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EXAMPLES OF EXPANDING MAPS WITH SOME SPECIAL PROPERTIES

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Let I be the unit interval $[0,1]$ of the real line. For integers $k \geq 1$ and $n \geq 2$, we construct simple piecewise monotonic expanding maps $F_{k,n}$ in $C^0(I,I)$ with the following three properties: (1) The positive integer n is an expanding constant for $F_{k,n}$ for all k; (2) The topological entropy of $F_{k,n}$ is greater than or equal to $\log n$ for all k; (3) $F_{k,n}$ has periodic points of least period $2^k \cdot 3$, but no periodic point of least period $2^{k-1}(2m+1)$ for any positive integer m. This is in contrast to the fact that there are expanding (but not piecewise monotonic) maps in $C^0(I,I)$ with very large expanding constants which have exactly one fixed point, say, at $x = 1$, but no other periodic point.

1.

Let I be the unit interval $[0,1]$ of the real line and let $f \in C^0(I,I)$. For any positive integer n, let f^n denote the n^{th} iterate of f: $f^1 = f$ and $f^n = f \circ f^{n-1}$ for $n > 1$. A point $x_0 \in I$

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is called a periodic point of f if $f^m(x_0) = x_0$ for some positive integer m and the smallest such positive integer m is called the least period of x_0 under f.

The continuous map f is said to be piecewise monotonic if I can be divided into finite number of subintervals $I_1, I_2, \ldots, I_j$ on which f is either strictly increasing or strictly decreasing. The separating points $c_1, c_2, \ldots, c_{j-1}$ at which f has a local minimum or maximum are called the turning points of f. If there is a constant $\lambda > 1$ such that $|f(x) - f(y)| \geq \lambda|x - y|$ whenever both x and y belong to some interval on which f is monotonic, then we call f an expanding map and, in this case, call λ an expanding constant for f.

In [10], Zhang shows that if f is a piecewise monotonic expanding map in $C^0(I,I)$ with an expanding constant $\geq \lambda$, then f must have periodic points of least period $2^k \cdot 3$ for some nonnegative integer k. If f has the additional property that it has exactly one turning point, then in [3] (see also [6]), Byers shows that the integer k above depends only on λ, but not on the map f itself. Based on these results, it is natural to ask the following question: If f is a piecewise monotonic expanding map in $C^0(I,I)$ with an expanding constant $\geq \lambda$, must f have periodic points of least period $2^k \cdot 3$ for some nonnegative integer k with k depending only on λ, but not on the map f itself? On the other hand, if $g \in C^0(I,I)$ has a periodic point of least period $2^k(2m+1)$ for some integers $k \geq 0$ and $m \geq 1$, then it is well known [2] (see also [7]) that the topological entropy (see [1] for definition) of g is greater than or equal to $(\log \lambda_m)/2^k$, where λ_m is the (unique) positive zero of the polynomial $x^{2m+1} - 2x^{2m-1} - 1$. The converse is false as is shown in [5] and [8]. However, the counterexamples given in [5, p. 407] are rather complicated and those given in [8] are not piecewise monotonic.

In this note, we answer the above question negatively by indicating how to construct, for all integers $k \geq 1$ and $n \geq 2$, simple examples of piecewise monotonic expanding maps $F_{k,n}$ in $C^0(I,I)$ with an expanding constant $\geq n$ which have periodic points of least period $2^k \cdot 3$, but no periodic point of least period $2^{k-1}(2m+1)$ for any positive integer m. Furthermore, these maps $F_{k,n}$ turn out to have their topological entropy ([9], [10]) greater than or equal to $\log n$ for all positive integers k. As a side product, these maps $F_{k,n}$ also provide simple counterexamples to the converse of the above-mentioned result of Block et al on the lower bound of topological entropy.

2.

Let $h_{[a,b],[c,d]}$ denote the linear map from $[a,b]$ to $[c,d]$ which carries a into c and b into d. For any integer $n \geq 2$, let $a_n = 1/n^2$, $b_n = 1 - 2/n^2$, $c_n = 1 - 1/n^2$, $d_n = 1 - 1/n^3$, and define a continuous map g_n in $C^0(I,I)$ such that $g_n(0) = 0$, $g_n(a_n) = c_n$, $g_n(b_n) = d_n$, $g_n(c_n) = a_n$, $g_n(d_n) = b_n$, $g_n(1) = 1$, and g_n is linear on each component of the complement of the set $\{a_n, b_n, c_n, d_n\}$ in $[0,1]$.

Suppose f is a continuous map in $C^0(I,I)$ with fixed points 0 and 1. Define a continuous map $g_{n*}f \in C^0(I,I)$ by letting

$$(g_{n*}f)(x) = g_n(x) \qquad \text{for } x \in I - (a_n,b_n)$$

and

$$(g_{n*}f)(x) = (h_{I,[c_n,d_n]} \circ f \circ h_{[a_n,b_n],I})(x) \qquad \text{for } x \in [a_n,b_n].$$

In the sequel, when we write $g_{n*}f$, we always mean that f is a continuous map in $C^0(I,I)$ with fixed points 0 and 1. Now we can state the following result which can be easily proved (see also [4]).

LEMMA 1. *(1) If there is a periodic point of* $g_{n*}f$ *with least period* $m > 1$, *then* m *is even and there is a periodic point of* f *with least period* $m/2$.

(2) If x_0 *is a periodic point of* f *with least period* $m > 0$, *then* $y = h_{I,[a_n,b_n]}(x_0)$ *is a periodic point of* $g_{n*}f$ *with least period* $2m$.

Since the composition of two linear maps is also linear and the slope of the resulting linear map equals the product of the slopes of the respective (original) linear maps, the following result is obvious.

LEMMA 2. *If* f *is a piecewise monotonic expanding map with an expanding constant* $\geq n^4$, *then* $g_n * f$ *is a piecewise monotonic expanding map with an expanding constant* $\geq n$.

3.

Now, for any two integers $k \geq 1$ and $n \geq 2$, let $\beta = \beta(k,n) = n^{4^k}+1$ and let $f_{k,n}$ be the continuous map in $C^0(I,I)$ such that $f_{k,n}(j/\beta) = 0$ for all even j with $0 \leq j \leq \beta$, $f_{k,n}(j/\beta) = 1$ for all odd j with $0 \leq j \leq \beta$, and $f_{k,n}$ is linear on each component of the complement of the set $\{j/\beta \mid 0 \leq j \leq \beta\}$ in $[0,1]$.

For all integers $1 \leq j \leq k$, let $\alpha = \alpha(j,k,n) = n^{4^{(k-j)}}$ and let $F_{0,n} = f_{k,n}$ and $F_{j,n} = g_\alpha * F_{j-1,n}$. Then, by Lemmas 1 and 2, and by induction, we obtain the following result.

THEOREM 3. *For any integers* $k \geq 1$ *and* $n \geq 2$, *let* $F_{k,n}$ *be the continuous map in* $C^0(I,I)$ *defined above. Then* $F_{k,n}$ *is a piecewise monotonic expanding map with the integer* n *as an expanding constant which has periodic points of least period* $2^k \cdot 3$, *but no periodic point of least period* $2^{k-1}(2m+1)$ *for any positive integer* m .

We remark that it is quite easy to construct simple examples of continuous expanding (but not piecewise monotonic) maps in $C^0(I,I)$ with very large expanding constants which have exactly one fixed point, say, at $x = 1$, but no other periodic point.

Now since $F_{k,n}$ (defined as in the above theorem) is piecewise monotonic expanding with n as an expanding constant, it is easy [10] to see that, for any positive integer m, $(F_{k,n})^m$ is piecewise monotonic expanding with n^m as an expanding constant. Consequently, if $x,y \in I$

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belong to a subinterval of I on which $(F_{k,n})^m$ is monotonic then $1 \geq |(F_{k,n})^m(x) - (F_{k,n})^m(y)| \geq n^m|x-y|$. So, $|x-y| \leq n^{-m}$. Thus, if u_m denotes the number of maximal subintervals of I on which $(F_{k,n})^m$ is monotonic, then $u_m \geq n^m$. It then follows easily from a result of Misiurewicz and Szlenk [9] that the topological entropy of $F_{k,n}$ is greater than or equal to $\limsup_{m\to\infty} (\log u_m)/m \geq \log n$. Therefore, we have proved the following result.

THEOREM 4. *For any integers* $k \geq 1$ *and* $n \geq 2$, *let* $F_{k,n}$ *be the continuous map in* $C^0(I,I)$ *defined as in Theorem 3. Then the topological entropy of* $F_{k,n}$ *is greater than or equal to* $\log n$ *for all* k .

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