*

(ii) *an FAP of order n less a point, a line and all its points, or a line and all its points but one,*

(iii) *an FAP of order $n-1$ with an additional 'point at infinity',*

(iv) *an FPP of order $n-2$ or $n-1$, perhaps less a point in the latter case,*

(v) *an FPP of order $n+1$ less three lines and all their points, or perhaps retaining one of these points, not the point of intersection, if these lines are concurrent,*

(vi) *an FPP of order n, less a line and all its points but one, and less either one, two, or all but one of the points (and therefore also the line in this last case) of a second line on this point, while retaining the point,*

(vii) *an FPP of order n less an $(n+1)$-arc or an $(n+2)$-arc, the latter only if n is even,*

(viii) *an FPP of order n less all points save one of each of two lines, with the point of intersection and the lines themselves deleted.*

Furthermore, if $p = n^2$, $n \geqslant 8$, or $p < n^2 - 1$, $n \geqslant 4$, we have the following additional possibilities:

(ix) $p = 11$ *and L is the Nwankpa plane,*

(x) $p = 12$ *and L is the Nwankpa–Shrikhande plane.*

(xi) $p = 13$ *and L is the extended Nwankpa–Shrikhande plane, a Steiner Triple System on 13 points or the unique FLS on 13 points and 20 lines with one 6-point and six 4-lines,*

(xii) $p = 46$ *and L is one of the problematic block designs* $(46, 69, 9, 6, 1)$.

THEOREM 1. *If L is a nontrivial FLS with $p \leqslant n^2$ points, $p \neq n^2-1$, $n \geqslant 4$ and line range $\{n-1, n+1\}$, then L is one of the following:*

(i) *an FAP of order $n-1$,*
(ii) *an FPP of order n less two lines and all their points except their meet,*
(iii) *an FPP of order $n-2$,*
(iv) *one of the problematic block designs* (46, 69, 9, 6, 1),
(v) *the extended Nwankpa–Shrinkhande plane,*
(vi) *a Steiner Triple System on 13 points (see Batten and Totten* (1979)), *or*
(vii) *the Nwankpa plane.*

THEOREM 2. *If L is an FLS with $p = n^2-1$ points, $n \geqslant 23$ and line range $\{n-1, n+1\}$, then L is*

(i) *an FPP of order $n+1$ less three concurrent lines and all their points, or*
(ii) *an FPP of even order n less an $(n+2)$-arc.*

2. Background from linear spaces

The points and lines of an FLS will normally be denoted by the symbols u_α and x_σ respectively, and for brevity's sake, we shall write b_α for $b(u_\alpha)$ and a_σ for $a(x_\sigma)$.

We have the following easy formulas.

P1. By counting the points lying on the lines through any fixed point u_α, we have

$$p-1 = \sum_\sigma (a_\sigma - 1)\, r_{\sigma\alpha},$$

where

$$r_{\sigma\alpha} = \begin{cases} 1 & \text{if } u_\alpha \in x_\sigma, \\ 0 & \text{if } u_\alpha \notin x_\sigma. \end{cases}$$

P2. If u_α and x_σ are nonincident, there are precisely $b_\alpha - a_\sigma$ lines passing through u_α and missing x_σ.

Theorems A and B below are results from the papers by Totten (1976) and by de Witte (1979b) mentioned in Section 1. Theorem C is due to Bose and Shrikhande (1963) for $n \geqslant 4$, $n \neq 6$ and due to de Witte (1976) for $n = 6$. Theorem D is a result of Mullin and Vanstone (1976); and Theorem E is the main theorem of Batten and Totten (1979).

THEOREM A. *If $p = n^2+n$, $n \geqslant 3$ and each line of the FLS L has n or $n+1$ points, then L is an FPP of order n less a single point, an FAP of order $n+1$ less a line and all its points, or the Nwankpa–Shrikhande plane.*

THEOREM B. *If L is a nontrivial FLS with $p \leqslant (n+1)^2$ points and $n \geqslant 3$, in which each line has n points, then L is an FPP of order $n-1$ or an FAP of order n, or an STS on 13 or 15 points, or a Steiner Quadruple System on 25 points, or one of the problematic block designs* $(46, 69, 9, 6, 1)$.

THEOREM C. *Let L be an FLS with n^2-1 points all of degree $n+1$, n^2+n+1 lines of which $n(n-1)/2$ are of degree $n+1$ and $(n+2).(n+1)/2$ of degree $n-1$, where $n \geqslant 4$ is even. Then L is an FPP of order n less $n+2$ points no three of which are collinear.*

THEOREM D. *Let α be any positive integer and n be any integer satisfying $n \geqslant \alpha+2$. Let L be a finite linear space with $p = n^2-\alpha n$ points, each of degree $n+1$, and $q \leqslant n^2+n-\alpha$ lines, at least $n-\alpha$ of which have degree at least n. If $n > (\alpha^4+2\alpha^3+2\alpha^2+3\alpha)/2$, then L can be embedded in an affine plane of order n.*

Letting $\alpha = 2$ and $n \to n+1$, this becomes applicable to our case $p = n^2-1$ below.

THEOREM E. *If L is an FLS with $p \leqslant (n+1)^2$, $n \geqslant 5$, in which every line has either n or $n+1$ points, then L is one of:*

(i) *an FPP (resp. FAP) of order $n-1$ or n (resp. n or $n+1$),*
(ii) *a punctured FPP (resp. FAP) of order n (resp. $n+1$),*
(iii) *an FAP of order n with an additional 'point at infinity',*
(iv) *an FAP of order $n+1$ less a line and all its points, but possibly one,*
(v) *an FPP of order $n+2$ less three non-concurrent lines and all their points,*
(vi) *the problematic block design* $(46, 69, 9, 6, 1)$.

Furthermore, if $p < (n+1)^2$, $n \geqslant 3$ and $p \neq 15$ (see de Witte (1979b), *then we need only add to this list the following:*

(vii) *the Nwankpa–Shrikhande plane,*
(viii) *a Steiner Triple System on 13 points,*
(ix) *the unique FLS on 13 points and 20 lines with one 6-point (the rest having degree five) and six 4-lines.*

3. Line range $(n-1, n+1\}$

All three theorems are proved simultaneously.

We first of all suppose that L is a nontrivial linear space with $p \leqslant n^2$ points and q lines such that each line has degree $n-1$ or $n+1$. In case $p \leqslant n^2$, $n \geqslant 4$, Theorem B allows us to suppose that there are $(n+1)$-lines. Moreover, P1 implies that not all lines are $(n+1)$-lines, and P1 and P2 together imply $p \geqslant (n+1)(n-2)+1 = n^2-n-1$.

(i) $\underline{p < n^2-3}$. Suppose that $p < n^2-3$. In this case, no b_α is $\geqslant n+2$ by P1. Since some $b_\alpha = n+1$, we may suppose that $p = n^2-n-1+2m$, $0 \leqslant 2m < n-2$, m an integer. Let x be an $(n+1)$-line. Then for $u_\alpha \notin x$, $b_\alpha = n+1$ where m of these are $(n+1)$-lines and $n+1-m$ are $(n-1)$-lines. If $m = 0$, then all lines distinct from x are $(n-1)$-lines and all lines meet x. Let $u_\beta \in x$. By P1,

$$p-1 = (b_\beta - 1)(n-2)+n = (n+1)(n-2).$$

In case $n = 4$, $L \setminus x$ is a 6-arc, $b_\alpha = 4$ for all points u_α of x, and so L is the Nwankpa plane implying $n-2 \mid n$ so $n \leqslant 4$. If $m = 1$, then except for perhaps one point on x, all points have b_α as above. If there is an exceptional point u of x, it lies on all the $(n+1)$-lines, and on no $(n-1)$-lines by P1. Then $L \setminus \{u\}$ is a linear space with n^2-n points, $b_\alpha = n+1$ for all u_α, one of these an n-line and n of these $(n-1)$-lines. By Theorem A, L is an FPP of order n less two lines and all their points except their meet (u), or L is the Nwankpa–Shrikhande plane along with one additional point on all the 4-lines.

So we suppose $m \geqslant 1$ and $b_\alpha = n+1$ for all α, m of these being $(n+1)$-lines and $n+1-m$ $(n-1)$-lines. Clearly then, all $(n+1)$-lines meet any given line. Let x be any $(n+1)$-line and l any $(n-1)$-line. The number of $(n+1)$-lines meeting l is $(n-1)m = mn-m$. The number of $(n+1)$-lines meeting x is $(m-1)(n+1)-1 = mn-n+m$. Hence $2m = n \not< n-2$ and we have a contradiction.

(ii) $\underline{p = n^2-3}$. In this case, using P1 and assuming for the moment that $n \geqslant 6$, we obtain $b_\alpha = n+2$, all $(n-1)$-lines; $b_\alpha = n+1$, $\frac{1}{2}(n-2)$ $(n+1)$-lines, $\frac{1}{2}(n+4)$ $(n-1)$-lines; or $b_\alpha = n$, $n-2$ $(n+1)$-lines, 2 $(n-1)$-lines. Any n-point u must be on all $(n+1)$-lines and is therefore unique. Let v be an $(n+1)$-point. Then v can be on at most one $(n+1)$-line, forcing $\frac{1}{2}(n-2) \leqslant 1$ or $n \leqslant 4$, and so a contradiction. Hence n-points do not exist.

If all points have degree $n+1$, counting incidences of $(n+1)$-lines in two different ways implies

$$n+1 \mid \tfrac{1}{2}(n-2)\cdot(n^2-3) = \tfrac{1}{2}n^2(n+1) - \tfrac{3}{2}n(n+1)+3$$

so that $n+1 \mid 3$, a contradiction. Hence, let x be an $(n+1)$-line and u an $(n+2)$-point not on x. Then u is on an $(n-1)$-line l missing x, and each point of l is an $(n+2)$ point.

If there is an $(n+2)$-point v not on l, then as above, v is on an $(n-1)$-line l' missing x. All lines meeting but distinct from l also meet x; these are $(n+1)(n-1) = n^2-1$ in number. Only $(n-1)^2$ of these meet l', leaving $n^2-1-(n-1)^2 = 2n-2$ lines meeting x but not l'. But n^2-1 lines meet l' and x so that the number of lines meeting x is at least $n^2-1+2n-2+1 = n^2+2n-2$. The precise number of lines meeting x is $n(n+1)+1 = n^2+n+1$ yielding $n^2+2n-2 \leqslant n^2+n+1$ or $n \leqslant 3$. Hence v does not exist, and all points of $L \setminus l$ are $(n+1)$-points.

Since no $(n+1)$-line meets l, each $(n+1)$-point is on at most two $(n+1)$-lines forcing $\frac{1}{2}(n-2) \leqslant 2$ or $n \leqslant 6$.

If $n = 6$, $|L \backslash l| = 28$, each point on two 7-lines and five 4-lines. Also all 7-lines meet. Label these

$$\{1,2,3,4,5,6,7\},\quad \{1,8,9,10,11,12,13\},\quad \{2,8,14,15,16,17,18\},$$
$$\{3,9,14,19,20,21,22\},\quad \{4,10,15,19,23,24,25\},\quad \{5,11,16,20,23,26,27\},$$
$$\{6,12,17,21,24,26,28\},\quad \{7,13,18,22,25,27,28\}.$$

The points of l correspond to parallel classes each containing seven 4-point lines. Without loss of generality, we may choose 1, 14 and 23 to be collinear, forcing the line $\{1,14,23,28\}$. Choosing, 2, 9 and 24, we find that $\{2,9,24,27\}$ is a line. Choosing 3 and 10 we are forced to have $\{3,10,18,26\}$ a line. The first two points of the next three lines force the remaining points: $\{4,11,17,22\}$, $\{5,12,15,25\}$, $\{6,13,16,19\}$. This leaves 7 and 8 with 20 and 21 while 20 and 21 are on a 7-point line. So L does not exist.

Suppose $n = 4$. In addition to the above possibilities for b_α, we have $b_\alpha = n-1$, all lines being $(n+1)$-lines.

If x is a unique 5-line, then each point not on x is on a unique 3-line missing x, thus partitioning L into disjoint subsets of order three with one of order five, while this is impossible as $p = 13$. Suppose L has at least two 5-lines. Since a 6-point is only on 3-lines, any two 5-lines meet each other.

Suppose finally that some point u has at least two, and therefore precisely two or three, 5-lines on it. If u has three 5-lines, then every other point is a 5-point, on one 5-line and four 3-lines. Hence $L \backslash \{u\}$ is the Nwankpa–Shrikhande plane or the FAP of order four less one line and all its points. Suppose no point is a 3-point, and hence suppose that u is a 4-point on the two 5-lines x and y. There can be no 5-lines not on u, and so points not on x or y are 6-points. Points of x and y distinct from $x \cap y$ are 5-points. Let 1 denote the point $x \cap y$ and 2, 3, 4, 5 the points not on x or y. We may assume that $\{1,2,3\}$ and $\{1,4,5\}$ are the 3 lines on 1. What other points then are on a line with 2 and 4? Certainly, none of 1, 3, 5. Let 6 denote the third point on 24. So 6 is on x or y. But each line on 6 then, meets both x and y and hence 24 has four points, a contradiction.

(iii) $\underline{p = n^2-2}$. If $p = n^2-2$ then by P1, $b_\alpha = n+1$ for all α, with $\frac{1}{2}(n-1)$ of these being $(n+1)$-lines and $\frac{1}{2}(n+3)$ $(n-1)$-lines. Counting incidences of $(n+1)$-lines in two different ways implies

$$n+1 \,\big|\, (n^2-2)\cdot\tfrac{1}{2}(n-1) = \tfrac{1}{2}n^2(n+1)-(n+1)\,n+1$$

and

$$\tfrac{1}{2}(n+1) \,\big|\, 1, \quad \text{impossible.}$$

(iv) $\underline{p = n^2-1}$. By P1, $b_\alpha = n, n-1$ $(n+1)$-lines and one $(n-1)$-line, $b_\alpha = n+1, \frac{1}{2}n$ of these $(n+1)$-lines and $\frac{1}{2}n+1$ of these $(n-1)$-lines, or $b_\alpha = n+2$, one of these an

$(n+1)$-line and the rest $(n-1)$-lines. If some point is on *all* $(n+1)$-lines, let l be an $(n-1)$-line not on this point u. Each point of l is joined to u by an $(n+1)$-line. Thus, there is precisely one $(n-1)$-line h on u and each point of $h\setminus\{u\}$ is on an $(n+1)$-line not on u—a contradiction. So no n-points exist.

Suppose that the $(n+1)$-line x misses the $(n-1)$-line l. Then $b_\alpha = n+2$ for all $u_\alpha \in l$ and so there are n $(n-1)$-lines on each point of l meeting x. Let s be the number of points u_β of x with $b_\beta = n+1$. Then the number of $(n-1)$-lines on points of x which meet l is both

$$\leqslant (\tfrac{1}{2}n+1)s+(n+1-s)(n-1) \quad \text{and} \quad \geqslant \tfrac{1}{2}ns+(n+1-s)(n-1).$$

So

$$s+\tfrac{1}{2}ns+n^2-1-sn+s \geqslant n^2-n \geqslant \tfrac{1}{2}ns+n^2-1-sn+s$$

so $s+1 \geqslant (\frac{1}{2}n-1)(s-2) \geqslant 1$ implying $n \leqslant 10$, a contradiction. So each $(n+1)$-line meets each $(n-1)$-line.

Assume now that all $(n+1)$-lines meet each other. Then $b_\alpha = n+1$ for all α, and since all lines meet a fixed $(n+1)$-line, $q = n^2+n+1$ where $(n+2)(n+1)/2$ of these are of degree $n-1$ and $n(n-1)/2$ are of degree $n+1$. By Theorem C, L is an FPP of even order n less an $(n+2)$-arc.

Now let x and y be $(n+1)$-lines which do not meet. So all points on x and y have $b_\alpha = n+2$. Let $u \notin x, y$. Then u is on at least $n+1$ $(n-1)$-lines and so $b(u) = n+2$. Hence $b_\alpha = n+2$ for all α and no two $(n+1)$-lines meet. Thus there are $(n^2-1)/(n+1) = n-1$ $(n+1)$-lines and $q = (n+1)(n+1)+n-1 = n^2+3n$. By Theorem D, L is embeddable in an FAP of order $n+1$. Hence L is an FAP of order $n+1$ less two nonintersecting $(n+1)$-lines, or equivalently, an FPP of order $n+1$ less three concurrent lines and all their points.

(v) $\underline{p = n^2}$. If $p = n^2$ then by P1, $b_\alpha = n+1$ for all α where $\frac{1}{2}(n+1)$ of these are $(n+1)$-lines and $\frac{1}{2}(n+1)$ are $(n-1)$-lines. Counting incidences of $(n-1)$-lines in two different ways gives $n-1 \mid n^2\cdot\frac{1}{2}(n+1)$ while $(n, n-1) = 1$ if $n \geqslant 2$ and $(\frac{1}{2}(n+1), n-1) = 1$ if $n \geqslant 4$.

The requirement that $n \geqslant 23$ in case $p = n^2-1$ is rather disappointing. The condition has not been improved upon as far as I know, in the graph theory context. It may, however, be possible to improve it in the context of linear geometry.

4. Line range $\{n-1, n, n+1\}$

We may now assume that lines of lengths $n-1, n, n+1$ exist. Then by P1, $p \geqslant n^2-n-1$.

(i) $\underline{n^2-n-1 \leqslant p \leqslant n^2-2}$. For $n \geqslant 3$, $n^2-n-1 \leqslant n^2-4$, and for values of p in this range, P1 implies that there are no $(n+2)$-points.

If $p = n^2-3$, by P1, $b_\alpha = n, n-2$ $(n+1)$-lines, 2 $(n-1)$-lines; $b_\alpha = n, n-3$ $(n+1)$-lines, 2 n-lines, 1 $(n-1)$-line; $b_\alpha = n$, $n-4$ $(n+1)$-lines, 4 n-lines; $b_\alpha = n+1$, s_α n-lines, $\frac{1}{2}(n-s_\alpha-2)$ $(n+1)$-lines, $\frac{1}{2}(n-s_\alpha+4)$ $(n-1)$-lines where s_α is a nonnegative integer; or $b_\alpha = n+2$, all $(n-1)$-lines.

Suppose there is an $(n+2)$-point v. For any fixed $(n-1)$-line x, there is a line l on v parallel to x, which therefore contains only $(n+2)$-points. Suppose there is an $(n+2)$-point w not on l. Then w is also on a line l' parallel to x and whose points are all $(n+2)$-points; furthermore, $l \cap l' = \emptyset$ as otherwise there is a point on at least $n+3$ lines. Those lines meeting and distinct from l all meet x, and there are $(n-1)(n+1) = n^2-1$ of them. Only $(n-1)(n-1)$ of these meet l', leaving $2n-2$ lines on x missing l'. But there are also n^2-1 lines meeting x and l' so that the number of lines meeting x is at least $n^2-1+2n-2+1 = n^2+2n-2$, while x is on at most $n(n+1)+1 = n^2+n+1$ lines. This implies $n^2+2n-2 \leqslant n^2+n+1$ or $n \leqslant 3$, and a contradiction. Hence w does not exist.

Let u be any point not on l. Then u is on at least three $(n-1)$-lines implying $b(u) = n+1$.

The linear space $L \setminus l$ contains an n-line h. Since for all points u of $L \setminus l$, b(u) = n + 1, we get a partition of $L \setminus l$ into $(n-2)$-, $(n-1)$- and n-lines, noting that all $(n+1)$-lines meet h. As $(n+1)$-lines exist, the partition has at least $n+1$ lines and so the total number of points in $L \setminus l$ is at least $n+n(n-2) = n^2-n$. However, the precise number is $n^2-3-(n-1) = n^2-n-2$. So $n^2-n-2 \geqslant n^2-n$, implying $-2 \geqslant 0$ and a contradiction.

So no $(n+2)$-point exists for $p = n^2-3$, $n \geqslant 4$.

Finally suppose that $p = n^2-2$. By P1, $b_\alpha = n$, $n-2$ $(n+1)$-lines, 1 n-line, 1 $(n-1)$-line; $b_\alpha = n$, $n-3$ $(n+1)$-lines, 3 n-lines; $b_\alpha = n+1$, s_α n-lines, $\frac{1}{2}(n-s_\alpha-1)$ $(n+1)$-lines, $\frac{1}{2}(n-s_\alpha+3)$ $(n-1)$-lines, s_α a nonnegative integer; or $b_\alpha = n+2$, 1 n-line, $n+1$ $(n-1)$-lines.

If v is an $(n+2)$-point, let x be a fixed $(n+1)$-line and l the line on v parallel to x.

If l is an n-line, then there are $n(n+1)$ $(n-1)$-lines meeting l and x. So $n(n+1) \leqslant (n+1) \,.\, \max\{1, \frac{1}{2}(n-\bar{s}_\alpha+3)\}$ where $\bar{s}_\alpha$ is the minimum value of the s_α's varying over the $(n+1)$-points of x. Clearly, therefore, $2n \leqslant n-\bar{s}_\alpha+3$ or $n-3 \leqslant -\bar{s}_\alpha$. The left-hand side is positive for $n \geqslant 4$ while the right-hand side is always nonpositive. So l is in fact an $(n-1)$-line.

Suppose that there is an $(n+2)$-point w not on l. Then there is a line l' on w parallel to x, and also to l as otherwise we would have an (n + 3)-point. As for l, l' is an $(n-1)$-line. So there are $(n-1)(n+1)$ lines on l and x, and the same number on l' and x, while only $(n-1)^2$ of the lines on l and x meet l'. So at least $n^2-1+2n-2+1$ lines meet x. But at most $n(n+1)+1$ lines meet x, and so $n^2+n+1 \geqslant n^2+2n-2$ or $3 \geqslant n$, a contradiction. Therefore w does not exist.

Suppose there is an n-point u not on l. Then at most one line on u misses l and since no $(n+1)$-lines meet l, we must have $n = 4$ and u is on three 4-lines and one 5-line y.

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Since $\frac{1}{2}(4-s_x-1)$ is an integer, s_x can never be zero and every point of $L\backslash l$ is on either one or three n-lines. Since all $(n+1)$-lines meet while no point is on more than one $(n+1)$-line, y is unique. Hence each point of l is on one 4-line and five 3-lines. Each point of $y\backslash\{u\}$ is on one 5-line, one 4-line and three 3-lines. The six points of $L\backslash l\cup y$ are on three 4-lines and two 3-lines.

It is easy to check that in fact the linear space $\{u\}\cup(L\backslash(y\cup l))$ is the Fano plane. Label these points $1,2,\ldots,7=u$. Adding one point to each line of this configuration on 7, we obtain the points 8, 9 and 10 of l, making them collinear. This gives us lines $\{6,7,3,8\}$, $\{2,7,5,10\}$, $\{1,7,4,9\}$ say. Now we add a point to each line of the Fano plane not on 7 and make all of these points collinear and collinear with 7. Let this line be $\{11,12,13,14,7\}$ and the others so formed be $\{1,2,3,11\}$, $\{1,6,5,12\}$, $\{3,4,5,13\}$, $\{2,6,4,14\}$. Finally, each point of l must be on an additional four 3-point lines, all of which must meet $y\backslash\{u\}$. Starting with the point 8, it must go on distinct lines with 1, 2, 4 and 5. Because of the lines we already have, 8 and 2 can be joined only to 12 or 13. If we choose 12, this forces the following lines: $\{8,4,11\}$, $\{8,5,14\}$, $\{8,1,13\}$, $\{9,2,13\}$, $\{9,5,11\}$ and $\{9,6,13\}$ and a contradiction, as 9 and 13 cannot be on two lines. If we choose $\{8,2,13\}$ as a line, it is easy to see that a similar problem arises. Hence our linear space does not exist.

So we may assume that all points off l are $(n+1)$-points.

We show now that no n-lines miss l. Otherwise, let h be such an n-line. We obtain a partition then of $L\backslash l$ into lines parallel to h. Since all $(n+1)$-lines meet h but miss l, there are at least $n+1$ lines in the partition. Lines of $L\backslash l$ have $n-2$, $n-1$ or n points, so that the minimum value of $|L\backslash l|$ is $n+n(n-2)=n^2-n$. But $|L\backslash l|=n^2-2-(n-1)=n^2-n-1$. So $n^2-n-1\geqslant n^2-n$ which is impossible.

Letting l and x be as above, we now count the number of $(n-1)$-lines meeting both l and x. There are precisely $n(n-1)$ such lines on l meeting x. Therefore,

$$n^2-n\leqslant(n+1)\cdot\tfrac{1}{2}(n-\bar{s}_x+3),$$

where $\bar{s}$ is the minimum value of the s_x's varying over the points of x. So

$$2n^2-2n\leqslant n^2+4n+3-\bar{s}_x(n+1)\quad\text{or}\quad n^2-6n-3\leqslant-\bar{s}_x(n+1).$$

The left-hand side is positive for $n>6$ while the right-hand side is always nonpositive, so that it suffices to consider $4\leqslant n\leqslant 6$.

If $n=6$, we have $-3\leqslant-7\bar{s}_x$ forcing $\bar{s}_x=0$. But this contradicts the fact that $\frac{1}{2}(n-\bar{s}_x+3)$ is an integer.

If $n=5$, $\frac{1}{2}(5-s_x-1)$ an integer implies that s_x is 0, 2 or 4 for each s_x. As n-lines exist, some s_x's are not zero.

Let $u\in l$ with h an n-line on u. Since each point of $h\backslash l\cap h$ is on one n-line, it must be on a second. So at least $n-1$ n-lines meet h in $k\backslash l\cap h$ and are distinct from h. But there are only $n-2$ n-lines in $L\backslash h$, so we have a contradiction.

If $n = 4$, $\frac{1}{2}(4 - s_\alpha - 1)$ an integer implies s_α is 1 or 3 for every s_α. If a point u is on three n-lines, it is the unique intersection of the n-lines on l, and so points distinct from u of lines on u not meeting l are on no n-lines, a contradiction. If each point of $L \backslash l$ is on precisely one n-line, this yields at most $(n-1) \cdot n$ points so that $n^2 - n \geqslant n^2 - 2$ or $n \leqslant 2$, a contradiction.

We may now suppose that for $n^2 - n - 1 \leqslant p \leqslant n^2 - 2$, no $(n+2)$-points exist. Since $(n+1)$-lines exist, it follows that the number of lines in L is at most $(n+1)n + 1 = n^2 + n + 1$. Invoking Theorem 2.3 of McCarthy and Vanstone (1977), we see that L is embeddable in a linear space $\overline{L}$ with line range $\{n-1, n+1\}$, with at least one $(n+1)$-line and such that no point is on more than $n+1$ lines. If $|\overline{L}| \leqslant n^2 - 2$, the previous part of the paper implies that $\overline{L}$ is an FPP of order n less two lines and all their points except their meet, or the extended Nwankpa–Shrikhande plane, or the Nwankpa plane. If $|\overline{L}| \geqslant n^2 - 1$; McCarthy and Vanstone (1977) implies that it is embeddable in an FPP of order n.

Since $n^2 - n - 1 \leqslant |L| \leqslant n^2 - 2$ and L must contain $(n-1)$-, n- and $(n+1)$-lines, the sole possibility for L is an FPP of order n less all points save one of each of two lines with the point of intersection deleted, and consequently the lines removed.

(ii) $\underline{p = n^2 - 1}$. Here, $b = n$, $n-1$ $(n+1)$-lines, 1 $(n-1)$-line; $b_\alpha = n+1$, s_α n-lines, $\frac{1}{2}(n - s_\alpha)$ $(n+1)$-lines, $\frac{1}{2}(n - s_\alpha + 2)$ $(n-1)$-lines, s_α a nonnegative integer; $b_\alpha = n+2$, 1 $(n+1)$-line, $n+1$ $(n-1)$-lines; $b_\alpha = n+2$, 2 n-lines, n $(n-1)$-lines.

Suppose there are two disjoint $(n+1)$-lines x and y. So all points of x and y have $b_\alpha = n+2$ and are therefore of the third type mentioned above. Let $u \notin x, y$. Then u is also the third type of point as it is only on $(n-1)$-lines. We therefore have a partition of L into $n-1$ $(n+1)$-lines, and all other lines are $(n-1)$-lines. By Theorem 2, L is an FPP of order $n+1$ less three concurrent lines and all their points. We therefore assume that all $(n+1)$-lines meet each other.

Suppose now that there is an n-line l missing an $(n+1)$-line x. Points of l must be of the fourth type, so that any point u of x is on at most two $(n+1)$-lines. If u is an $(n+1)$-point, u must be on at least $n-4$ and at most $n-2$ n-lines, all meeting l. There are precisely n n-lines meeting l and x, and no $(n+2)$-point of x is on n-lines, so there is a second $(n+1)$-point of x; also on at least $n-4$ n-lines meeting l. This gives $2(n-4) \leqslant n$ *or* $n \leqslant 8$, a contradiction. So all points of x must be of type three, but as mentioned, this means no n-lines meet x, a contradiction.

Now suppose that l is an $(n-1)$-line missing the $(n+1)$-line x. Let s be the number of $(n+1)$-points of x, and a the number of points of l of type three. Thus the number of $(n+1)$-lines on the $(n+1)$-points of x is at least $a+1$ and at most $a+s+1$. So the number of $(n-1)$-lines on these $(n+1)$-points is at least $a+2s$ and at most $a+3s$.

Hence, the number of n-lines on these points is at most

$$sn + 1 - [a + 1 + (a + 2s)] = sn - 2a - 2s.$$

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Now the number of n-lines meeting l (all of which meet x) is $2(n-1-a)$ and therefore

$$sn-2a-2s \geqslant 2n-2-2a,$$

$$s(n-2) \geqslant 2n-2 = 2(n-2)+2,$$

$$s > 2.$$

Also, the number of $(n-1)$-lines on x is at most $(n+1-s)(n+1)+a+3s$ while the number on l meeting x is $an+(n-1-a)(n-1)$. Therefore,

$$n^2+2n+1-sn+a+2s \geqslant n^2-2n+1+a,\ 4n \geqslant s(n-2) \quad \text{and so} \quad s \leqslant 4.$$

Hence $s = 3$ or 4.

Now the number of n-lines on l meeting x is $2(n-1-a)$. Let b be the number of n-lines on $(n+1)$-points of x which do not meet l. So $sn+1-2(n-1-a)-b$ of the lines on the $(n+1)$-points of x are either $(n+1)$-lines distinct from x, or $(n-1)$-lines. So $\frac{1}{2}(sn+1-2(n-1-a)-b-2s)$ of these are $(n+1)$-lines and at least $\frac{1}{2}(sn+1-2(n-1-a)-b-2s)-s$ of them meet l. So

$$sn+3-2n+2a-b-4s \leqslant 2a,$$

$$n(s-2) < 4s+b-3,$$

$$n \leqslant 4+(b+5)/(s-2)$$

which is $\leqslant 12$, a contradiction.

Thus *all* lines meet $(n+1)$-lines. If there is an n-point u, it is unique. Let l be the $(n-1)$-line on u, and $v \in l \setminus \{u\}$. Any second line on v meets all $(n+1)$-lines on u and hence can only be an n-line. So each point of $L \setminus l$ is on at least $n-2$ n-lines and so has $b_x = n+1$, 1 $(n+1)$-line, $n-2$ n-lines, and 2 $(n-1)$-lines. And in the new linear space $L \setminus l$, each point has $b_x = n+1$, 1 n-line, n $(n-1)$-lines. By Theorem E, $L \setminus l$ is an FAP of order n less a line and all its points but one. Therefore L is an FPP of order n less a line and all its points but one, and less two additional points of a line on this remaining point.

We assume that there are no n-points. Clearly, any point not on a fixed $(n+1)$-line has $b_x = n+1$. Suppose some point v is on all $(n+1)$-lines. If this number is more than one, then $b(v)$ is still $n+1$. So suppose v is on a unique $(n+1)$-line x. If $b(v) = n+2$, all other lines on it are $(n-1)$-lines, and since each $u_x \notin x$ is on a unique $(n-1)$-line, this line is $u_x v$. So all other points of x are *only* on n-lines, which is a contradiction. Hence all points have $b_x = n+1$.

Finally, fix an n-line l. We get a partition into a n-lines and b $(n-1)$-lines yielding $an+b(n-1) = n^2-1$. Thus $n \mid b-1$ and $n-1 \mid a > 0$. So $a = n-1$ and $b = 1$, and $a+b = n$ so that $(n+1)$-lines cannot exist, a contradiction.

(iii) $\underline{p = n^2}$. In this last case $b_x = n$, $n-1$ $(n+1)$-lines, 1 n-line; $b_x = n+1$, s_x n-lines, $\frac{1}{2}(n+1-s_x)$ $(n+1)$-lines, $\frac{1}{2}(n+1-s_x)$ $(n-1)$-lines, s_x a nonnegative integer;

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$b_\alpha = n+2$, 3 n-lines, $n-1$ $(n-1)$-lines; or $b_\alpha = n+2$, 1 $(n+1)$-line, 1 n-line, n $(n-1)$-lines.

If an n-point u exists, then $L\setminus\{u\}$ has only n- or $(n-1)$-lines. By Theorem E, $L\setminus\{u\}$ is a punctured FAP of order n. Hence, L is an FPP of order n less a line and all its points but one, and less an additional point. So we suppose that n-points do not exist.

Suppose there are two parallel $(n+1)$-lines x and y; so all points of x and y are of type four above. Let u be a point not on x or y. If $b(u) = n+1$, u is on no $(n+1)$-lines and therefore is only on n-lines. Since each point of x is on precisely one n-line, u is a unique $(n+1)$-point. Thus every other point is an $(n+2)$-point, and since it can be on at most two n-lines, it is of the fourth type mentioned above. So u is on *all* n-lines, and $L\setminus\{u\}$ has only $(n+1)$- or $(n-1)$-lines. By Theorem 2, L is an FPP of order $n+1$ less three concurrent lines and all their points but one, not the point of intersection.

So suppose all points have $b_\alpha = n+2$. If some point is of type three above, we have an $(n-1)$- or n-line l missing x, and either 2n or $3(n-1)$ n-lines meeting l and x while x meets precisely $n+1$ n-lines. This forces $n \leqslant 2$, and a contradiction. So all points are of the fourth type, and it follows that we have a partition of L into $(n+1)$-lines, implying $n+1 \mid n$, a contradiction.

Hence, we may assume that all $(n+1)$-lines meet each other.

Suppose there is an $(n+2)$-point not on the fixed $(n+1)$-line x. Hence it is on an n- or $(n-1)$-line l missing x. Let x be the number of points of type four on l, and s the number of $(n+1)$-points on x. There are precisely $a+1$ $(n+1)$-lines on $(n+1)$-points of x. Since this includes x, there are precisely $a+s$ $(n-1)$-lines in all on these points. Therefore the total number of n-lines on $(n+1)$-points of x is $sn+1-(a+1+a+s) = sn-2a-s$. The total number of n-lines on l but not equal to l is either $2(n-a)$ or $a+3(n-1-a)$, the smaller of which is $2(n-a)$. All of these lines meet x. Therefore,

$$2(n-a) \leqslant sn-2a-s+(n+1-s),$$

$$n-1 \leqslant s(n-2),$$

$$1 < (n-1)/(n-2) \leqslant s.$$

So $s \geqslant 2$.

Also, the number of $(n-1)$-lines on l but distinct from l is either $na+(n-a)(n-1)$ or $a(n-1)+(n-1-a)(n-2)$, the smaller of which is the latter. All of these meet x. The total number of $(n-1)$-lines on x is $a+s+(n+1-s)n$. So

$$n^2-3n+2+a \leqslant n^2+n+a+s-sn,$$

$$s(n-1) \leqslant 4n-2 = 2(n-1)$$

and so $s \leqslant 2$. So there are precisely two $(n+1)$-points on x.

Suppose there are (at least) two $(n+1)$-points $v, v' \notin x$. There are, at least, $2(n-1)-1$ distinct lines joining v and v' to $(n+2)$-points of x. At most $n-1$ of these can be n-lines and the rest, at least $2n-3-(n-1) = n-2$, are $(n-1)$-lines. Since any $(n+1)$-point has the same number of $(n+1)$-lines as $(n-1)$-lines, v and v' are on at least $n-3$ $(n+1)$-lines. So v and v' are on at least $2(n-1)-1+n-3 = 3n-6$ lines, while they are on at most $(n+1)+n = 2n+1$. So $3n-6 \leqslant 2n+1$ or $n \leqslant 7$, a contradiction. So there is at most one $(n+1)$-point not on x.

Suppose w is an $(n+1)$-point not on x, and u and v are $(n+1)$-points of x. If w is only on n-lines, then $L \setminus \{w\}$ has n^2-1 points, and $(n-1)$-, n- and $(n+1)$-lines, and we may apply the last case. Since there are precisely two $(n+1)$-points of x, w is on at most two $(n+1)$-lines. Each point of $L \setminus \{w\}$ is on a line parallel to x, giving a partition P of $L \setminus x \cup \{w\}$ into n- and $(n-1)$-lines (recall that all $(n+1)$-lines meet). Letting a be the number of n-lines and b the number of $(n-1)$-lines in P, we get

$$n+1+an+b(n-1) = n^2-1, \quad an+b(n-1) = n^2-n-2,$$

so $n \mid b-2, n-1 \mid a+2 > 0$. So $a = n-3$ and $b = 2$. So including x, there are n lines in P. This means that all $(n+1)$-lines besides x, meet w. Also, any n-line on w and not in P meets each line of P. x meets either $n-1+n-3+n-1$ (one $(n+1)$-line on w) n-lines or $2(n-3)+n-1$ (two $(n+1)$-lines on w) n-lines. All lines on w meet x so x is on either $3n-5-(n-1) = 2n-4$ or $3n-7-(n-3) = 2n-4$ n-lines missing w and therefore meeting all lines of P. Let l be an n-line of P. There are either $2(n-1)$ or $2(n-2)$ n-lines meeting l, distinct from l. But at least either $n-2$ or $n-4$ respectively of these, pass through w. So we have a contradiction, and w does not exist.

Suppose then, that u and v on x are the only $(n+1)$-points. Again we get a partition into a n-lines and b $(n-1)$-lines where

$$n+1+an+b(n-1) = n^2, \quad an+b(n-1) = n^2-n-1$$

so that $n \mid b-1, n-1 \mid a+1 > 0$. Thus $a = n-2$ and $b = 1$. So there are n lines again in this partition P. Thus x is a unique $(n+1)$-line and any n-line not in P meets every line of P. The number of n-lines meeting x is $2(n-1)+n-1 = 3n-3$. The number meeting an n-line of P, and not equal to l is $2n$, so we have a contradiction.

We may therefore suppose that *no* point not on x is an $(n+2)$-point. Suppose some point u of x is of the fourth type. Let l be an $(n-1)$-line on u, and v a point of $l \setminus \{u\}$. So v is on an $(n+1)$-line $y \neq x$, $u \notin y$. Since $b(u) = n+2$, there is a line h on u missing y and $h \neq x$. But then each point of h has $b_x = n+2$, a contradiction.

So all points have $b_x = n+1$, forcing $q = n^2+n+1$. By de Witte (1979a), L is a projective plane of order n less *an* $(n+1)$-arc.

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