APPROXIMATE CONTINUITY AND DIFFERENTIATION

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1. Introduction. The relation between the notions of measurability and continuity of a function has received a great deal of attention. The best-known result in this connection is the Vitali-Lusin theorem. Various versions of it can be found in (**2**; **3**; **5**; **8**; **9**). We prove one in this paper (Theorem 3.5) under very weak assumptions and state the classical one in Corollary 3.6. Functions satisfying the property stated in the theorem are frequently called quasi-continuous.

Here we are interested in the notion of approximate continuity, which we call μ-continuity for short and which is closely connected to differentiation. It was first introduced by Lebesgue (**4**) and it has so far required knowledge of density theorems in order to prove its relation to measurability. In this paper, making use of a certain type of Vitali property, we prove first the relation between μ-measurable functions and those μ-continuous almost everywhere (Theorems 3.8 and 3.9). Then, in §4, with Theorem 3.8 as the main tool, we derive several theorems about density and differentiation which extend results found in (**1**; **3**; **7**; **8**).

The spaces that we consider are very general topological spaces. The conditions are on the relation between the measure and the topology through families of neighbourhoods of points. Our measures are Caratheodory (outer) measures and we refer to standard texts such as (**3**; **8**) for the parts of the general theory used in this paper.

2. Notation and terminology. Throughout this paper we shall use the following notation and terminology.

2.1 ω denotes the set of all natural numbers.

2.2 $A \sim B = \{x: x \in A \text{ and } x \notin B\}$.

2.3 $\sigma F = \bigcup_{\alpha \in F} \alpha$.

2.4 $f^{-1}V = \{x: f(x) \in V\}$.

2.5 μ is an (outer) measure on X iff μ is a function on the family of all subsets of X such that

(i) $\mu 0 = 0$,

(ii) $0 \leqslant \mu A \leqslant \sum_{n \in \omega} \mu B_n$ whenever $A \subset \bigcup_{n \in \omega} B_n \subset X$.

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2.6 For μ a measure on X, a set A is μ-measurable iff $A \subset X$ and for every $B \subset X$ we have $\mu B = \mu(B \cap A) + \mu(B \sim A)$.

2.7 For μ a measure on X, a function f on X to a topological space Y is μ-measurable iff, for every open V in Y, $f^{-1}V$ is a μ-measurable set.

2.8 For μ almost all x, $P(x)$ is true iff $\mu\{x: P(x) \text{ is not true}\} = 0$.

2.9 A is a G_δ iff there is a sequence B such that $A = \bigcap_{n \in \omega} B_n$ and B_n is open for every $n \in \omega$.

2.10 There are arbitrarily small neighbourhoods W of x for which $P(W)$ is true iff for every neighbourhood U of x there is a neighbourhood W of x with $W \subset U$ for which $P(W)$ is true.

3. Approximate continuity.

3.1. General hypothesis.

Throughout this section we assume that X and Y are topological spaces; f is a function on X to Y; for every $x \in X$, $N(x)$ is a family of open sets containing x and forming a basis for the neighbourhoods of x; and μ is a measure on X satisfying the following conditions:

(i) $\mu X < \infty$,

(ii) For every $A \subset X$ and $\epsilon > 0$ there exists an open set U such that $A \subset U$ and $\mu U < \mu A + \epsilon$.

Condition (i) above can easily be replaced by the following without affecting the validity of the results:

(ia) There exists a sequence A such that

$$X = \bigcup_{n \in \omega} A_n.$$

A_n is μ-measurable and $\mu A_n < \infty$ for every $n \in \omega$.

3.2. Definitions.

$$\mu - \lim_{t \to x} f(t) = y$$

iff for every $\epsilon > 0$ and neighbourhood V of y there exists a neighbourhood U of x such that for every $W \in N(x)$ with $W \subset U$ we have

$$\mu(W \sim f^{-1}V) \leqslant \epsilon \cdot \mu W$$

f is μ-continuous at x iff

$$\mu - \lim_{t \to x} f(t) = f(x).$$

3.3. Lemma. *If $\{x\}$ is μ-measurable and $\mu\{x\} > 0$ then f is μ-continuous at x.*

Proof. Given $\epsilon > 0$, choose open U with $x \in U$ and

$$\mu U < (1 + \epsilon)\, \mu\{x\}.$$

Then, for any neighbourhood V of $f(x)$, if $x \in W \subset U$ we have

$$\mu(W \sim f^{-1}V) \leqslant \mu(W \sim \{x\}) = \mu W - \mu\{x\} \leqslant \mu U - \mu\{x\} < \epsilon \cdot \mu\{x\} \leqslant \epsilon \cdot \mu W.$$

3.4. THEOREM. *If, for every $x \in X$, $N(x)$ is the family of all open neighbourhoods of x then*

$$\{x: \mu - \lim_{t \to x} f(t) \text{ exists}\} = A \cup B$$

where A is a G_δ and $B \subset \{x: \mu\{x\} > 0\}$.

Proof. For each $n \in \omega$, let

$$A_n = \Big\{x: \text{for every } y \in Y \text{ there is a neighbourhood } V \text{ of } y \text{ for which there exist arbitrarily small } W \in N(x) \text{ with } \mu(W \sim f^{-1}V) > \frac{1}{n+1}\mu W\Big\}.$$

$$E = \{x: \mu - \lim_{t \to x} f(t) \text{ exists}\}.$$

Then

$$X \sim E = \bigcup_{n \in \omega} A_n.$$

Let $\bar{A}_n$ denote the closure of A_n and $P = \{x: \mu\{x\} > 0\}$. We first show that

(i) $$A_n \supset \bar{A}_n \sim P.$$

Let $x \in \bar{A}_n \sim P$. Given $y \in Y$ and $U \in N(x)$, first choose $z \in U \cap A_n$ then a neighbourhood V of y, $W \in N(z)$ with $W \subset U$, and $\epsilon > 0$ so that

$$\mu(W \sim f^{-1}V) > \left(\frac{1}{n+1} + \epsilon\right) \mu W.$$

Next, since $x \notin P$, we have $\mu\{x\} = 0$ and hence there is an open W' with $x \in W' \subset U$ and

$$\frac{1}{n+1}\mu W' < \epsilon \cdot \mu W.$$

Then $x \in W \cup W' \subset U$ and

$$\mu((W \cup W') \sim f^{-1}V) \geqslant \mu(W \sim f^{-1}V) > \left(\frac{1}{n+1} + \epsilon\right) \mu W$$

$$> \frac{1}{n+1}(\mu W + \mu W') \geqslant \frac{1}{n+1}\mu(W \cup W')$$

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so that $x \in A_n$. This completes the proof of (i). Now, let

$$A = \bigcap_{n\epsilon\omega} (X \sim \bar{A}_n)$$

$$B = P \cap E.$$

Then A is a G_δ and

$$A = X \sim \bigcup_{n\epsilon\omega} \bar{A}_n \subset X \sim \bigcup_{n\epsilon\omega} A_n = E \subset X \sim \bigcup_{n\epsilon\omega} (\bar{A}_n \sim P) = \bigcap_{n\epsilon\omega} (X \sim \bar{A}_n) \cup P$$
$$= A \cup P.$$

Thus,

$$E = A \cup B.$$

3.5. THEOREM. *Suppose f is a μ-measurable function, $\epsilon > 0$, and Y has a countable base. Then there exists a set $C \subset X$ such that $\mu(X \sim C) < \epsilon$ and f is continuous on C.*

Proof. Let us denote the elements of the countable base of Y by V_i, $i \in \omega$. Then, the $f^{-1}V_i$ being μ-measurable, we can choose open U_i so that $f^{-1}V_i \subset U_i$ and

$$\mu(U_i \sim f^{-1}V_i) = \mu U_i - \mu(f^{-1}V_i) < \frac{\epsilon}{2^{i+1}}.$$

Let

$$A = \bigcup_{i\epsilon\omega} (U_i \sim f^{-1}V_i)$$

$$C = X \sim A.$$

Then $\mu A < \epsilon$ and f is continuous on C since for every $i \in \omega$

$$C \cap f^{-1}V_i = C \cap U_i,$$

that is, $f^{-1}V_i$ is open in C and hence for any open W in Y, $f^{-1}W$ is open in C.

3.6. COROLLARY. *Suppose f is a μ-measurable function, $\epsilon > 0$, Y has a countable base and, in addition to the general hypothesis* 3.1, *μ is such that open sets are μ-measurable. Then there exists a closed set $C \subset X$ such that $\mu(X \sim C) < \epsilon$ and f is continuous on C.*

Note. We cannot replace condition (i) by (ia) in hypothesis 3.1 without affecting the validity of 3.6.

In order to prove the next theorems we need stronger conditions on μ. We state them in the following definition suggested by the Vitali covering theorem.

3.7. DEFINITION. *μ has property (V) on S iff $S \subset X$, hypothesis* 3.1 *is satisfied, and there exists $\lambda < \infty$ such that, given any $A \subset S$ and family F of open sets containing, for every $x \in A$, arbitrarily small elements of $N(x)$, there exists a countable $F' \subset F$ with the following properties:*

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(i) $\mu(A \sim \sigma F') = 0$,
(ii) *for every* $B \subset \sigma F'$,

$$\sum_{W \in F'} \mu(B \cap W) \leqslant \lambda \cdot \mu B.$$

3.8. THEOREM. *Suppose μ has property (V) on S, Y has a countable base, and f is a μ-measurable function. Then, for μ-almost all $x \in S$, f is μ-continuous at x.*

Proof. For each $n \in \omega$, let $A_n = \Big\{x: x \in S$ and there exists a neighbourhood V of $f(x)$ for which there are arbitrarily small $W \in N(x)$ with

$$\mu(W \sim f^{-1}V) > \frac{1}{n+1}\mu W\Big\}.$$

Then

$$\{x: x \in S \text{ and } f \text{ is not } \mu\text{-continuous at } x\} = \bigcup_{n \in \omega} A_n$$

and hence it is enough to show that $\mu A_n = 0$ for $n \in \omega$. To this end, let $\epsilon > 0$ and, by 3.5 above, let $C \subset X$, $\mu(X \sim C) < \epsilon$ and f be continuous on C. Let $A' = A_n \cap C$ and, for each $x \in A'$, let V_x be a neighbourhood of $f(x)$ for which there are arbitrarily small $W \in N(x)$ with

$$\mu(W \sim f^{-1}V_x) > \frac{1}{n+1}\mu W.$$

Let $U_x \in N(x)$ and such that

$$C \cap U_x \subset f^{-1}V_x.$$

Let

$$F = \Big\{W: \text{for some } x \in A', W \in N(x), W \subset U_x \text{ and } \mu(W \sim f^{-1}V_x) > \frac{1}{n+1}\mu W\Big\}.$$

Then, since μ has property (V) on S, there is a $\lambda < \infty$ and a countable $F' \subset F$ such that $\mu(A' \sim \sigma F') = 0$ and for every $B \subset \sigma F'$

$$\sum_{W \in F'} \mu(B \cap W) \leqslant \lambda \cdot \mu B.$$

Let

$$B = \sigma F' \sim C.$$

Then $\mu B < \epsilon$ and for every $W \in F'$ there is an $x \in A'$ with $x \in W \subset U_x$ and

$$B \cap W = W \sim C \supset W \sim f^{-1}V_x$$

so that

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$$\mu(B \cap W) \geqslant \mu(W \sim f^{-1}V_x) > \frac{1}{n+1}\mu W.$$

Therefore

$$\lambda \cdot \epsilon > \lambda \cdot \mu B \geqslant \sum_{W \epsilon F'} \mu(B \cap W) > \frac{1}{n+1} \sum_{W \epsilon F'} \mu W \geqslant \frac{1}{n+1}\mu(\sigma F') \geqslant \frac{1}{n+1}\mu A'.$$

Hence

$$\mu A_n \leqslant \mu(A_n \cap C) + \mu(A_n \sim C) < \lambda(n+1)\epsilon + \epsilon.$$

Letting $\epsilon \to 0$ we conclude $\mu A_n = 0$.

3.9. THEOREM. *Suppose μ has property (V) on X, open sets and singletons in X are μ-measurable, and, for μ-almost all x, f is μ-continuous at x. Then f is a μ-measurable function.*

Proof. Let V be an open set in Y. To see that $f^{-1}V$ is μ-measurable, let

$$C = \{x: f \text{ is } \mu\text{-continuous at } x \text{ and } \mu\{x\} = 0\},$$
$$A = C \cap f^{-1}V.$$

Since $\{x: \mu\{x\} > 0\}$ is countable, we see that $X \sim C$, and hence $f^{-1}V \sim C$, is the union of a countable set and a set of measure zero. Thus $X \sim C$ and $f^{-1}V \sim C$ are μ-measurable. Hence, we only have to show that A is μ-measurable. Let $\epsilon > 0$ and

$$F = \{W: \text{for some } x \in A,\ W \in \mathcal{N}(x), \text{ and } \mu(W \sim f^{-1}V) \leqslant \epsilon\cdot\mu W\}.$$

Since μ has property (V), there exist $\lambda < \infty$ and countable $F' \subset F$ such that $\mu(A \sim \sigma F') = 0$ and

$$\sum_{W \epsilon F'} \mu W \leqslant \lambda\, .\mu(\sigma F').$$

Let

$$B = (\sigma F' \sim A) \cap C.$$

Then for any $W \in F'$

$$B \cap W = (W \sim A) \cap C \subset W \sim f^{-1}V$$

so that

$$\mu(B \cap W) \leqslant \epsilon\cdot\mu W.$$

Therefore

$$\mu B \leqslant \sum_{W \epsilon F'} \mu(B \cap W) \leqslant \epsilon\cdot \sum_{W \epsilon F'} \mu W \leqslant \epsilon\cdot\lambda\cdot\mu(\sigma F') \leqslant \epsilon\cdot\lambda\cdot\mu X.$$

Since ϵ is arbitrary and C, $\sigma F'$ and $(A \sim \sigma F')$ are μ-measurable, we conclude that for every $n \in \omega$ there exists a μ-measurable D_n such that $A \subset D_n$ and

$$\mu(D_n \sim A) \leqslant \frac{1}{n+1}.$$

Let

$$E = \bigcap_{n \in \omega} D_n.$$

Then E is μ-measurable, $A \subset E$, and $\mu(E \sim A) = 0$. Hence A is μ-measurable.

3.10. Corollary. *Suppose Y has a countable base, μ has property (V) on X, open sets and singletons in X are μ-measurable. Then f is a μ-measurable function iff, for μ-almost all x, f is μ-continuous at x.*

4. Differentiation. In this section we consider some applications of Theorem 3.8 to problems of density and differentiation. We first prove the following theorem which generalizes the classical result for Lebesgue measures in Euclidean n-space as well as results of Besicovitch **(1)** and Morse and Randolph **(7)**.

4.1. Remark. *Note that, for every $x \in X$, $N(x)$ is directed by $\subset$ and we can therefore take limits with respect to it.*

4.2. Theorem. *Let μ have property (V) on S and A be a μ-measurable set. Then, for μ-almost all $x \in S \sim A$,*

$$\lim_{W \in N(x)} \frac{\mu(A \cap W)}{\mu W} = 0.$$

Proof. Let f be the characteristic function of A. Then, by 3.8, for μ-almost all $x \in S$, f is μ-continuous at x. Let $B = \{x : x \in S \sim A$ and f is μ-continuous at $x\}$, $\epsilon > 0$, and V a neighbourhood of 0 that excludes 1. Then for every $x \in B$ there exists $U \in N(x)$ such that for every $W \in N(x)$ with $W \subset U$ we have

$$\mu(W \cap A) = \mu(W \sim f^{-1}V) \leqslant \epsilon \cdot \mu W.$$

Now, since μ has property (V) on S, we conclude

$$\mu\{x : x \in S \text{ and } \mu W = 0 \text{ for some } W \in N(x)\} = 0.$$

Hence, for μ-almost all $x \in B$ and therefore for μ-almost all $x \in S \sim A$,

$$\frac{\mu(W \cap A)}{\mu W} \leqslant \epsilon.$$

The above theorem can be extended provided property (V) is replaced by the following.

4.3. Definition. *μ has property (V') on S iff hypothesis 3.1 is satisfied; open sets are μ-measurable; for any $A \subset S$ and family F containing, for every $x \in A$, arbitrarily small $W \in N(x)$ there is a countable, disjoint $F' \subset F$ with $\mu(A \sim \sigma F') = 0$.*

4.4. Theorem. *Let μ and ν have property (V') on S, and A be a μ-measurable set. Then, for μ-almost all $x \in S \sim A$,*

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$$\lim_{W \in N(x)} \frac{\nu(A \cap W)}{\mu W} = 0.$$

Proof. Let ϕ be the restriction of ν to A, that is, for every $T \subset X$

$$\phi T = \nu(T \cap A).$$

Then A is ϕ-measurable and ϕ has property (V') on S. We shall show that $\mu + \phi$ has property (V) on S. Let $B \subset S$ and F be a family of open sets containing, for every $x \in B$, arbitrarily small $W \in N(x)$. Choose countable, disjoint subfamilies F' and F'' so that $\mu(B \sim \sigma F') = 0 = \phi(B \sim \sigma F'')$. Let $G = F' \cup F''$. Then G is countable, $(\mu + \phi)$ $(B \sim \sigma G) = 0$ and for any $C \subset \sigma G$

$$\sum_{W \in G} (\mu + \phi)(C \cap W) \leqslant \sum_{W \in F'} (\mu + \phi)(C \cap W) + \sum_{W \in F''} (\mu + \phi)(C \cap W) \leqslant 2(\mu + \phi)C.$$

Therefore, by 4.2, for $\mu + \phi$ almost all $x \in S \sim A$

$$\lim_{W \in N(x)} \frac{(\mu + \phi)(A \cap W)}{(\mu + \phi)W} = 0.$$

But for μ-almost all $x \in S \sim A$

$$\lim_{W \in N(x)} \frac{\mu(A \cap W)}{(\mu + \phi)W} \leqslant \lim_{W \in N(x)} \frac{\mu(A \cap W)}{\mu W} = 0.$$

Hence, for μ-almost all $x \in S \sim A$,

$$\lim_{W \in N(x)} \frac{\phi(A \cap W)}{(\mu + \phi)W} = 0.$$

But $\phi(A \cap W) = \phi W$. Therefore, for μ-almost all $x \in S \sim A$

$$\lim_{W \in N(x)} \frac{\phi W}{\mu W} = 0.$$

4.5. COROLLARY. *Suppose μ has property (V') on S; ν satisfies hypothesis* 3.1, *is absolutely continuous with respect to μ, and open sets are ν-measurable; A is a μ-measurable set. Then, for μ-almost all $x \in S \sim A$,*

$$\lim_{W \in N(x)} \frac{\nu(A \cap W)}{\mu W} = 0.$$

Proof. Note that if $\mu B = 0$ then there is a G_δ set B' such that $B \subset B'$ and $\mu B' = 0$. But B' is ν-measurable and, since ν is absolutely continuous with respect to μ, $\nu B' = 0$. Hence $\nu B = 0$. Thus ν has property (V') on S also and 4.4 applies.

4.6. THEOREM. *Let μ have property (V') on S, ν satisfy hypothesis* 3.1, *and open sets be ν-measurable; $\mu A = 0$ and $\nu(X \sim A) = 0$. Then, for μ-almost all $x \in S \sim A$,*

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$$\lim_{W \epsilon N(x)} \frac{\nu W}{\mu W} = 0.$$

Proof. Let $0 < K < \infty$ and

$$B = \left\{ x : x \in S \sim A \text{ and } \overline{\lim_{W \epsilon N(x)}} \frac{\nu W}{\mu W} > K \right\}.$$

Given $\epsilon > 0$, since $\nu B = 0$, let B' be open, $B \subset B'$ and $\nu B' < \epsilon$. Let

$$F = \{W\text{: for some } x \in B,\ W \in N(x),\ W \subset B', \text{ and } \nu W > K\mu W\}.$$

Then there is a countable disjoint $F' \subset F$ with $\mu(B \sim \sigma F') = 0$.
Since $\sigma F' \subset B'$ we have

$$\epsilon \geqslant \nu(\sigma F') = \sum_{W \epsilon F'} \nu W > K \sum_{W \epsilon F'} \mu W = K\mu(\sigma F') \geqslant K\mu B.$$

Since ϵ is arbitrary we must have $\mu B = 0$.

4.7. THEOREM. *Let μ have property (V') on S; ν satisfy hypothesis* 3.1 *and open sets be ν-measurable; A be a μ-measurable set. Then, for μ-almost all $x \in S \sim A$,*

$$\lim_{W \epsilon N(x)} \frac{\nu(A \cap W)}{\mu W} = 0.$$

Proof. Let $\nu = \nu_1 + \nu_2$ where ν_1 is absolutely continuous and ν_2 is singular with respect to μ. Then apply 4.5 and 4.6.

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